Torsion-free vacuum biconformal spaces

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Abstract

Details of the calculation, dropping the matter assumptions in JMP1997. The gauge choice here is also more powerful.

Contents

1 The conformal group

We begin with a review of the conformal group, focussing on bosonic representations. Consider a compactified, pseudo-Euclidean space with metric

$$\eta_{ab} = diag\left(1, \ldots, 1, -1, \ldots, -1\right)$$

having p positive and q negative eigenvalues, with $p + q = n$. For Euclidean spaces $(p = n)$, the compactification is accomplished with a single point. For Lorentzian spaces $(q = 1)$, the compactification requires a light cone at infinity. The conformal group is defined as the set of transformations leaving η_{ab} unchanged except for a possible overall function. It may be shown that $SO\left(p+1, q+1\right)$ provides a linear representation of this group. While the considerations below apply to any values of p and q,

Geometrically, we interpret the $\frac{(n+1)(n+2)}{2}$ transformations of the conformal group in the following way. First, $\frac{n(n-1)}{2}$ of the transformations correspond to rotations, Lorentz transformations, or, in general, pseudo-rotations in the underlying pseudo-Euclidean space. Next is a single dilatation, which is accomplished by a simple scaling. These together comprise the homothetic (or Weyl) subgroup. The remaining $2n$ transformations include n translations and n special conformal transformations. The special conformal transformations are translations of the compactifying point inverse to the origin.

When the Lie algebra of the conformal group is expressed in differential forms, we have connection 1-forms, ω^a_b for pseudo-rotations, ω^0_0 for dilatations, ω^a for translations and ω_a for translations. These satisfy the Maurer-Cartan structure equations,

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + 2\Delta^{ad}_{cb}\omega_d\omega^c && (1)\\
\mathbf{d}\omega^a &= \omega^b\omega^a_b + \omega^0_0\omega^a && (2)\\
\mathbf{d}\omega_a &= \omega^b_a\omega_b + \omega_a\omega^0_0 && (3)\\
\mathbf{d}\omega^0_0 &= \omega^a\omega_a && (4)
\end{aligned}$$

where

$$\Delta^{ad}_{cb} \equiv \frac{1}{2}\left(\delta^a_c\delta^d_b - \eta^{ad}\eta_{cb}\right)$$

and the antisymmetric wedge product is assumed between differential forms.

Flat biconformal spaces are now formed as the quotient of the conformal group by the homothetic group. The quotient provides a projection in the sense that each homothetic equivalence class projects to one point of the base manifold. We therefore have a fiber bundle with homothetic fibers and a $2n$-dim base manifold. The base manifold is spanned by ω^a and ω_a, now called the solder form and co-solder form, respectively.

Flat biconformal space is a scale-invariant symplectic manifold. The dilatational gauge field, ω_0^0, called the Weyl vector, guarantees scale invariance, while the corresponding structure equation,

$$\mathbf{d}\omega_0^0 = \omega^a \omega_a \tag{5}$$

is manifestly symplectic – the left side is closed while the right side is necessarily non-degenerate.

Curved biconformal spaces are now generated by passing to the Cartan structure equations. Starting from the Maurer-Cartan equations, generalize the connection. The use of a general connection in the Maurer-Cartan equations leads to

$$\begin{aligned}
\mathbf{d}\omega_b^a &= \omega_b^c \omega_c^a + 2\Delta_{cb}^{ad} \omega_d \omega^c + \mathbf{\Omega}_b^a & (6)\\
\mathbf{d}\omega^a &= \omega^b \omega_b^a + \omega_0^0 \omega^a + \mathbf{\Omega}^a & (7)\\
\mathbf{d}\omega_a &= \omega_a^b \omega_b + \omega_a \omega_0^0 + \mathbf{\Omega}_a & (8)\\
\mathbf{d}\omega_0^0 &= \omega^a \omega_a + \mathbf{\Omega}_0^0 & (9)
\end{aligned}$$

where the 2-forms $\mathbf{\Omega}_b^a, \mathbf{\Omega}^a, \mathbf{\Omega}_a$, and $\mathbf{\Omega}_0^0$, called curvatures, characterize the failure of the new connection to satisfy the original equations. These curvatures are subject to two conditions:

1. The curvatures must be horizontal. This means they must be 2-forms in ω^a and ω_a only. The condition is equivalent to the demand that the integral of the connection along any curve in the bundle is independent of lifting on the bundle.

2. The equations must remain integrable. The integrability condition is found by taking the exterior derivative of each structure equation, and results in the (four) Bianchi identities:

$$\begin{aligned}
0 &= \mathbf{d}\mathbf{\Omega}_b^a + \mathbf{\Omega}_b^c \omega_c^a - \omega_b^c \mathbf{\Omega}_c^a + 2\Delta_{cb}^{ad} \mathbf{\Omega}_d \omega^c + 2\Delta_{cb}^{ad} \omega_d \mathbf{\Omega}^c & (10)\\
0 &= \mathbf{d}\mathbf{\Omega}^a + \mathbf{\Omega}^b \omega_b^a - \omega_0^0 \mathbf{\Omega}^a - \omega^b \mathbf{\Omega}_b^a + \mathbf{\Omega}_0^0 \omega^a & (11)\\
0 &= \mathbf{d}\mathbf{\Omega}_a + \mathbf{\Omega}_a^b \omega_b - \omega_a^b \mathbf{\Omega}_b + \mathbf{\Omega}_a \omega_0^0 - \omega_a \mathbf{\Omega}_0^0 & (12)\\
0 &= \mathbf{d}\mathbf{\Omega}_0^0 + \mathbf{\Omega}^a \omega_a - \omega^a \mathbf{\Omega}_a & (13)
\end{aligned}$$

The 2-forms $\mathbf{\Omega}_b^a, \mathbf{\Omega}^a, \mathbf{\Omega}_a, \mathbf{\Omega}_0^0$ are called the Lorentz curvature, torsion, co-torsion and dilatational curvature, respectively.

The Cartan equations for several familiar submanifolds easily found from these structure equations. If the co-torsion is zero, we may consistently set the co-solder form to zero as well. This reduces the $2n$-dim geometry to an n-dim geomerty, on which we retain the structure equations for a Weyl geometry with torsion. Renaming

$$\begin{aligned} \omega^a &= \mathbf{e}^a \\ \omega_0^0 &= \mathbf{W} \\ \mathbf{\Omega}^a &= \mathbf{T}^a \\ \mathbf{\Omega}_b^a &= \mathbf{R}_b^a \end{aligned}$$

we have

$$\begin{aligned} \mathbf{d}\omega_b^a &= \omega_b^c \omega_c^a + \mathbf{R}_b^a & (14) \\ \mathbf{de}^a &= \mathbf{e}^b \omega_b^a + \mathbf{We}^a + \mathbf{T}^a & (15) \\ \mathbf{dW} &= \mathbf{\Omega}_0^0 & (16) \end{aligned}$$

If, in addition, the torsion vanishes, we have the structure equations of a Weyl geometry,

$$\begin{aligned} \mathbf{d}\omega_b^a &= \omega_b^c \omega_c^a + \mathbf{R}_b^a & (17) \\ \mathbf{de}^a &= \mathbf{e}^b \omega_b^a + \mathbf{We}^a & (18) \\ \mathbf{dW} &= \mathbf{\Omega}_0^0 & (19) \end{aligned}$$

Finally, if the dilatational curvature vanishes we may choose a gauge in which the Weyl vector vanishes, leaving the structure equations of an n-dim pseudo-Riemannian geometry,

$$\begin{aligned} \mathbf{d}\omega_b^a &= \omega_b^c \omega_c^a + \mathbf{R}_b^a & (20) \\ \mathbf{de}^a &= \mathbf{e}^b \omega_b^a & (21) \end{aligned}$$

The remaining piece of the Lorentz curvature,

$$\mathbf{R}_b^a = \frac{1}{2} R^a_{bcd} \mathbf{e}^c \mathbf{e}^d$$

is the usual Riemann curvature 2-form. This reduction guides us in finding the Einstein equation within more general biconformal spaces.

Notice that without the restriction to the $\omega_b = 0$ submanifolds, each of the curvatures has far more components. Generically,

$$\mathbf{\Omega}^A_B = \frac{1}{2}\mathbf{\Omega}^A_{Bcd}\omega^c\omega^d + \mathbf{\Omega}^{Ac}_{Bd}\omega_c\omega^d + \frac{1}{2}\mathbf{\Omega}^{Acd}_B\omega_c\omega_d$$

where $\binom{A}{B} \in \left\{\binom{a}{b}, \binom{a}{}, \binom{}{a}, \binom{0}{0}\right\}$. This is the reason that the reduction of the field equations presented below is somewhat lengthy. We will call the three terms of this expansion the spacetime curvature, cross-curvature, and momentum curvature, respectively.

We note that in certain dimensions biconformal spaces may be supersymmetrized. When $n = 4$, the supersymmetric structure equations become

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + 4\Delta^{ca}_{bd}\omega_c\omega^d + P^{\alpha\beta}\left[\sigma^a{}_b\right]_{AB}\chi^A_\alpha\psi^B_\beta && (22)\\
\mathbf{d}\omega^a &= \omega^c\omega^a_c + \omega\omega^a - \frac{1}{2}P^{\alpha\beta}\left[\gamma^a\right]_{(AB)}\psi^A_\alpha\psi^B_\beta && (23)\\
\mathbf{d}\omega_a &= \omega^c_a\omega_c + \omega_a\omega - \frac{1}{2}P^{\alpha\beta}\left[\gamma_a\right]_{(AB)}\chi^A_\alpha\chi^B_\beta && (24)\\
\mathbf{d}\omega &= 2\omega^a\omega_a + \frac{1}{2}P^{\alpha\beta}Q_{AB}\chi^A_\alpha\psi^B_\beta && (25)
\end{aligned}$$

These equations describe a superspace of 16 dimensions (8 bosonic + 8 fermionic). The bosonic sector agrees with eqs.(6-9) if we replace $\sqrt{2}\omega^a \to \omega^a$ and $\sqrt{2}\omega_a \to \omega_a$. Details of the supersymmetric case may be found in [ANDERSONWHEELER]

The remainder of our considerations refer to the purely bosonic case.

2 Field equations in biconformal space

In addition to the structure equations, we impose field equations derived from an action functional. The most general such functional which is of zero conformal weight and is linear in the biconformal curvatures is [WEHNERWHEELER]:

$$S = \int \left(\alpha\mathbf{\Omega}^a_b + \beta\mathbf{\Omega}^0_0\delta^a_b + \gamma\omega_b\omega^a\right)\varepsilon_{ac\cdots d}\varepsilon^{be\cdots f}\omega_e\cdots\omega_f\omega^c\cdots\omega^d$$

See [WW] for details.

The field equations follow by varying S with respect to each of the connection 1-forms. Generically we have

$$\delta\omega^A_B = f^A_{Ba}\omega^a + g^{Aa}_B\omega_a$$

where the coefficients of the variation, $f^A_{Ba}\omega^a$ and g^{Aa}_B, are arbitrary. Each of the four connection types therefore leads to two independent field equations.

The field equations following from the linear action are:

$$\begin{aligned}
\beta(\Omega^a{}_{ba} - 2\Omega_{ca}{}^d\delta^{ca}_{db}) &= 0 && (26)\\
\beta(\Omega_a{}^{ba} - 2\Omega^{cd}{}_a\delta^{ab}_{dc}) &= 0 && (27)\\
\alpha(-\Delta^{af}_{eg}\Omega^b{}_{ab} + 2\Delta^{cf}_{eb}\delta^{ab}_{dg}\Omega_{ac}{}^d) &= 0 && (28)\\
\alpha(-\Delta^{gf}_{eb}\Omega_a{}^{ab} + 2\Delta^{af}_{ed}\delta^{gc}_{ab}\Omega^{bd}{}_c) &= 0 && (29)\\
\alpha\Omega^a_{bac} + \beta\Omega^0_{0bc} &= 0 && (30)\\
2(\alpha\Omega^{ec}_{cd} + \beta\Omega^{0e}_{0d})\delta^{ad}_{eb} + \Lambda^a_b &= 0 && (31)\\
\alpha\Omega^{bac}_a + \beta\Omega^{0bc}_0 &= 0 && (32)\\
2(\alpha\Omega^{ce}_{dc} + \beta\Omega^{0e}_{0d})\delta^{ad}_{eb} + \Lambda^a_b &= 0 && (33)
\end{aligned}$$

and we have defined

$$\Lambda^a_b \equiv (\alpha(n-1) - \beta + \gamma n^2)\delta^a_b \qquad (34)$$

Equations (6-9) and (26-33) define a gravitational field theory on biconformal space. We seek a description of the torsion-free solutions to these equations.

Our derivation proceeds in two parts. First, the field equations, vanishing torsion, $\mathbf{\Omega}^a = 0$, and the Bianchi identities, provide algebraic constraints on the biconformal curvatures. These may be manipulated to simplify the form of the curvatures. Second, the structure equations may be partially integrated to further specify the solution. These parts are presented in the next two sections.

3 Algebraic part of the solution

Setting the torsion to zero,

$$\mathbf{\Omega}^a = 0$$

the structure equations reduce to

$$\begin{aligned}
\mathbf{\Omega}^a_b &= \mathbf{d}\omega^a_b - \omega^c_b\omega^a_c - 2\Delta^{ad}_{cb}\omega_d\omega^c &\quad (35)\\
0 &= \mathbf{d}\omega^a - \omega^b\omega^a_b - \omega^0_0\omega^a &\quad (36)\\
\mathbf{\Omega}_a &= \mathbf{d}\omega_a - \omega^b_a\omega_b - \omega_a\omega^0_0 &\quad (37)\\
\mathbf{\Omega}^0_0 &= \mathbf{d}\omega^0_0 - \omega^a\omega_a, &\quad (38)
\end{aligned}$$

while the first four field equations (eqs. 26-29) reduce to:

$$\begin{aligned}
-2\beta\Omega_{ca}{}^{d}\delta^{ca}_{db} &= 0 &\quad (39)\\
\beta\Omega_a{}^{ba} &= 0 &\quad (40)\\
2\alpha\Delta^{cf}_{eb}\delta^{ab}_{dg}\Omega_{ac}{}^{d} &= 0 &\quad (41)\\
-\alpha\Delta^{gf}_{eb}\Omega_a{}^{ab} &= 0 &\quad (42)
\end{aligned}$$

The remaining field equations are unchanged.

3.1 Bianchi identity for vanishing torsion

First, vanishing torsion leaves the corresponding Bianchi identity algebraic,

$$0 = \omega^b\mathbf{\Omega}^a_b - \mathbf{\Omega}^0_0\omega^a \quad (43)$$

This has components

$$\begin{aligned}
0 &= \Omega^a_{[bcd]} - \delta^a_{[b}\Omega^0_{0cd]}\\
0 &= \Omega^{ac}_{bd} - \Omega^{ac}_{db} - \delta^a_b\Omega^{0c}_{0d} + \delta^a_d\Omega^{0c}_{0b}\\
0 &= \Omega^{acd}_b - \delta^a_b\Omega^{0cd}_0 &\quad (44)
\end{aligned}$$

There are three independent trace equations. The ab trace of the first gives

$$\Omega^a_{bac} - \Omega^a_{cab} = -(n-2)\,\Omega^0_{0bc} \quad (45)$$

the ab trace of the second gives

$$\Omega^{ac}_{da} = -(n-1)\,\Omega^{0c}_{0d} \quad (46)$$

while the ab trace of the third gives

$$\Omega^{0cd}_0 = 0 \quad (47)$$

from which it immediately follows that

$$\Omega_{b}^{acd} = 0 \tag{48}$$

Consider eq.(46). Lower the first index and cycle the indices.

$$\begin{aligned}
0 &= \eta_{cf}\Omega_{bd}^{fa} - \eta_{cf}\Omega_{db}^{fa} - \eta_{cb}\Omega_{0d}^{0a} + \eta_{cd}\Omega_{0b}^{0a} \\
0 &= \eta_{df}\Omega_{cb}^{fa} - \eta_{df}\Omega_{bc}^{fa} - \eta_{dc}\Omega_{0b}^{0a} + \eta_{db}\Omega_{0c}^{0a} \\
0 &= \eta_{bf}\Omega_{dc}^{fa} - \eta_{bf}\Omega_{cd}^{fa} - \eta_{bd}\Omega_{0c}^{0a} + \eta_{bc}\Omega_{0d}^{0a} \qquad (49)
\end{aligned}$$

Now add the first two and subtract the third, using the antisymmetry of $\eta_{cf}\Omega_{bd}^{fa}$ on cb :

$$0 = 2\eta_{df}\Omega_{cb}^{fa} - 2\eta_{bc}\Omega_{0d}^{0a} + 2\eta_{bd}\Omega_{0c}^{0a} \tag{50}$$

or

$$\begin{aligned}
\Omega_{cb}^{da} &= \eta^{de}\eta_{bc}\Omega_{0e}^{0a} - \delta_{b}^{d}\Omega_{0c}^{0a} \\
&= -2\Delta_{bc}^{de}\Omega_{0e}^{0a} \qquad (51)
\end{aligned}$$

As a check, we note that the trace reproduces eq.(46)

$$\begin{aligned}
\Omega_{cb}^{ba} &= \eta^{be}\eta_{bc}\Omega_{0e}^{0a} - \delta_{b}^{b}\Omega_{0c}^{0a} \\
&= \Omega_{0c}^{0a} - n\Omega_{0c}^{0a} \qquad (52)
\end{aligned}$$

as required. Summarizing:

$$\begin{aligned}
\Omega_{b}^{acd} &= 0 &(53)\\
\Omega_{0}^{0cd} &= 0 &(54)\\
\Omega_{cd}^{ab} &= -2\Delta_{dc}^{ae}\Omega_{0e}^{0b} &(55)\\
\Omega_{bac}^{a} - \Omega_{cab}^{a} &= -(n-2)\,\Omega_{0bc}^{0} &(56)
\end{aligned}$$

Notice that, with the exception of the spacetime term, Ω_{bcd}^{a}, the curvature is totally determined by the dilatation.

The remaining Bianchi identities are still differential relations among the curvatures, rather than algebraic constraints. This completes the most useful consequences of the Bianchi identites.

3.2 Field equations for curvature

Next we look at the field equations for the curvatures (eqs. 30,31, and 33). Eq.(32) is identically satisfied by the Bianchi identities.

3.2.1 Eq.(30)

The symmetric and antisymmetric parts of eq.(30) are

$$\begin{aligned} \Omega^a_{bac} + \Omega^a_{cab} &= 0 & (57) \\ \alpha \left(\Omega^a_{bac} - \Omega^a_{cab}\right) &= -2\beta\Omega^0_{0bc} & (58) \end{aligned}$$

Combining the second of these with eq.(56) yields

$$\left[2\beta - (n-2)\,\alpha\right]\Omega^0_{0bc} = 0 \tag{59}$$

which, in turn, holds if and only if one of the factors vanishes. Thus, we have:

Case 1 (Generic): For generic coupling constants in the action we must have

$$\Omega^0_{0bc} = 0 \tag{60}$$

Case 2: (Exceptional): An exceptional case occurs if the coupling constants are related by

$$2\beta = (n-2)\,\alpha \tag{61}$$

We will examine both cases after studying the remaining field equations.

3.2.2 Equations (31) and (33)

Now consider eqs.(31 and (33). The difference between these shows that

$$\Omega^{ab}_{ca} = \Omega^{ba}_{ac} \tag{62}$$

From the previous Bianchi result, eq.(55),

$$\Omega^{ab}_{cd} = -2\Delta^{ae}_{dc}\Omega^{0b}_{0e} = -\delta^a_d\Omega^{0b}_{0c} + \eta^{ae}\eta_{dc}\Omega^{0b}_{0e}$$

we find the traces

$$\begin{aligned} \Omega^{ab}_{ca} &= -n\Omega^{0b}_{0c} + \Omega^{0b}_{0c} \\ \Omega^{ba}_{ac} &= -\delta^b_c\Omega^{0a}_{0a} + \eta^{be}\eta_{ca}\Omega^{0a}_{0e} \end{aligned}$$

and the double trace

$$\Omega^{ab}_{ba} = -(n-1)\Omega^{0b}_{0b}$$

From eq.(62), we have the equality of the two traces,

$$\begin{aligned} -n\Omega^{0b}_{0c} + \Omega^{0b}_{0c} &= -\delta^b_c \Omega^{0a}_{0a} + \eta^{be}\eta_{ca}\Omega^{0a}_{0e} \\ \eta_{dc}\Omega^{0a}_{0a} &= (n-1)\,\eta_{db}\Omega^{0b}_{0c} + \eta_{ca}\Omega^{0a}_{0d} \end{aligned}$$

so symmetrizing and antisymmetrizing, we find

$$n\eta_{db}\Omega^{0b}_{0c} + n\eta_{cb}\Omega^{0b}_{0d} = 2\eta_{dc}\Omega^{0a}_{0a} \tag{63}$$

$$(n-2)\left(\eta_{db}\Omega^{0b}_{0c} - \eta_{cb}\Omega^{0b}_{0d}\right) = 0 \tag{64}$$

so for $n \neq 2$ we get simply

$$\Omega^{0a}_{0b} = \frac{1}{n}\delta^a_b \Omega^{0c}_{0c} \tag{65}$$

Defining a function

$$\kappa \equiv \frac{1}{n}\Omega^{0c}_{0c} \tag{66}$$

we have simply

$$\Omega^{0a}_{0b} = \kappa\delta^a_b \tag{67}$$

For $n = 2$ the antisymmetric part remains undetermined, so we may write

$$\Omega^{0a}_{0b} = \kappa\delta^a_b + \Delta^{ac}_{db}\Omega^{0d}_{0c} \tag{68}$$

Case 1: $\mathbf{n > 2}$ Now, using eq.(55) from the torsion Bianchi identity, we find the full cross-curvature,

$$\begin{aligned} \Omega^{ab}_{cd} &= -2\Delta^{ae}_{dc}\Omega^{0b}_{0e} && (69) \\ &= -2\kappa\Delta^{ab}_{dc} && (70) \end{aligned}$$

From the remaining field equation involving the cross term of the curvature, eq.(31),

$$2\left(\alpha\Omega^{ec}_{cd} + \beta\Omega^{0e}_{0d}\right)\delta^{ad}_{eb} + \Lambda^a_b = 0 \tag{71}$$

we now find

$$(n-1)\left(\alpha\left(n-1\right) - \beta\right)\kappa\delta^a_b + \Lambda^a_b = 0 \tag{72}$$

where

$$\Lambda^a_b \equiv \left(\alpha(n-1) - \beta + \gamma n^2\right)\delta^a_b \tag{73}$$

Therefore,

$$\kappa = -\frac{1}{(n-1)} - \frac{\gamma n^2}{(n-1)\left(\alpha\left(n-1\right)-\beta\right)} \tag{74}$$

$$= -\frac{1}{(n-1)}\left(1+\frac{\gamma n^2}{\alpha\left(n-1\right)-\beta}\right) \tag{75}$$

There is one exceptional case, when

$$\alpha\left(n-1\right)-\beta = 0$$

In this case,

$$0 = \Lambda^a_b \equiv \left(\alpha(n-1)-\beta+\gamma n^2\right)\delta^a_b = \gamma n^2 \delta^a_b \tag{76}$$

so the exceptional case, eq.(61), can only occur if $\gamma = 0$ in the original Lagrangian.

Collecting results for $n > 2$ we have:

$$\Omega^{ab}_{cd} = -2\kappa\Delta^{ab}_{dc} \tag{77}$$

$$\Omega^{0a}_{0b} = \kappa\delta^a_b \tag{78}$$

$$\Omega^{acd}_{b} = 0 \tag{79}$$

$$\Omega^{0cd}_{0} = 0 \tag{80}$$

$$\kappa = -\frac{1}{(n-1)}\left(1+\frac{\gamma n^2}{\alpha\left(n-1\right)-\beta}\right) \tag{81}$$

and either the generic case

$$\begin{aligned} \Omega^a_{bac} &= 0 \\ \Omega^0_{0bc} &= 0 \end{aligned}$$

or the special case,

$$\begin{aligned} \alpha\left(n-1\right)-\beta &= 0 \\ \gamma &= 0 \end{aligned}$$

Case 2: $n = 2$ If $n = 2$ then

$$\Omega^{0a}_{0b} = \frac{1}{2}\delta^a_b\Omega^{0c}_{0c} + \Delta^{ac}_{db}\Omega^{0d}_{0c} \tag{82}$$

replaces eq.(78).

Check the $n = 2$ case: Define a function

$$\kappa \equiv \frac{1}{2}\Omega^{0c}_{0c} \tag{83}$$

then

$$\Omega^{0a}_{0b} = \kappa\delta^a_b + \Delta^{ac}_{db}\Omega^{0d}_{0c} \tag{84}$$

Now, using eq.(55) from the torsion Bianchi identity, we find the full cross-curvature,

$$\begin{aligned}
\Omega^{ab}_{cd} &= -2\Delta^{ae}_{dc}\Omega^{0b}_{0e} && (85)\\
&= -2\Delta^{ae}_{dc}\left(\kappa\delta^b_e + \Delta^{bg}_{fe}\Omega^{0f}_{0g}\right) && (86)\\
&= -2\kappa\Delta^{ab}_{dc} - 2\Delta^{ae}_{dc}\Delta^{bg}_{fe}\Omega^{0f}_{0g} && (87)
\end{aligned}$$

From the remaining field equation involving the cross term of the curvature, eq.(31),

$$2\left(\alpha\Omega^{ec}_{cd} + \beta\Omega^{0e}_{0d}\right)\delta^{ad}_{eb} + \Lambda^a_b = 0 \tag{88}$$

we now find

$$\begin{aligned}
2\left(\alpha\left(-2\kappa\Delta^{ec}_{dc} - 2\Delta^{em}_{dc}\Delta^{cg}_{fm}\Omega^{0f}_{0g}\right) + \beta\left(\kappa\delta^e_d + \Delta^{eg}_{fd}\Omega^{0f}_{0g}\right)\right)\delta^{ad}_{eb} + \Lambda^a_b &= 0 && (89)\\
\left(-4\alpha\kappa\Delta^{ec}_{dc} - 4\alpha\Delta^{em}_{dc}\Delta^{cg}_{fm}\Omega^{0f}_{0g} + 2\beta\left(\kappa\delta^e_d + \Delta^{eg}_{fd}\Omega^{0f}_{0g}\right)\right)\delta^{ad}_{eb} + \Lambda^a_b &= 0 && (90)
\end{aligned}$$

The extra terms reduce to

$$-4\alpha\Delta^{em}_{dc}\Delta^{cg}_{fm}\Omega^{0f}_{0g}\delta^{ad}_{eb} + 2\beta\Delta^{eg}_{fd}\Omega^{0f}_{0g}\delta^{ad}_{eb} = (-\alpha+\beta)\,\Delta^{ag}_{fb}\Omega^{0f}_{0g}$$

so the full expression is

$$(\alpha-\beta)\,\kappa\delta^a_b + (-\alpha+\beta)\,\Delta^{ag}_{fb}\Omega^{0f}_{0g} + \Lambda^a_b = 0$$

and the symmetric and antisymmetric parts now give two equations,

$$\begin{aligned}
(\alpha-\beta)\,\kappa\delta^a_b + \Lambda^a_b &= 0 && (91)\\
(-\alpha+\beta)\,\Delta^{ag}_{fb}\Omega^{0f}_{0g} &= 0 && (92)
\end{aligned}$$

Therefore, κ has the same constant value as before and the antisymmetric part of the cross-term vanishes,

$$\begin{aligned}
\kappa &= -\left(1 + \frac{4\gamma}{\alpha-\beta}\right) && (93)\\
\Omega^{0a}_{0b} &= \kappa\delta^a_b && (94)
\end{aligned}$$

unless $\alpha = \beta$. In this one exceptional case,

$$\alpha = \beta$$

In this case, we find

$$0 = \Lambda^a_b \equiv 4\gamma\delta^a_b \tag{95}$$

so the exceptional case, $\alpha = \beta$, can only occur if $\gamma = 0$ in the original Lagrangian. In this one special case, both κ and the antisymmetric part of the cross-term of the dilatational curvature remain arbitrary.

This completes the algebraic part of the solution of the curvature equations.

3.2.3 Co-torsion field equations

Next, we turn to the remaining field equations which now contain only co-torsion terms when we set the torsion to zero:

$$\begin{aligned}
2\beta\Omega_{ca}{}^{d}\delta^{ca}_{db} &= 0 &(96)\\
\beta\Omega_{a}{}^{ba} &= 0 &(97)\\
2\alpha\Delta^{cf}_{eb}\delta^{ab}_{dg}\Omega_{ac}{}^{d} &= 0 &(98)\\
-\alpha\Delta^{gf}_{eb}\Omega_{a}{}^{ab} &= 0 &(99)
\end{aligned}$$

The second and fourth equations show that $\Omega_{a}{}^{ab} = 0$. Now consider the remaining two:

$$\begin{aligned}
\beta\left(\Omega_{ab}{}^{a} - \Omega_{ba}{}^{a}\right) &= 0 &(100)\\
\alpha\Delta^{cf}_{eb}\left(\delta^b_g\Omega_{ac}{}^{a} - \delta^a_g\Omega_{ac}{}^{b}\right) &= 0 &(101)
\end{aligned}$$

Expanding,

$$\Omega_{ge}{}^{f} - \delta^f_g\Omega_{ae}{}^{a} - \eta_{eb}\eta^{cf}\left(\Omega_{gc}{}^{b} - \delta^b_g\Omega_{ac}{}^{a}\right) = 0 \tag{102}$$

we trace fg:

$$-\left(n-2\right)\Omega_{ae}{}^{a} = \eta_{eb}\eta^{cf}\Omega_{fc}{}^{b} \tag{103}$$

Summarizing:

$$\begin{aligned}
\Omega_{a}{}^{ab} &= 0 &(104)\\
\beta\left(\Omega_{ab}{}^{a} - \Omega_{ba}{}^{a}\right) &= 0 &(105)\\
\alpha\Delta^{cf}_{eb}\left(\delta^b_g\Omega_{ac}{}^{a} - \delta^a_g\Omega_{ac}{}^{b}\right) &= 0 &(106)\\
-\left(n-2\right)\Omega_{ae}{}^{a} &= \eta_{eb}\eta^{cf}\Omega_{fc}{}^{b} &(107)
\end{aligned}$$

3.3 Summary of algebraic part:

We first summarize the generic case. For all n and generic values of the constants α, β and γ, we have:

Curvature:

$$\begin{aligned} \Omega_b^{acd} &= 0 & (108) \\ \Omega_{cd}^{ab} &= -2\kappa\Delta_{dc}^{ab} & (109) \\ \Omega_{cab}^{a} &= 0 & (110) \end{aligned}$$

Torsion:

$$\mathbf{\Omega}^a = 0 \tag{111}$$

Dilatation:

$$\begin{aligned} \Omega_0^{0cd} &= 0 & (112) \\ \Omega_{0b}^{0a} &= \kappa\delta_b^a & (113) \\ \Omega_{0bc}^{0} &= 0 & (114) \end{aligned}$$

Co-torsion:

$$\begin{aligned} \Omega_a^{\ ab} &= 0 & (115) \\ \alpha\Delta_{eb}^{cf}\left(\Omega_{gc}^{\ \ b} - \delta_g^b\Omega_{ac}^{\ \ a}\right) &= 0 & (116) \\ \beta\left(\Omega_{cb}^{\ \ c} - \Omega_{ba}^{\ \ a}\right) &= 0 & (117) \\ -\left(n-2\right)\Omega_{ae}^{\ \ a} &= \eta_{eb}\eta^{cf}\Omega_{fc}^{\ \ b} & (118) \end{aligned}$$

where

$$\kappa = -\frac{1}{(n-1)}\left(1 + \frac{\gamma n^2}{\alpha\left(n-1\right) - \beta}\right) \tag{119}$$

There is one special case, when $2\beta - (n-2)\,\alpha = 0$. In this case, the spacetime dilatational curvature, Ω^0_{0bc}, is undetermined and

$$\begin{aligned} \Omega^a_{bac} + \Omega^a_{cab} &= 0 & (120) \\ \alpha\left(\Omega^a_{bac} - \Omega^a_{cab}\right) &= -2\beta\Omega^0_{0bc} & (121) \end{aligned}$$

4 Differential part of the solution

Now we look at the structure equations. We begin with the generic case in its entirety. In a subsequent Section we consider the special cases above.

First, we substitute what we know of the curvatures into the structure equations:

$$\begin{aligned} \frac{1}{2}\Omega^a_{bcd}\omega^c\omega^d - 2\kappa\Delta^{ac}_{db}\omega_c\omega^d &= \mathbf{d}\omega^a_b - \omega^c_b\omega^a_c - 2\Delta^{ad}_{cb}\omega_d\omega^c & (122) \\ 0 &= \mathbf{d}\omega^a - \omega^b\omega^a_b - \omega^0_0\omega^a & (123) \\ \boldsymbol{\Omega}_a &= \mathbf{d}\omega_a - \omega^b_a\omega_b - \omega_a\omega^0_0 & (124) \\ \kappa\omega_a\omega^a &= \mathbf{d}\omega^0_0 - \omega^a\omega_a, & (125) \end{aligned}$$

with the additional condition

$$\Omega^a_{bac} = 0 \qquad (126)$$

for the curvature and

$$\begin{aligned} \Omega_a{}^{ab} &= 0 & (127) \\ \alpha\Delta^{cf}_{eb}\left(\Omega_{gc}{}^b - \delta^b_g\Omega_{ac}{}^a\right) &= 0 & (128) \\ \beta\left(\Omega_{cb}{}^c - \Omega_{ba}{}^a\right) &= 0 & (129) \\ -(n-2)\,\Omega_{ae}{}^a &= \eta_{eb}\eta^{cf}\Omega_{fc}{}^b & (130) \end{aligned}$$

for the co-torsion. The coefficient κ is constant.

4.0.1 Dilatation

Let's look at the dilatation. First, in the generic case (where $\Omega^0_{0bc} = 0$) we have

$$\begin{aligned} \kappa\omega_a\omega^a &= \mathbf{d}\omega^0_0 - \omega^a\omega_a & (131) \\ \mathbf{d}\omega^0_0 &= (1-\kappa)\,\omega^a\omega_a & (132) \end{aligned}$$

with Bianchi identity,

$$\begin{aligned}
0 &= \mathbf{d}\left(\left(1-\kappa\right)\omega^a\ \omega_a\right) && (133)\\
&= \left(1-\kappa\right)\mathbf{d}\omega^a\omega_a - \left(1-\kappa\right)\omega^a\mathbf{d}\omega_a && (134)\\
&= \left(1-\kappa\right)\left(\omega^b\omega^a_b + \ \omega^0_0\omega^a\right)\ \omega_a \\
&\quad -\left(1-\kappa\right)\omega^a\left(\mathbf{\Omega}_a + \omega^b_a\omega_b + \omega_a\ \omega^0_0\right) && (135)\\
&= -\left(1-\kappa\right)\omega^a\mathbf{\Omega}_a && (136)
\end{aligned}$$

so that

$$-\left(1-\kappa\right)\omega^a\mathbf{\Omega}_a = 0 \tag{137}$$

Evidently, we have another special case if $\kappa = 1$. Using eq.(75) for κ, this only happens when

$$\begin{aligned}
n-1 &= -1-\frac{\gamma n^2}{\alpha\left(n-1\right)-\beta} && (138)\\
\alpha\left(n-1\right)-\beta &= -\gamma n && (139)
\end{aligned}$$

Once again we have a special choice of the constants in the action. We will therefore treat the $\kappa = 1$ case at the end with the other special cases.

Continuing with the generic case. From eq.(137) and $\kappa \neq 1$, it follows that

$$\begin{aligned}
\Omega_{[abc]} &= 0 && (140)\\
\Omega_{[ac]}^{\ \ b} &= 0 && (141)\\
\Omega_a^{\ bc} &= 0 && (142)
\end{aligned}$$

In addition we have

$$\begin{aligned}
\alpha\Delta^{cf}_{eb}\left(\Omega_{gc}^{\ \ b} - \delta^b_g\Omega_{ac}^{\ \ a}\right) &= 0 && (143)\\
-\left(n-2\right)\Omega_{ae}^{\ \ a} &= \eta_{ea}\eta^{bc}\Omega_{bc}^{\ \ a} && (144)
\end{aligned}$$

4.0.2 The solder form and its involution

Next consider the torsion equation. With the torsion vanishing, the solder form ω^a is in involution,

$$\mathbf{d}\omega^a = \omega^b \omega_b^a + \omega_0^0 \omega^a$$

Therefore, by the Fröbenius theorem, there exist n coordinates such that

$$\omega^a \equiv \mathbf{e}^a = \mathbf{e}_\mu{}^a \mathbf{d}x^\mu$$

where we introduce the usual notation for the solder form to denote this particular class of coordinate choices. Holding x^μ constant so that $\omega^a = 0$ gives a set of submanifolds spanned by the co-solder form, ω_a. On these submanifolds, the structure equations reduce to

$$\begin{aligned} \mathbf{d}\omega_b^a &= \omega_b^c \omega_c^a & (145) \\ \mathbf{d}\omega_a &= \omega_a^b \omega_b + \omega_a \omega_0^0 & (146) \\ \mathbf{d}\omega_0^0 &= 0 & (147) \end{aligned}$$

These equations describe a flat Weyl geometry. By performing a suitable local Lorentz transformation and local dilatation, we may choose a basis for this geometry with vanishing connection,

$$\begin{aligned} \omega_b^a &= 0 \\ \omega_0^0 &= 0 \end{aligned} \qquad (148)$$

The first and third equations are satisfied, while the second becomes

$$\mathbf{d}\omega_a = 0$$

with solution

$$\omega_a = \mathbf{d}\theta_a = \partial^\alpha \theta_a \mathbf{d}y_\alpha$$

We denote this restriction of the co-solder form by $\mathbf{f}_a$. Notice that the exterior derivative is restricted to the y_α coordinates since we are still on the submanifold. Since we got to this form by making a Lorentz transformation and a dilatation, holding x fixed, the transformations required may be different at different values of x^μ. The functions θ_a therefore depend on all $2n$ coordinates.

Having made a partial gauge choice, we may now allow x^μ to vary. Each connection or basis 1-form will acquire a piece proportional to $\mathbf{e}^a$. The forms therefore extend back to a basis for the full biconformal space in the form

$$
\begin{aligned}
\omega^a_b &= \omega^a_{bc}\mathbf{e}^c && (149)\\
\omega^a &= \mathbf{e}^a && (150)\\
&= e_\alpha{}^{\ a}(x,y)\,\mathbf{d}x^\alpha && (151)\\
\omega_a &= \mathbf{f}_a + b_{ab}\mathbf{e}^b && (152)\\
&= \partial^\alpha\theta_a(x,y)\,\mathbf{d}y_\alpha + b_{a\beta}(x,y)\,\mathbf{d}x^\beta && (153)\\
\omega^0_0 &= W_c\mathbf{e}^c = W_\beta\mathbf{d}x^\beta && (154)
\end{aligned}
$$

Both $e_\alpha{}^{\ a}$ and $f_a{}^{\ \alpha}$ must be invertible.

We can immediately restrict the functional dependence of the solder form by examining its structure equation with this form of the connection:

$$
\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega^a_b + \omega^0_0\mathbf{e}^a\\
&= \omega^a_{bc}\mathbf{e}^b\mathbf{e}^c + W_b\mathbf{e}^b\mathbf{e}^a
\end{aligned}
$$

Since the right side is quadratic in the solder form, the $\partial^\alpha\mathbf{e}^a(x,y)\,\mathbf{d}y_\alpha$ contribution to the exterior derivative on the left vanishes. Therefore, the solder form depends on x only, $\mathbf{e}^a = \mathbf{e}^a(x)$.

We can simplify the form of the co-solder form as well. First, observe that there is total freedom in the choice of the coordinate functions θ_a. Extracting an inverse solder form $e_a{}^{\ \mu}(x)$ and a constant, a, from both θ_a and $b_{a\beta}$,

$$
\begin{aligned}
\omega_a &= \partial^\alpha\theta_a(x,y)\,\mathbf{d}y_\alpha + b_{a\beta}(x,y)\,\mathbf{d}x^\beta\\
&= ae_a{}^{\ \mu}\left(\partial^\alpha\theta_\mu\mathbf{d}y_\alpha + b_{\mu\beta}(x,y)\,\mathbf{d}x^\beta\right)
\end{aligned}
$$

we rewrite the y_α derivative of θ_μ as an exterior derivative on the full biconformal space,

$$
\begin{aligned}
\omega_a &= ae_a{}^{\ \mu}\left(\partial^\alpha\theta_\mu\mathbf{d}y_\alpha + b_{\mu\beta}(x,y)\,\mathbf{d}x^\beta\right) && (155)\\
&= ae_a{}^{\ \mu}\left(\mathbf{d}\theta_\mu - \partial_\beta\theta_\mu\mathbf{d}x^\beta + b_{\mu\beta}\mathbf{d}x^\beta\right) && (156)\\
&= ae_a{}^{\ \mu}\left(\mathbf{d}\theta_\mu + (b_{\mu\beta} - \partial_\beta\theta_\mu)\,\mathbf{d}x^\beta\right) && (157)
\end{aligned}
$$

we see that coordinate freedom on the $\mathbf{e}^a = 0$ submanifold leads to a change in $b_{\alpha\beta}$. We use this freedom below to simplify the form of $b_{\alpha\beta}$.

4.0.3 Back to the structure equations

In the following subsections, we work systematically through the structure equations to arrive at a final form for the connection.

The solder form structure equation We begin with eq.(123) for the solder form,

$$\mathbf{de}^a = \mathbf{e}^b \omega_b^a + \omega_0^0 \mathbf{e}^a \tag{158}$$

as follows. Let

$$\omega_b^a = \alpha_b^a + \beta_b^a \tag{159}$$

where we define α_b^a by

$$\mathbf{de}^a = \mathbf{e}^b \alpha_b^a \tag{160}$$

so that α_b^a is the usual x-dependent spin connection for the x-dependent solder form $e_\alpha{}^a(x)\mathbf{d}x^\alpha$. Then we have

$$0 = \mathbf{e}^b \beta_b^a + \omega_0^0 \mathbf{e}^a \tag{161}$$

Expanding in components, we find

$$\beta_{bc}^a = -2\Delta_{cb}^{ad} W_d \tag{162}$$

Checking,

$$\begin{aligned}
\mathbf{e}^b \beta_b^a + \omega_0^0 \mathbf{e}^a &= -\mathbf{e}^b 2\Delta_{cb}^{ad} W_d \mathbf{e}^c + W_b \mathbf{e}^b \mathbf{e}^a \\
&= -\mathbf{e}^b \left(\delta_c^a \delta_b^d - \eta^{ad}\eta_{cb}\right) W_d \mathbf{e}^c + W_b \mathbf{e}^b \mathbf{e}^a \\
&= -W_b \mathbf{e}^b \mathbf{e}^a + \eta^{ad}\eta_{cb} W_d \mathbf{e}^b \mathbf{e}^c + W_b \mathbf{e}^b \mathbf{e}^a \\
&= 0
\end{aligned} \tag{163}$$

Thus,

$$\omega_b^a = \alpha_b^a - 2\Delta_{db}^{ac} W_c \mathbf{e}^d \tag{164}$$

The dilatation equation Next, consider eq.(125) for the dilatation,

$$\mathbf{d}\omega_0^0 = (1-\kappa)\,\omega^a \omega_a \tag{165}$$

Expanding the right side,

$$\begin{aligned}
\mathbf{d}\omega_0^0 &= (1-\kappa)\,\omega^a \omega_a & (166)\\
\mathbf{d}\left(W_\beta \mathbf{d}x^\beta\right) &= (1-\kappa)\left(\mathbf{e}_\alpha{}^a \mathbf{d}x^\alpha\right) a e_a{}^\mu \left(\mathbf{d}\theta_\mu + \left(b_{\mu\beta} - \partial_\beta \theta_\mu\right)\mathbf{d}x^\beta\right) & (167)\\
&= (1-\kappa)\, a\left(\mathbf{d}x^\mu \mathbf{d}\theta_\mu + \left(b_{\mu\beta} - \partial_\beta\theta_\mu\right)\mathbf{d}x^\mu \mathbf{d}x^\beta\right) & (168)
\end{aligned}$$

Choosing $a = (1-\kappa)^{-1} = const.$, setting

$$\tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \partial_\beta \theta_\alpha$$

and expanding the exterior derivative,

$$\partial^\alpha W_\beta \mathbf{d}\theta_\alpha \mathbf{d}x^\beta + \partial_\alpha W_\beta \mathbf{d}x^\alpha \mathbf{d}x^\beta = \mathbf{d}x^\beta \mathbf{d}\theta_\beta + \tilde{b}_{\alpha\beta}\mathbf{d}x^\alpha \mathbf{d}x^\beta \tag{169}$$

we equate the coefficients of like terms

$$\begin{aligned} \partial_\alpha W_\beta - \partial_\beta W_\alpha &= \tilde{b}_{\alpha\beta} - \tilde{b}_{\beta\alpha} & (170) \\ \partial^\alpha W_\beta &= -\delta^\beta_\alpha & (171) \end{aligned}$$

The second of these, eq.(171), may be integrated immediately to give

$$W_\beta = -\theta_\beta + g_\beta(x) \tag{172}$$

This form is independent of the choice for the θ_β coordinates. Substituting this solution into eq.(170),

$$\partial_\alpha g_\beta - \partial_\beta g_\alpha = b_{\alpha\beta} - \partial_\beta \theta_\alpha - b_{\beta\alpha} + \partial_\alpha \theta_\beta \tag{173}$$

Rearranging,

$$b_{\alpha\beta} - b_{\beta\alpha} = \partial_\beta (\theta_\alpha - g_\alpha) - \partial_\alpha (\theta_\beta - g_\beta) \tag{174}$$

we see that the antisymmetric part of $b_{\alpha\beta}$ is the curl of $\theta_\alpha - g_\alpha$. Since the choice of θ_α is arbitrary, we may choose it to make this curl vanish by setting the coordinate on the $\mathbf{e}^a = 0$ submanifolds to be

$$y_\alpha = \theta_\alpha - g_\alpha$$

With this choice $b_{\alpha\beta}$ is symmetric and the Weyl vector takes the simple form

$$W_\beta = -y_\beta$$

As an aside, we note that this transformation is a symplectic transformation that relates the minimally coupled form of the momentum to the Newtonian momentum. See [WHEELER JMP], where g_α is interpreted as the electromagnetic vector potential, and the argument above shows that it is possible to choose phase space coordinates to remove it.

Dropping the tilde on $b_{\alpha\beta}$, and using the solder form and inverse solder form to exchange coordinate and orthonormal indices so that

$$\begin{aligned} b_{ab} &= e_a{}^{\alpha} e_b{}^{\beta} b_{\alpha\beta} \\ W_a &= e_a{}^{\alpha} W_{\alpha} \end{aligned}$$

the connection takes the form

$$\begin{aligned} \omega^a_b &= \alpha^a_b - 2\Delta^{ac}_{db} W_c \mathbf{e}^d && (175) \\ \omega^a &= \mathbf{e}^a(x) && (176) \\ \omega_a &= a\left(\mathbf{f}_a + \mathbf{b}_a\right) \\ &= a e_a{}^{\beta}(x)\ \mathbf{d}y_\beta + a b_{ab}\mathbf{e}^b && (177) \\ \omega^0_0 &= W_c \mathbf{e}^c = -y_\beta \mathbf{d}x^\beta && (178) \\ a &= (1-\kappa)^{-1} && (179) \\ b_{ab} &= b_{ba} && (180) \end{aligned}$$

This form is unchanged if we perform any x-dependent Lorentz transformation or scaling.

The curvature equation Now consider the curvature equation

$$\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d - 2\kappa\Delta^{ac}_{db}\omega_c\mathbf{e}^d = \mathbf{d}\omega^a_b - \omega^c_b\omega^a_c - \Delta^{ad}_{cb}\ \omega_d\mathbf{e}^c \quad (181)$$

Define the curvature of $\alpha^a_b\left(e\left(x\right)\right)$ in the usual way,

$$\mathbf{R}^a_b = \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c \quad (182)$$

In particular, $\mathbf{R}^a_b$ depends only on x. Expanding,

$$\begin{aligned} \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d - 2\kappa\Delta^{ac}_{db}\omega_c\mathbf{e}^d &= \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c - 2\Delta^{ac}_{db}\omega_c\mathbf{e}^d \\ &\quad + \mathbf{d}\beta^a_b - \alpha^c_b\beta^a_c - \beta^c_b\alpha^a_c - \beta^c_b\beta^a_c && (183) \\ &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\omega_c\mathbf{e}^d \\ &\quad + \mathbf{d}\beta^a_b - \alpha^c_b\beta^a_c - \beta^c_b\alpha^a_c - \beta^c_b\beta^a_c && (184) \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\left(1-\kappa\right)\Delta^{ac}_{db}\omega_c\mathbf{e}^d \\ &\quad + \mathbf{d}\beta^a_b - \alpha^c_b\beta^a_c - \beta^c_b\alpha^a_c - \beta^c_b\beta^a_c \end{aligned}$$

The exterior derivative is

$$\begin{aligned}
\mathbf{d}\beta^a_b &= \mathbf{d}\left(-2\Delta^{ac}_{db} e_c \;\; {}^\mu W_\mu \mathbf{e}^d\right) \\
&= \mathbf{d}\left(2\Delta^{ac}_{db} e_c \;\; {}^\mu y_\mu \mathbf{e}^d\right) \\
&= 2\Delta^{ac}_{db} \mathbf{d} e_c \;\; {}^\mu y_\mu \mathbf{e}^d + 2\Delta^{ac}_{db} e_c \;\; {}^\mu \mathbf{d} y_\mu \mathbf{e}^d + 2\Delta^{ac}_{db} e_c \;\; {}^\mu y_\mu \mathbf{d}\mathbf{e}^d
\end{aligned}$$

Then since $\omega_a = (1-\kappa)^{-1}(\mathbf{f}_a + \mathbf{b}_a)$,

$$\begin{aligned}
\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\left(e_c \;\; {}^\mu \mathbf{d} y_\mu + \mathbf{b}_c\right)\mathbf{e}^d \\
&\quad + 2\Delta^{ac}_{db} \mathbf{d} e_c \;\; {}^\mu y_\mu \mathbf{e}^d + 2\Delta^{ac}_{db} e_c \;\; {}^\mu \mathbf{d} y_\mu \mathbf{e}^d \\
&\quad + 2\Delta^{ac}_{db} e_c \;\; {}^\mu y_\mu \mathbf{d}\mathbf{e}^d - \alpha^c_b\left(-2\Delta^{ae}_{dc} e_e \;\; {}^\mu W_\mu \mathbf{e}^d\right) \\
&\quad - \left(-2\Delta^{ce}_{db} e_e \;\; {}^\mu W_\mu \mathbf{e}^d\right)\alpha^a_c \\
&\quad - \left(-2\Delta^{ce}_{db} e_e \;\; {}^\mu W_\mu \mathbf{e}^d\right)\left(-2\Delta^{af}_{gc} e_f \;\; {}^\mu W_\mu \mathbf{e}^g\right)
\end{aligned} \tag{185}$$

The $\mathbf{d}y_\mu$ dependent terms,

$$0 = -2\Delta^{ac}_{db} e_c \;\; {}^\mu \mathbf{d} y_\mu \mathbf{e}^d + 2\Delta^{ac}_{db} e_c \;\; {}^\mu \mathbf{d} y_\mu \mathbf{e}^d \tag{186}$$

cancel identically. Expanding the remainder, and using the antisymmetry of the spin connection, $\eta_{dc}\alpha^c_b = -\eta_{bc}\alpha^c_d$,

$$\begin{aligned}
\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathbf{b}_c\mathbf{e}^d \\
&\quad + 2\Delta^{ac}_{db} \mathbf{d} e_c \;\; {}^\mu y_\mu \mathbf{e}^d + 2\Delta^{ac}_{db} e_c \;\; {}^\mu y_\mu \mathbf{d}\mathbf{e}^d \\
&\quad - 2\alpha^c_b \Delta^{ae}_{dc} e_e \;\; {}^\mu y_\mu \mathbf{e}^d - 2\Delta^{ce}_{db} e_e \;\; {}^\mu y_\mu \mathbf{e}^d \alpha^a_c \\
&\quad - 4\Delta^{ce}_{db} e_e \;\; {}^\mu y_\mu \mathbf{e}^d \Delta^{af}_{gc} e_f \;\; {}^\nu y_\nu \mathbf{e}^g \\
&= \mathbf{R}^a_b - \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{b}_c\mathbf{e}^d \\
&\quad + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{d} e_c \;\; {}^\mu y_\mu \mathbf{e}^d \\
&\quad + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right) e_c \;\; {}^\mu y_\mu \mathbf{d}\mathbf{e}^d \qquad (187) \\
&\quad - \alpha^e_b e_e \;\; {}^\mu y_\mu \mathbf{e}^a - \eta^{ae}\eta_{bc} e_e \;\; {}^\mu y_\mu \alpha^c_d \mathbf{e}^d \\
&\quad - e_b \;\; {}^\mu y_\mu \mathbf{e}^c \alpha^a_c - \eta^{ca}\eta_{db} e_e \;\; {}^\mu y_\mu \mathbf{e}^d \alpha^e_c \\
&\quad - e_b \;\; {}^\mu y_\mu \mathbf{e}^d e_d \;\; {}^\nu y_\nu \mathbf{e}^a + \eta^{ef}\eta_{db} e_e \;\; {}^\mu y_\mu \mathbf{e}^d e_f \;\; {}^\nu y_\nu \mathbf{e}^a \\
&\quad - \eta_{db}\eta^{af} e_e \;\; {}^\mu y_\mu \mathbf{e}^d e_f \;\; {}^\nu y_\nu \mathbf{e}^e \qquad (188)
\end{aligned}$$

Setting $y_a = e_a \;\; {}^\mu y_\mu$, this becomes

$$\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d = \mathbf{R}^a_b - \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{b}_c\mathbf{e}^d \tag{189}$$

$$
\begin{aligned}
& +\left(\delta^a_d\delta^c_b-\eta^{ac}\eta_{db}\right)\left(\mathbf{d}e_c^{\ \ \mu}-\alpha^e_c e_e^{\ \ \mu}\right)y_\mu\mathbf{e}^d && (190)\\
& +\left(\delta^a_d\delta^c_b-\eta^{ac}\eta_{db}\right)y_c\left(\mathbf{d}\mathbf{e}^d-\mathbf{e}^e\alpha^d_e\right) && (191)\\
& -\left(\delta^c_b\delta^a_d-\eta_{db}\eta^{ac}\right)y_c y_e\mathbf{e}^e\mathbf{e}^d && (192)\\
& +\left(\delta^c_b\delta^a_d-\eta_{db}\eta^{ca}\right)\frac{1}{2}\eta_{ce}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^e\mathbf{e}^d && (193)\\
= \;& \mathbf{R}^a_b-2\Delta^{ac}_{db}\Big(\mathbf{b}_c\mathbf{e}^d+\left(e_c^{\ \ \nu}y_\mu\mathbf{\Gamma}^\mu_\nu\right)\mathbf{e}^d && (194)\\
& +y_c y_e\mathbf{e}^e\mathbf{e}^d-\frac{1}{2}\eta_{ce}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^e\mathbf{e}^d\Big) && (195)
\end{aligned}
$$

where we use

$$
\begin{aligned}
\mathbf{d}e_c^{\ \ \mu}-\alpha^e_c e_e^{\ \ \mu}+e_c^{\ \ \nu}\Gamma^\mu_{\nu\alpha}\mathbf{d}x^\alpha &= 0\\
\Gamma^\mu_{\nu\alpha}\mathbf{d}x^\alpha &= \mathbf{\Gamma}^\mu_\nu
\end{aligned}
$$

in the last step. Thus, $\Gamma^\mu_{\nu\alpha}$ is the Christoffel connection corresponding to solder form $\mathbf{e}^a$ and spin connection α^e_c. For simplicity, define

$$
\mathbf{c}_a=c_{ac}\mathbf{e}^c=\mathbf{b}_a+e_a^{\ \ \nu}y_\mu\mathbf{\Gamma}^\mu_\nu+y_a y_c\mathbf{e}^c-\frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c \tag{196}
$$

Then we have simply

$$
\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d=\mathbf{R}^a_b-2\Delta^{ac}_{db}\mathbf{c}_c\mathbf{e}^d \tag{197}
$$

or, in components,

$$
\begin{aligned}
\Omega^a_{bcd} &= R^a_{bcd}-2\Delta^{ae}_{db}c_{ec}+2\Delta^{ae}_{cb}c_{ed} && (198)\\
&= R^a_{bcd}-\left(\delta^a_d\delta^e_b-\eta_{db}\eta^{ae}\right)c_{ec}+\left(\delta^a_c\delta^e_b-\eta_{cb}\eta^{ae}\right)c_{ed} && (199)\\
&= R^a_{bcd}-\delta^a_d c_{bc}+\eta_{db}c^a_{\ \ c}+\delta^a_c c_{bd}-\eta_{cb}c^a_{\ \ d} && (200)
\end{aligned}
$$

and can immediately solve for $\mathbf{c}_a$ from the remaining curvature field equation:

$$
\begin{aligned}
0 &= \Omega^c_{bcd} && (201)\\
&= R_{bd}+(n-2)\,c_{bd}+\eta_{db}c^c_{\ \ c} && (202)
\end{aligned}
$$

The trace gives $c=-\frac{1}{2(n-1)}R$, so solving for c_{ab},

$$
c_{bd}=-\frac{1}{n-2}\left(R_{bd}-\frac{1}{2\,(n-1)}\eta_{bd}R\right)\equiv\mathcal{R}_{bd}
$$

This object is called the Eisenhart tensor. With this, the full spacetime curvature 2-form is the Weyl (conformal) curvature 2-form of the spacetime submanifold:

$$\begin{aligned}\boldsymbol{\Omega}^a_b &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathcal{R}_c\mathbf{e}^d = \mathbf{C}^a_b && (203)\\ \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \frac{1}{2}C^a_{bcd}\mathbf{e}^c\mathbf{e}^d && (204)\end{aligned}$$

This result provides a convenient form for the Weyl curvature in terms of the Riemann curvature 2-form and the Eisenhart 1-form.

We also have the form of $\mathbf{b}_a$,

$$\mathbf{b}_a = \mathcal{R}_a - e_a{}^{\nu} y_\mu \boldsymbol{\Gamma}^\mu_\nu - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c \qquad (205)$$

The symmetry of the components, $b_{ab} = b_{ba}$ is immediate.

We have now solved for the entire connection:

$$\begin{aligned}\omega^a_b &= \alpha^a_b - 2\Delta^{ac}_{db}W_c\mathbf{e}^d && (206)\\ \omega^a &= \mathbf{e}^a(x) && (207)\\ \omega_a &= a\left(\mathbf{f}_a + \mathbf{b}_a\right) && (208)\\ \omega^0_0 &= W_c\mathbf{e}^c = -y_\beta \mathbf{d}x^\beta && (209)\end{aligned}$$

where

$$\begin{aligned}a &= (1-\kappa)^{-1} && (210)\\ \mathbf{f}_a &= e_a{}^{\beta}(x)\ \mathbf{d}y_\beta && (211)\\ \mathbf{b}_a &= \mathcal{R}_a - e_a{}^{\nu} y_\mu \boldsymbol{\Gamma}^\mu_\nu - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c && (212)\end{aligned}$$

All that is left is to solve the co-torsion field and structure equations.

Alternate version using covariant derivatives The calculation of the previous section is complicated enough that it is worthwhile checking it independently. We can do this by expressing the derivatives as covariant derivatives.

Look at $\mathbf{D}_{(x,\alpha)}\beta^a_b$:

$$\begin{aligned}\mathbf{D}_{(x,\alpha)}\beta^a_b &= \mathbf{d}_x\ \beta^a_b - \alpha^c_b\beta^a_c - \ \beta^c_b\alpha^a_c\\ &= \mathbf{d}_x\left(-2\Delta^{ad}_{cb}W_d\mathbf{e}^c\right) - \alpha^c_b\left(-2\Delta^{ad}_{ec}W_d\mathbf{e}^e\right)\end{aligned}$$

$$
\begin{aligned}
& -\left(-2\Delta^{cd}_{eb}W_d\mathbf{e}^e\right)\alpha^a_c && (213)\\
\mathbf{D}_{(x,\alpha)}\beta^a_b \;=\; & -\left(\delta^a_e\delta^d_b-\eta^{ad}\eta_{eb}\right)\left(\mathbf{d}_xW_d\mathbf{e}^e+W_d\mathbf{d}\mathbf{e}^e\right)\\
& +\alpha^c_b\left(\delta^a_e\delta^d_c-\eta^{ad}\eta_{ec}\right)W_d\mathbf{e}^e\\
& +\left(\delta^c_e\delta^d_b-\eta^{cd}\eta_{eb}\right)W_d\mathbf{e}^e\alpha^a_c && (214)\\
=\; & -\delta^a_e\delta^d_b\mathbf{d}_xW_d\mathbf{e}^e+\eta^{ad}\eta_{eb}\mathbf{d}_xW_d\mathbf{e}^e-\delta^a_e\delta^d_bW_d\mathbf{d}\mathbf{e}^e\\
& +\eta^{ad}\eta_{eb}W_d\mathbf{d}\mathbf{e}^e+\alpha^c_b\delta^a_e\delta^d_cW_d\mathbf{e}^e-\alpha^c_b\eta^{ad}\eta_{ec}W_d\mathbf{e}^e\\
& +\delta^c_e\delta^d_bW_d\mathbf{e}^e\alpha^a_c-\eta^{cd}\eta_{eb}W_d\mathbf{e}^e\alpha^a_c && (215)\\
=\; & \eta^{ad}\eta_{eb}\mathbf{d}_xW_d\mathbf{e}^e-\mathbf{d}_xW_b\mathbf{e}^a\\
& -W_b\mathbf{d}\mathbf{e}^a+\eta^{ad}\eta_{eb}W_d\mathbf{d}\mathbf{e}^e\\
& +\alpha^d_bW_d\mathbf{e}^a+\eta_{bc}\alpha^c_e\eta^{ad}W_d\mathbf{e}^e\\
& +W_b\mathbf{e}^c\alpha^a_c+\eta_{eb}W_d\mathbf{e}^e\eta^{ca}\alpha^d_c\\
=\; & \eta^{ad}\eta_{eb}\left(\mathbf{d}_xW_d-W_c\alpha^c_d\right)\mathbf{e}^e-\left(\mathbf{d}_xW_b-W_d\;\alpha^d_b\right)\mathbf{e}^a\\
& -W_b\left(\mathbf{d}\mathbf{e}^a-\mathbf{e}^c\alpha^a_c\right)+\eta^{ad}\eta_{eb}W_d\left(\mathbf{d}\mathbf{e}^e-\mathbf{e}^f\;\alpha^e_f\right) && (216)\\
=\; & \eta^{ad}\eta_{eb}\left(\mathbf{d}_xW_d-W_c\alpha^c_d\right)\mathbf{e}^e-\delta^a_e\delta^d_b\left(\mathbf{d}_xW_d-W_c\alpha^c_d\right)\mathbf{e}^e\\
=\; & -2\Delta^{ad}_{cb}\left(\mathbf{D}_{(x,\alpha)}W_d\right)\mathbf{e}^c && (217)
\end{aligned}
$$

where we have used:

$$
\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\alpha^a_c && (218)\\
\beta^a_b &= \beta^a_{bc}\mathbf{e}^c=-2\Delta^{ad}_{cb}W_d\mathbf{e}^c && (219)\\
\mathbf{D}_{(x,\alpha)}W_d &= \mathbf{d}_xW_d-W_c\alpha^c_d && (220)\\
&= -\mathbf{d}_xe_d{}^{\beta}y_\beta+e_c{}^{\beta}y_\beta\alpha^c_d && (221)
\end{aligned}
$$

We also will need

$$
\begin{aligned}
\beta^c_b\beta^a_c &= 4\Delta^{cf}_{eb}W_f\Delta^{ag}_{dc}W_g\mathbf{e}^e\mathbf{e}^d\\
&= \left(\delta^c_e\delta^f_b-\eta^{cf}\eta_{eb}\right)W_f\left(\delta^a_d\delta^g_c-\eta^{ag}\eta_{dc}\right)W_g\mathbf{e}^e\mathbf{e}^d\\
&= \left(\delta^g_e\delta^f_b\delta^a_d-\eta^{gf}\eta_{eb}\delta^a_d-\delta^f_b\eta^{ag}\eta_{de}+\delta^f_d\eta_{eb}\eta^{ag}\right)W_fW_g\mathbf{e}^e\mathbf{e}^d\\
&= \left(W_d\mathbf{e}^d\right)\left(W_b\mathbf{e}^a-\eta_{eb}\eta^{ag}W_g\mathbf{e}^e\right)-\left(\eta^{fg}W_fW_g\right)\eta_{eb}\mathbf{e}^e\mathbf{e}^a\\
&= 2\left(W_d\mathbf{e}^d\right)\Delta^{ac}_{eb}W_c\mathbf{e}^e-\left(\eta^{fg}W_fW_g\right)\eta_{eb}\mathbf{e}^e\mathbf{e}^a\\
&= 2\left(W_d\mathbf{e}^d\right)\Delta^{ac}_{eb}W_c\mathbf{e}^e-\frac{1}{2}\left(\eta^{fg}W_fW_g\right)2\Delta^{ac}_{eb}\eta_{cd}\mathbf{e}^d\mathbf{e}^e
\end{aligned}
$$

$$= \quad 2\Delta^{ac}_{eb}\left(\left(W_d\mathbf{e}^d\right)W_c - \frac{1}{2}\left(\eta^{fg}W_fW_g\right)\eta_{cd}\mathbf{e}^d\right)\mathbf{e}^e \tag{222}$$

where we have used

$$\begin{aligned} 2\Delta^{ac}_{eb}\eta_{cd}\mathbf{e}^d\mathbf{e}^e &= \left(\delta^a_e\delta^c_b - \eta^{ac}\eta_{eb}\right)\eta_{cd}\mathbf{e}^d\mathbf{e}^e \\ &= \eta_{bd}\mathbf{e}^d\mathbf{e}^a + \eta_{bd}\mathbf{e}^d\mathbf{e}^a \\ &= 2\eta_{bd}\mathbf{e}^d\mathbf{e}^a \end{aligned} \tag{223}$$

Now,

$$\begin{aligned} \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b + \left[-2\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d + \mathbf{d}_{(y)}\beta^a_b\right] \\ &\quad + \left[-2\Delta^{ac}_{db}b_{ce}\mathbf{e}^e\mathbf{e}^d + \mathbf{D}_{(x,\alpha)}\beta^a_b - \beta^c_b\beta^a_c\right] \end{aligned} \tag{224}$$

First, the mixed terms of the curvature equation must cancel:

$$\begin{aligned} \left[-2\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d + \mathbf{d}_{(y)}\beta^a_b\right] &= -2\Delta^{ac}_{db}e_c^{\ \ \beta}\mathbf{d}y_\beta\mathbf{e}^d + \mathbf{d}_{(y)}\left(2\Delta^{ac}_{db}e_c^{\ \ \beta}y_\beta\mathbf{e}^d\right) \\ &= 2\Delta^{ac}_{db}\left(-e_c^{\ \ \beta}\mathbf{d}y_\beta\mathbf{e}^d + e_c^{\ \ \beta}\mathbf{d}y_\beta\mathbf{e}^d\right) && (225) \\ &= 0 && (226) \end{aligned}$$

This is satisfied identically, so the curvature equation is purely spacetime. The final bracket is given by

$$\begin{aligned} [-]_2 &= -2\Delta^{ac}_{db}\mathbf{b}_c\mathbf{e}^d - 2\Delta^{ac}_{db}\left(\mathbf{D}_{(x,\alpha)}W_c\right)\mathbf{e}^d \\ &\quad -2\Delta^{ac}_{db}\left(\left(W_e\mathbf{e}^e\right)W_c - \frac{1}{2}\left(\eta^{fg}W_fW_g\right)\eta_{ce}\mathbf{e}^e\right)\mathbf{e}^d \\ &= -2\Delta^{ac}_{db}\mathbf{b}_c\mathbf{e}^d \\ &\quad -2\Delta^{ac}_{db}\left(\mathbf{D}_{(x,\alpha)}W_c + \left(W_e\mathbf{e}^e\right)W_c - \frac{1}{2}\left(\eta^{fg}W_fW_g\right)\eta_{ce}\mathbf{e}^e\right)\mathbf{e}^d \end{aligned} \tag{227}$$

Without loss of generality, we define a new 1-form $\mathbf{c}_a$ as

$$\mathbf{c}_a = \mathbf{b}_a + \mathbf{D}_{(x,\alpha)}W_a + \left(W_e\mathbf{e}^e\right)W_a - \frac{1}{2}\left(\eta^{fg}W_fW_g\right)\eta_{ae}\mathbf{e}^e \tag{228}$$

or equivalently,

$$\mathbf{b}_a = \mathbf{c}_a - \mathbf{D}_{(x,\alpha)}W_a - \left(W_e\mathbf{e}^e\right)W_a + \frac{1}{2}\left(\eta^{fg}W_fW_g\right)\eta_{ae}\mathbf{e}^e \tag{229}$$

Then the full curvature structure equation becomes

$$
\begin{aligned}
\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathbf{c}_c\mathbf{e}^d \\
\Omega^a_{bcd} &= R^a_{bcd} - 2\left(\Delta^{ae}_{db}c_{ec} - \Delta^{ae}_{cb}c_{ed}\right) \qquad (230)
\end{aligned}
$$

The field equation is

$$\Omega^a_{bac} = 0 = \Omega^a_{cab} \qquad (231)$$

which is sufficient to determine $\mathbf{c}_a$. We have

$$\Omega^c_{bcd} = R_{bd} + (n-2)\,c_{bd} + \eta_{bd}c^c_c = 0 \qquad (232)$$

or

$$-(n-2)\,c_{bd} - \eta_{bd}c^c_c = R_{bd} \qquad (233)$$

Inverting,

$$2\,(n-1)\,c = -R \qquad (234)$$

so

$$
\begin{aligned}
c_{ab} &= -\frac{1}{(n-2)}\left(R_{ab} - \frac{1}{2\,(n-1)}R\eta_{ab}\right) \qquad (235)\\
&\equiv \mathcal{R}_{ab} \qquad (236)
\end{aligned}
$$

Using this form, we can find the full curvature:

$$
\begin{aligned}
\Omega^a_{bcd} &= R^a_{bcd} - 2\left(\Delta^{ae}_{db}c_{ec} - \Delta^{ae}_{cb}c_{ed}\right) \qquad (237)\\
&= R^a_{bcd} - \frac{1}{(n-2)}\left(\delta^a_d R_{bc} - \eta_{db}R^a{}_c - \delta^a_c R_{bd} + \eta_{cb}R^a{}_d\right) \qquad (238)\\
&\quad -\frac{R}{(n-1)\,(n-2)}\left(\delta^a_c\eta_{bd} - \delta^a_d\eta_{bc}\eta_{ed}\right) \qquad (239)\\
&= C^a_{bcd} \qquad (240)
\end{aligned}
$$

The curvature is fully spacelike and equal to the Weyl curvature tensor. We may also write

$$
\begin{aligned}
C^a_{bcd} &= R^a_{bcd} - 2\left(\Delta^{ae}_{db}\mathcal{R}_{ec} - \Delta^{ae}_{cb}\mathcal{R}_{ed}\right) \qquad (241)\\
\mathbf{C}^a_b &= \mathbf{R}^a_b - 2\Delta^{ae}_{db}\mathcal{R}_e\mathbf{e}^d \qquad (242)
\end{aligned}
$$

in agreement with the previous result.

The co-torsion We must still satisfy the following equations:

$$\alpha\Delta^{cf}_{eb}\left(\Omega_{gc}{}^{b}-\delta^{b}_{g}\Omega_{ac}{}^{a}\right) = 0 \tag{243}$$
$$-(n-2)\,\Omega_{ae}{}^{a} = \eta_{ea}\eta^{bc}\Omega_{bc}{}^{a} \tag{244}$$

where

$$\Omega_{[abc]} = 0 \tag{245}$$
$$\Omega_{[ac]}^{\ \ b} = 0 \tag{246}$$
$$\Omega_{a}{}^{bc} = 0 \tag{247}$$

In addition we have the structure equation for the co-torsion, eq.(124):

$$\frac{1}{2}\Omega_{abc}\mathbf{e}^{b}\mathbf{e}^{c}+\Omega_{ac}^{\ \ b}\omega_{b}\mathbf{e}^{c}=\mathbf{d}\omega_{a}-\omega_{a}^{b}\omega_{b}-\omega_{a}\omega_{0}^{0} \tag{248}$$

It is convenient to first derive some relations.

Some useful relations Working in the orthonormal basis, we will need certain exterior derivatives.

$$\begin{aligned}\mathbf{df}_{a} &= \mathbf{d}e_{a}{}^{\beta}\mathbf{d}y_{\beta} \\ &= \left(\alpha_{a}^{\beta}-\mathbf{\Gamma}_{a}^{\beta}\right)\mathbf{d}y_{\beta}\end{aligned}$$

and

$$\mathbf{df}_{a}-\alpha_{a}^{c}\mathbf{f}_{c}+\mathbf{\Gamma}_{a}^{c}\mathbf{f}_{c}=0$$

where

$$\mathbf{f}_{a} = e_{a}{}^{\beta}(x)\ \mathbf{d}y_{\beta} \tag{249}$$
$$\mathbf{b}_{a} = \mathcal{R}_{a}-e_{a}{}^{\nu}y_{\mu}\mathbf{\Gamma}_{\nu}^{\mu}-y_{a}y_{c}\mathbf{e}^{c}+\frac{1}{2}\eta_{ac}\left(\eta^{gh}y_{g}y_{h}\right)\mathbf{e}^{c} \tag{250}$$

We also need

$$\begin{aligned}\mathbf{d}y_{a} &= \mathbf{d}e_{a}{}^{\mu}y_{\mu}+e_{a}{}^{\mu}\mathbf{d}y_{\mu} \\ &= \left(-e_{a}{}^{\nu}\mathbf{\Gamma}_{\nu}^{\mu}+e_{b}{}^{\mu}\alpha_{a}^{b}\right)y_{\mu}+e_{a}{}^{\mu}\mathbf{d}y_{\mu} \\ &= e_{a}{}^{\mu}\left(\mathbf{d}y_{\mu}-y_{\alpha}\mathbf{\Gamma}_{\mu}^{\alpha}+y_{b}\alpha_{\mu}^{b}\right)\end{aligned}$$

which follows from the vanishing covariant derivative of the solder form and its inverse,

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a - \mathbf{e}^b\alpha^a_b &= 0 \\
\partial_\beta e_\alpha{}^a + e_\alpha{}^b\alpha^a_{b\beta} - e_\mu{}^a\Gamma^\mu_{\alpha\beta} &= 0 \\
\partial_\beta e_c{}^\alpha - e_b{}^\alpha\alpha^b_{c\beta} + e_c{}^\mu\Gamma^\alpha_{\mu\beta} &= 0 \\
\mathbf{d}e_a{}^\beta + e_a{}^\nu\mathbf{\Gamma}^\beta_\nu - e_b{}^\beta\alpha^b_a &= 0
\end{aligned}$$

Check:

$$\begin{aligned}
\mathbf{d}\mathbf{f}_a &= \mathbf{d}\left(e_a{}^\beta\mathbf{d}y_\beta\right) \\
&= \mathbf{d}e_a{}^\beta\mathbf{d}y_\beta \\
&= \left(e_b{}^\beta\alpha^b_a - e_a{}^\mu\mathbf{\Gamma}^\beta_\mu\right)\mathbf{d}y_\beta \\
&= \left(\alpha^\beta_a - \mathbf{\Gamma}^\beta_a\right)\mathbf{d}y_\beta \\
&= \left(\alpha^c_a - \mathbf{\Gamma}^c_a\right)\mathbf{f}_c
\end{aligned}$$

or

$$\mathbf{d}\mathbf{f}_a - \alpha^c_a\mathbf{f}_c + \mathbf{\Gamma}^c_a\mathbf{f}_c = 0$$

and

$$\begin{aligned}
\mathbf{d}y_a &= \mathbf{d}e_a{}^\mu y_\mu + e_a{}^\mu\mathbf{d}y_\mu \\
&= \left(-e_a{}^\nu\mathbf{\Gamma}^\mu_\nu + e_b{}^\mu\alpha^b_a\right)y_\mu + e_a{}^\mu\mathbf{d}y_\mu \\
&= e_a{}^\mu\left(\mathbf{d}y_\mu - y_b\mathbf{\Gamma}^b_\mu + y_b\alpha^b_\mu\right) \\
&= e_a{}^\mu\left(\mathbf{d}y_\mu - y_\alpha\mathbf{\Gamma}^\alpha_\mu + y_b\alpha^b_\mu\right) \\
&= \mathbf{f}_a - e_a{}^\mu y_\alpha\mathbf{\Gamma}^\alpha_\mu + y_b\alpha^b_a
\end{aligned}$$

Now we compute the co-torsion.

Vanishing cross torsion To begin, we compute the terms on the right side:

$$\begin{aligned}
\frac{1}{a}\mathbf{\Omega}_a &= \mathbf{d}\mathbf{f}_a + \mathbf{d}\mathbf{b}_a - \omega^b_a\left(\mathbf{f}_b+\mathbf{b}_b\right) - \left(\mathbf{f}_a+\mathbf{b}_a\right)\omega^0_0 \\
&= \alpha^c_a\mathbf{f}_c - \mathbf{\Gamma}^c_a\mathbf{f}_c - \alpha^b_a\mathbf{f}_b + 2\Delta^{bc}_{da}W_c\mathbf{e}^d\mathbf{f}_b - \mathbf{f}_a\omega^0_0 \\
&\quad +\mathbf{d}\mathbf{b}_a - \alpha^b_a\mathbf{b}_b + 2\Delta^{bc}_{da}W_c\mathbf{e}^d\mathbf{b}_b - \mathbf{b}_a\omega^0_0 \\
&= -\mathbf{\Gamma}^c_a\mathbf{f}_c + W_a\mathbf{e}^b\mathbf{f}_b - \eta^{bc}\eta_{da}W_c\mathbf{e}^d\mathbf{f}_b - \mathbf{f}_aW_c\mathbf{e}^c \\
&\quad +\mathbf{d}\mathbf{b}_a - \alpha^b_a\mathbf{b}_b + 2\Delta^{bc}_{da}W_c\mathbf{e}^d\mathbf{b}_b - \mathbf{b}_a\omega^0_0
\end{aligned}$$

Now look at

$$\begin{aligned}
\mathbf{d}\mathbf{b}_a &= \mathbf{d}\left(\mathcal{R}_a - e_a{}^{\nu} y_\mu \mathbf{\Gamma}^\mu_\nu - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{e}^c\right) \\
&= \mathbf{d}\mathcal{R}_a - \mathbf{d}e_a{}^{\nu} y_\mu \mathbf{\Gamma}^\mu_\nu - e_a{}^{\nu}\mathbf{d}y_\mu \mathbf{\Gamma}^\mu_\nu - e_a{}^{\nu} y_\mu \mathbf{d}\mathbf{\Gamma}^\mu_\nu \\
&\quad -\mathbf{d}y_a y_c \mathbf{e}^c - y_a \mathbf{d}y_\alpha \mathbf{d}x^\alpha + \eta_{ac}\left(\eta^{gh} y_g \mathbf{d}y_h\right)\mathbf{e}^c \\
&\quad +\frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{d}\mathbf{e}^c
\end{aligned}$$

Combining,

$$\begin{aligned}
\frac{1}{a}\mathbf{\Omega}_a &= -\mathbf{\Gamma}^c_a \mathbf{f}_c + W_a \mathbf{e}^b \mathbf{f}_b - \eta^{bc}\eta_{da} W_c \mathbf{e}^d \mathbf{f}_b - \mathbf{f}_a W_c \mathbf{e}^c \\
&\quad -e_a{}^{\nu}\mathbf{d}y_\mu \mathbf{\Gamma}^\mu_\nu - \mathbf{f}_a y_c \mathbf{e}^c - y_a \mathbf{d}y_\alpha \mathbf{d}x^\alpha + \eta_{ac}\eta^{gh} y_g \mathbf{f}_h \mathbf{e}^c \\
&\quad +\mathbf{d}\mathcal{R}_a - \mathbf{d}e_a{}^{\nu} y_\mu \mathbf{\Gamma}^\mu_\nu - e_a{}^{\nu} y_\mu \mathbf{d}\mathbf{\Gamma}^\mu_\nu \\
&\quad +e_a{}^{\mu} y_\alpha \mathbf{\Gamma}^\alpha_\mu y_c \mathbf{e}^c - y_b \alpha^b_a y_c \mathbf{e}^c \\
&\quad -\eta_{ac}\eta^{gh} y_g e_h{}^{\mu} y_\alpha \mathbf{\Gamma}^\alpha_\mu \mathbf{e}^c + \eta_{ac}\eta^{gh} y_g y_b \alpha^b_h \mathbf{e}^c \\
&\quad +\frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{d}\mathbf{e}^c - \alpha^b_a \mathbf{b}_b + 2\Delta^{bc}_{da} W_c \mathbf{e}^d \mathbf{b}_b - \mathbf{b}_a \omega^0_0
\end{aligned}$$

Look at the $\mathbf{f}_c$ terms only:

$$\begin{aligned}
\frac{1}{a}\Omega_{ac}{}^{b}\mathbf{f}_b \mathbf{e}^c &= -\mathbf{\Gamma}^c_a \mathbf{f}_c - \mathbf{f}_c \mathbf{\Gamma}^c_a + W_a \mathbf{e}^b \mathbf{f}_b - \eta^{bc}\eta_{da} W_c \mathbf{e}^d \mathbf{f}_b - \mathbf{f}_a W_c \mathbf{e}^c \\
&\quad -\mathbf{f}_a y_c \mathbf{e}^c - y_a \mathbf{f}_c \mathbf{e}^c + \eta_{ac}\eta^{gh} y_g \mathbf{f}_h \mathbf{e}^c \\
&= 0
\end{aligned}$$

This requires

$$\Omega_{ac}{}^{b} = 0$$

and all field equations are satisfied. Only the spacetime Bianchi, $\Omega_{[abc]} = 0$, remains.

Spacetime co-torsion The remaining terms are:

$$\begin{aligned}
\frac{1}{2a}\Omega_{abc}\mathbf{e}^b \mathbf{e}^c &= \mathbf{d}\mathcal{R}_a - \mathbf{d}e_a{}^{\nu} y_\mu \mathbf{\Gamma}^\mu_\nu - e_a{}^{\nu} y_\mu \mathbf{d}\mathbf{\Gamma}^\mu_\nu \\
&\quad +e_a{}^{\mu} y_\alpha \mathbf{\Gamma}^\alpha_\mu y_c \mathbf{e}^c - y_b \alpha^b_a y_c \mathbf{e}^c \\
&\quad -\eta_{ac}\eta^{gh} y_g e_h{}^{\mu} y_\alpha \mathbf{\Gamma}^\alpha_\mu \mathbf{e}^c + \eta_{ac}\eta^{gh} y_g y_b \alpha^b_h \mathbf{e}^c
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{d}\mathbf{e}^c - \alpha_a^b\mathbf{b}_b \\
& +2\Delta_{da}^{bc}W_c\mathbf{e}^d\mathbf{b}_b - \mathbf{b}_a\omega_0^0 \\
= \; & \mathbf{d}\mathcal{R}_a - e_b{}^{\nu}y_\mu\alpha_a^b\mathbf{\Gamma}_\nu^\mu - e_a{}^{\nu}y_\mu\mathbf{d}\mathbf{\Gamma}_\nu^\mu + e_a{}^{\beta}y_\mu\mathbf{\Gamma}_\beta^\nu\mathbf{\Gamma}_\nu^\mu \\
& +e_a{}^{\mu}y_\alpha\mathbf{\Gamma}_\mu^\alpha y_c\mathbf{e}^c - y_b\alpha_a^b y_c\mathbf{e}^c \\
& -\eta_{ac}\eta^{gh}y_g e_h{}^{\mu}y_\alpha\mathbf{\Gamma}_\mu^\alpha\mathbf{e}^c + \eta_{ac}\eta^{gh}y_g y_b\alpha_h^b\mathbf{e}^c \\
& +\frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^b\alpha_b^c \\
& -\alpha_a^b\mathcal{R}_b + \alpha_a^b e_b{}^{\nu}y_\mu\mathbf{\Gamma}_\nu^\mu + \alpha_a^b y_b y_c\mathbf{e}^c \\
& -\alpha_a^b\frac{1}{2}\eta_{bc}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c \\
& +2\Delta_{da}^{bc}W_c\mathbf{e}^d\mathcal{R}_b - 2\Delta_{da}^{bc}W_c\mathbf{e}^d e_b{}^{\nu}y_\mu\mathbf{\Gamma}_\nu^\mu \\
& -2\Delta_{da}^{bc}W_c\mathbf{e}^d y_b y_e\mathbf{e}^e + 2\Delta_{da}^{bc}W_c\mathbf{e}^d\frac{1}{2}\eta_{be}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^e \\
& +\mathcal{R}_a y_b\mathbf{e}^b - e_a{}^{\nu}y_\mu\mathbf{\Gamma}_\nu^\mu y_b\mathbf{e}^b \\
& -y_a y_c\mathbf{e}^c y_b\mathbf{e}^b + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c y_b\mathbf{e}^b
\end{aligned}
$$

Collecting terms,

$$
\begin{aligned}
\frac{1}{2a}\Omega_{abc}\mathbf{e}^b\mathbf{e}^c \; = \; & \mathbf{d}\mathcal{R}_a - \alpha_a^b\mathcal{R}_b - \eta^{bc}\eta_{da}W_c\mathbf{e}^d\mathcal{R}_b - \mathcal{R}_a W_b\mathbf{e}^b \\
& -e_a{}^{\beta}y_\mu\left(\mathbf{d}\mathbf{\Gamma}_\beta^\mu - \mathbf{\Gamma}_\beta^\nu\mathbf{\Gamma}_\nu^\mu\right) \\
& +\frac{1}{2}\left(\eta^{gh}y_g y_h\right)\left(\eta_{da}\mathbf{e}^d y_c\mathbf{e}^c + \eta_{ac}\mathbf{e}^c y_b\mathbf{e}^b - 2\eta_{da}\mathbf{e}^d y_e\mathbf{e}^e\right) \\
& +y_b y_c\alpha_a^b\mathbf{e}^c - y_b y_c\alpha_a^b\mathbf{e}^c \\
& -\eta_{ac}\eta^{bh}y_g y_b\alpha_h^g\mathbf{e}^c + y_a y_e\mathbf{e}^b\mathbf{\Gamma}_b^e \\
& -\eta_{ad}\eta^{bc}y_c y_e\mathbf{e}^d\mathbf{\Gamma}_b^e + \eta_{ad}\eta^{bc}y_c y_e\mathbf{e}^d\mathbf{\Gamma}_b^e
\end{aligned}
$$

The third, fourth and sixth lines each vanish identically while

$$\eta_{ac}\eta^{bh}y_g y_b\alpha_h^g\mathbf{e}^c = 0$$

by the antisymmetry of the spin connection, and

$$y_a y_e\mathbf{e}^b\mathbf{\Gamma}_b^e = 0$$

by the symmetry of the Christoffel conneciton. With these simplifications,

$$\frac{1}{2a}\Omega_{abc}\mathbf{e}^b\mathbf{e}^c \; = \; \mathbf{d}\mathcal{R}_a - \alpha_a^b\mathcal{R}_b - \eta^{bc}\eta_{da}W_c\mathbf{e}^d\mathcal{R}_b - \mathcal{R}_a W_b\mathbf{e}^b$$

$$\begin{aligned}
& -e_a{}^{\beta} y_\mu \left(\mathbf{d}\boldsymbol{\Gamma}^\mu_\beta - \boldsymbol{\Gamma}^\nu_\beta \boldsymbol{\Gamma}^\mu_\nu\right) \\
= & \ \mathbf{d}\mathcal{R}_a - \alpha^b_a \mathcal{R}_b + W_b \left(\mathbf{R}^b_a - 2\Delta^{bc}_{da}\mathcal{R}_c \mathbf{e}^d\right) \\
= & \ \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + W_b \mathbf{C}^b_a
\end{aligned}$$

The final Bianchi is identically satisfied:

$$\begin{aligned}
0 &= \frac{1}{2a}\Omega_{abc}\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \\
&= \mathbf{e}^a \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + \mathbf{e}^a W_b \mathbf{C}^b_a \\
&= \mathbf{D}_{(x,\alpha)}\left(\mathbf{e}^a \mathcal{R}_a\right) + W_b \mathbf{C}^b_a \mathbf{e}^a \\
&= 0
\end{aligned}$$

5 Summary

The connection takes the final form

$$\omega^a_b = \alpha^a_b - 2\Delta^{ac}_{db} W_c \mathbf{e}^d \tag{251}$$
$$\omega^a = \mathbf{e}^a(x) \tag{252}$$
$$\omega_a = a\left(\mathbf{f}_a + \mathbf{b}_a\right) \tag{253}$$
$$\omega^0_0 = W_c \mathbf{e}^c = -y_\beta \mathbf{d}x^\beta \tag{254}$$

where

$$a = (1-\kappa)^{-1} \tag{255}$$
$$\kappa = -\frac{1}{(n-1)}\left(1 + \frac{\gamma n^2}{(\alpha(n-1) - \beta)}\right) \tag{256}$$
$$\mathbf{f}_a = e_a{}^{\beta}(x)\ \mathbf{d}y_\beta \tag{257}$$
$$\mathbf{b}_a = \mathcal{R}_a - e_a{}^{\nu} y_\mu \boldsymbol{\Gamma}^\mu_\nu - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{e}^c \tag{258}$$

Notice that the strange looking $\boldsymbol{\Gamma}^\mu_\nu$ term in $\mathbf{b}_a$ may be written α-covariantly as

$$\begin{aligned}
D_\nu y_\mu &= \frac{\partial}{\partial x^\nu}\, y_\mu - y_\alpha \Gamma^\alpha_{\mu\nu} = -y_\alpha \Gamma^\alpha_{\mu\nu} \\
\mathbf{D} y_\mu &= \mathbf{d}_x\, y_\mu - y_\alpha \boldsymbol{\Gamma}^\alpha_\mu = -y_\alpha \boldsymbol{\Gamma}^\alpha_\mu
\end{aligned}$$

with the understanding that the μ index of $\mathbf{D}y_\mu$ labels n different functions, but does not transform as a vector. Is there a sense in which y_μ *does* transform as a vector? Recall that the submanifolds spanned by y_μ are flat Riemannian geometries. If we assume the corresponding manifolds are R^n, then the coordinate doubles as a vector. Then this term is fully covariant.

We also have

$$\mathcal{R}_{ab} \equiv -\frac{1}{(n-2)}\left(R_{ab} - \frac{1}{2(n-1)}R\eta_{ab}\right) \tag{259}$$

The curvatures are then

$$\begin{aligned}
\mathbf{\Omega}^a_b &= \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d - 2\kappa\Delta^{ac}_{db}\omega_c\mathbf{e}^d && (260)\\
&= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathcal{R}_c\mathbf{e}^d - 2a\kappa\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d && (261)\\
&= \mathbf{C}^a_b - 2a\kappa\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d && (262)\\
\mathbf{\Omega}^a &= 0 && (263)\\
\mathbf{\Omega}_a &= \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + W_b\mathbf{C}^b_a && (264)\\
&= \mathbf{d}\mathcal{R}_a - \alpha^b_a\mathcal{R}_b + W_b\mathbf{C}^b_a && (265)\\
\mathbf{\Omega}^0_0 &= -\kappa\mathbf{e}^a\omega_a = -a\kappa\mathbf{e}^a\mathbf{f}_a = -a\kappa\mathbf{d}x^\beta\mathbf{d}y_\beta && (266)
\end{aligned}$$

Finally, the structure equations take the form

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + 2\Delta^{ad}_{cb}\omega_d\mathbf{e}^c + \mathbf{C}^a_b && (267)\\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega^a_b + \omega^0_0\mathbf{e}^a && (268)\\
\mathbf{d}\omega_a &= \omega^b_a\omega_b + \omega_a\omega^0_0 + \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + W_b\mathbf{C}^b_a && (269)\\
\mathbf{d}\omega^0_0 &= \mathbf{e}^a\omega_a && (270)
\end{aligned}$$

where

$$\mathbf{C}^a_b = \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathcal{R}_c\mathbf{e}^d$$

is the Weyl curvature.

6 Existence and Ricci flatness of spacetime

We now return to our second assumption: the minimial condition necessary to guarantee the existence of a spacetime submanifold. This is provided by a second involution, this time of the co-solder form, ω_b. Setting $\omega_b = 0$ we have

$$
\begin{aligned}
0 &= \frac{1}{a}\omega_a \\
&= e_a{}^{\beta}(x)\ \mathbf{d}y_\beta + \mathcal{R}_a - e_a{}^{\nu} y_\mu \mathbf{\Gamma}^\mu_\nu \\
&\quad - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{e}^c
\end{aligned}
\tag{271}
$$

together with the reduced structure equations

$$
\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b \omega^a_c + \frac{1}{2} C^a_{bcd}\mathbf{e}^c\mathbf{e}^d \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega^a_b + \omega^0_0 \mathbf{e}^a = \mathbf{e}^b\alpha^a_b \\
0 &= \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + W_b \mathbf{C}^b_a \\
\mathbf{d}\omega^0_0 &= 0
\end{aligned}
\tag{272}
$$

To examine the consequences of these equations, we first examine eq.(271). Eq.(271) has become a differential equation for a hypersurface, $y_a = y_a(x)$. We first rewrite the derivative term as

$$
\begin{aligned}
e_a{}^{\beta}(x)\ \mathbf{d}y_\beta &= \mathbf{d}y_a - y_\beta \mathbf{d}e_a{}^{\beta} \\
&= \mathbf{d}y_a - y_\beta e_b{}^{\beta}\alpha^b_a + y_\beta e_a{}^{\nu}\mathbf{\Gamma}^\beta_\nu
\end{aligned}
$$

where α^a_b is the spin connection compatible with the solder form $\mathbf{e}^a(x)$. Rearranging, we have

$$
\mathbf{d}y_a = y_b\alpha^b_a + y_a y_c \mathbf{e}^c - \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{e}^c - \mathcal{R}_a \tag{273}
$$

Because y_a is the negative of the Weyl vector, eq.(273) is closely related to the change in the Ricci and Eisenhart tensors under a conformal transformation ([?]),

$$
\begin{aligned}
\tilde{R}_{ab} &= R_{ab} - \eta_{ab}\Box\phi - (n-2)\left[\phi_{;ab} - \phi_{;a}\phi_{;b} + \eta_{ab}\phi^{;c}\phi_{;c}\right] \\
\tilde{\mathcal{R}}_a &= \mathcal{R}_a + \mathbf{d}\phi_{;a} - \phi_{;b}\alpha^b_a - \phi_{;a}\phi_{;c}\mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh}\phi_{;g}\phi_{;h}\right)\mathbf{e}^c
\end{aligned}
$$

where $\phi_{;a} = D_a\phi = \partial_a\phi$ and n is the dimension. Specifically, notice that if we could replace y_a with $\phi_{;a}$ in eq.(273) we would have exactly the condition

$\overset{\heartsuit}{\mathcal{R}}_a = 0$, equivalent to the vacuum Einstein equation in the conformally transformed basis. Therefore, eq(273), together with

$$y_\alpha \mathbf{d}x^\alpha = y_c \mathbf{e}^c = \mathbf{d}\phi \tag{274}$$

guarantee the existence of a conformal gauge in which the vacuum Einstein equation holds.

We now show that the reduced structure equations, eqs.(272) provide the integrability conditions for eqs.(273) and (274). The integrability conditions are given by the Poincarè lemma, $\mathbf{d}^2 = 0$. Applying this first to eq.(273), we have

$$\begin{aligned} 0 &= \mathbf{d}^2 y_a \\ &= \mathbf{d}y_b \alpha_a^b + y_b \mathbf{d}\alpha_a^b + \mathbf{d}y_a y_c \mathbf{e}^c + y_a \mathbf{d}\left(y_c \mathbf{e}^c\right) \\ &\quad -\eta_{ac}\eta^{gh} y_g \mathbf{d}y_h \mathbf{e}^c - \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right) \mathbf{d}\mathbf{e}^c - \mathbf{d}\mathcal{R}_a \end{aligned}$$

Substituting for all occurrences of $\mathbf{d}y_b$ we find after some cancellations,

$$0 = y_b \mathbf{R}_a^b - \mathbf{D}\mathcal{R}_a - \mathcal{R}_a y_c \mathbf{e}^c - \left(\eta^{bc} y_b \mathcal{R}_c\right) \eta_{ad} \mathbf{e}^d + y_a \mathbf{d}\left(y_c \mathbf{e}^c\right)$$

Now, we substitute for the curvature 2-form,

$$\mathbf{R}_b^a = \mathbf{C}_b^a + 2\Delta_{db}^{ac} \mathcal{R}_c \mathbf{e}^d$$

This reduces the integrability condition to

$$0 = y_b \mathbf{C}_a^b - \mathbf{D}\mathcal{R}_a + y_a \mathbf{d}\left(y_c \mathbf{e}^c\right)$$

Turning now to the integrability condition for eq.(274), we have

$$\begin{aligned} 0 &= \mathbf{d}^2\phi \\ &= \mathbf{d}\left(y_c \mathbf{e}^c\right) \end{aligned}$$

Combining these two conditions as the pair

$$\begin{aligned} y_b \mathbf{C}_a^b - \mathbf{D}\mathcal{R}_a &= 0 \\ \mathbf{d}\left(y_c \mathbf{e}^c\right) &= 0 \end{aligned}$$

and recalling that $W_a = -y_a$, we see that the reduced structure equations, (272) provide exactly these conditions. Therefore, there exists a choice of the conformal gauge such that the Einstein equation holds on the $\omega_a = 0$ submanifold. This same choice reduces the Weyl vector to zero and the remaining structure equations to

$$\begin{aligned} \mathbf{d}\alpha^a_b &= \alpha^c_b \alpha^a_c + \frac{1}{2} C^a_{bcd} \mathbf{e}^c \mathbf{e}^d \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \alpha^a_b \end{aligned}$$

the first of which again shows that the Ricci tensor must vanish. In this gauge, the spacetime is a Ricci flat, purely Riemannian spacetime. Therefore, the reduced structure equations have as solutions the class of metrics conformally equivalent to the set of solutions to the vacuum Einstein equation.

We close this section with one further observation: the second condition, giving the integrability of y_a, is unnecessary. If we contract the solder form with $\mathbf{d}y_a$ we have

$$\begin{aligned} (\mathbf{d}y_a)\, \mathbf{e}^a &= \mathbf{d}(y_a \mathbf{e}^a) - y_a \mathbf{d}\mathbf{e}^a \\ &= \mathbf{d}(y_a \mathbf{e}^a) - y_a \mathbf{e}^b \alpha^a_b \end{aligned}$$

Substituting this into eq.(273) we find

$$\begin{aligned} \mathbf{d}(y_a \mathbf{e}^a) - y_a \mathbf{e}^b \alpha^a_b &= \left(y_b \alpha^b_a + y_a y_c \mathbf{e}^c - \frac{1}{2} \eta_{ac} \left(\eta^{gh} y_g y_h \right) \mathbf{e}^c - \mathcal{R}_a \right) \mathbf{e}^a \\ &= y_b \alpha^b_a \mathbf{e}^a - \mathcal{R}_a \mathbf{e}^a \end{aligned}$$

and therefore

$$\mathbf{d}(y_a \mathbf{e}^a) = 0$$

by the symmetry of the Eisenhart tensor.

The meaning of this additional result is clearest if we begin with the expression for the change in the Eisenhart tensor under a conformal transformation,

$$\overset{\heartsuit}{\mathcal{R}}_a = \mathcal{R}_a + \mathbf{d}\phi_{;a} - \phi_{;b} \alpha^b_a - \phi_{;a} \phi_{;c} \mathbf{e}^c + \frac{1}{2} \eta_{ac} \left(\eta^{gh} \phi_{;g} \phi_{;h} \right) \mathbf{e}^c$$

Treating $\phi_a = \phi_{;a}$ as a vector, we see that

$$\phi_b \mathbf{C}^b_a - \mathbf{D}\mathcal{R}_a = 0 \tag{275}$$

is the integrability condition for the existence of a vector field ϕ_a such that $\overset{\heartsuit}{\mathcal{R}}_a = 0$. Then, contracting with the solder form as above, we see that

$\mathbf{d}\left(\phi_a \mathbf{e}^a\right) = 0$ so that ϕ_a must be a gradient. Eq.(**??**) alone is therefore a sufficient condition for the existence of a conformal transformation to a Ricci flat spacetime. Szekeres [**?**] uses the spinor representation to show that eq.(**??**) may be written as a constraint on the curvatures which is independent of y_a. It therefore follows from the results of [**?**] that eq.(**??**) is also necessary. Incidently, our result shows the equivalence of certain well-known conditions: the C-spaces $\left(\phi_{;b}\mathbf{C}_a^b - \mathbf{D}\mathcal{R}_a = 0\right)$ of Szekeres [**?**] , the J-spaces $(\mathbf{D}\mathcal{R}_a = 0)$ of Thompson ([**?**]), and conformally Ricci flat spaces. This follows because Ricci-flatness implies the J- and C-conditions, while we have shown that the C-condition implies conformal Ricci flatness.

These results were published in [BCYMG].

7 Limiting cases

We briefly consider two limiting cases of this solution – vanishing curvatures and vanishing co-torsion.

7.1 Flat limit

The solution for vanishing curvature was first presented in [WHEELER JMP]. Since the curvatures are

$$\begin{aligned}
\mathbf{\Omega}_b^a &= \mathbf{R}_b^a - 2\kappa\Delta_{db}^{ac}\omega_c\mathbf{e}^d && (276)\\
&= \mathbf{C}_b^a + 2a\kappa\Delta_{db}^{ac}\mathbf{f}_c\mathbf{e}^d && (277)\\
\mathbf{\Omega}^a &= 0 && (278)\\
\mathbf{\Omega}_a &= \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + \mathbf{C}_a^b W_b && (279)\\
\mathbf{\Omega}_0^0 &= \kappa\omega_a\omega^a = -a\kappa\mathbf{d}x^\alpha\mathbf{d}y_\alpha && (280)
\end{aligned}$$

we set them to zero to find:

$$\begin{aligned}
0 &= \mathbf{C}_b^a + 2a\kappa\Delta_{db}^{ac}\mathbf{f}_c\mathbf{e}^d && (281)\\
&= \mathbf{R}_b^a + 2\kappa\Delta_{db}^{ac}\omega_c\mathbf{e}^d && (282)\\
0 &= \mathbf{\Omega}^a && (283)\\
0 &= \mathbf{D}_{(x,\alpha)}\mathcal{R}_a - y_b\mathbf{C}_a^b && (284)\\
0 &= \kappa\omega_a\omega^a = -a\kappa\mathbf{d}x^\alpha\mathbf{d}y_\alpha && (285)
\end{aligned}$$

Because of the independence of the ys, these require vanishing conformal curvature and vanishing $\mathbf{D}_{(x,\alpha)}\mathcal{R}_a$. The final equation also implies vanishing $a\kappa$. Since

$$a\kappa = \frac{\kappa}{1-\kappa} \tag{286}$$

we see that $a\kappa = 0$ implies $\kappa = 0$. Therefore, with

$$\kappa = -\frac{1}{(n-1)}\left(1 + \frac{\gamma n^2}{\alpha\,(n-1) - \beta}\right) \tag{287}$$

we find the constraint

$$0 = \gamma n^2 - \beta + \alpha\,(n-1) \tag{288}$$

on the initial action. In this case,

$$\Lambda^a_b \equiv (\alpha(n-1) - \beta + \gamma n^2)\delta^a_b = 0 \tag{289}$$

as required by the field equations.

Finally, the connection may be written as:

$$\begin{aligned}
\omega^a_b &= 2\Delta^{ac}_{db} y_c \mathbf{e}^d && (290)\\
\omega^a &= \mathbf{e}^a = \mathbf{d}x^a && (291)\\
\omega_a &= \mathbf{d}y_a - \left(y_a y_b - \frac{1}{2}\eta_{ab} y^2\right)\mathbf{e}^b && (292)\\
\omega^0_0 &= W_c \mathbf{e}^c = -y_\beta \mathbf{d}x^\beta && (293)
\end{aligned}$$

This agrees with the flat solution [WHEELER JMP].

We now explore two weaker conditions.

7.2 Torsion and Dilatation free solution

A weaker constraint is to set the dilatational curvature to zero as well as the torsion. In this case, we have the general torsion-free solution together with the condition

$$\mathbf{\Omega}^0_0 = -a\kappa \mathbf{d}x^\beta \mathbf{d}y_\beta = 0 \tag{294}$$

This can only be accomplished by setting

$$a\kappa = \frac{\kappa}{1-\kappa} = 0$$

and hence $\kappa = 0$ and $a = 1$. From the definition of κ, this condition holds if only if

$$0 = \alpha (n-1) - \beta + \gamma n^2 \tag{295}$$

implying a special subclass of linear Lagrange densities.

The connection then takes the final form

$$\begin{aligned}
\omega^a_b &= \alpha^a_b - 2\Delta^{ac}_{db} W_c \mathbf{e}^d \\
\omega^a &= \mathbf{e}^a(x) \\
\omega_a &= \mathbf{f}_a + \mathbf{b}_a \\
\omega^0_0 &= W_c \mathbf{e}^c = -y_\beta \mathbf{d}x^\beta
\end{aligned}$$

and the curvatures reduce to

$$\begin{aligned}
\mathbf{\Omega}^a_b &= \mathbf{C}^a_b && (296) \\
\mathbf{\Omega}^a &= 0 && (297) \\
\mathbf{\Omega}_a &= \mathbf{D}_{(x,\alpha)} \mathcal{R}_a + W_b \mathbf{C}^b_a && (298) \\
\mathbf{\Omega}^0_0 &= 0 && (299)
\end{aligned}$$

7.3 Co-torsion free solution

Now consider the limit of vanishing co-torsion. From the curvatures,

$$\begin{aligned}
\mathbf{\Omega}^a_b &= \mathbf{C}^a_b + 2a\kappa \Delta^{ac}_{db} \mathbf{f}_c \mathbf{e}^d && (300) \\
&= \mathbf{R}^a_b + 2\kappa \Delta^{ac}_{db} \omega_c \mathbf{e}^d && (301) \\
\mathbf{\Omega}^a &= 0 && (302) \\
\mathbf{\Omega}_a &= \mathbf{D}_{(x,\alpha)} \mathcal{R}_a + W_b \mathbf{C}^b_a && (303) \\
\mathbf{\Omega}^0_0 &= \kappa \omega_a \omega^a = -a\kappa \mathbf{d}x^\alpha \mathbf{d}y_\alpha && (304)
\end{aligned}$$

we set

$$0 = \mathbf{\Omega}_a = \mathbf{D}_{(x,\alpha)} \mathcal{R}_a - y_b \mathbf{C}^b_a \tag{305}$$

and since there is no y-dependence in the first term, we require $\mathbf{C}^b_a = 0$ and $\mathbf{D}_{(x,\alpha)} \mathcal{R}_a = 0$ seperately. The remaining curvatures then reduce to

$$\begin{aligned}
\mathbf{\Omega}^a_b &= 2a\kappa \Delta^{ac}_{db} \mathbf{f}_c \mathbf{e}^d && (306) \\
\mathbf{\Omega}^0_0 &= -a\kappa \mathbf{d}x^\alpha \mathbf{d}y_\alpha && (307)
\end{aligned}$$

The spacetime is conformally flat, so we can perform an x-dependent rescaling to flat space. Then the Weyl vector becomes

$$\omega_0^0 = W_\beta \mathbf{d}x^\beta = (-y_\beta + \partial_\beta \varphi)\, \mathbf{d}x^\beta \tag{308}$$

and we can choose Cartesian coordinates, $\mathbf{e}^a = \mathbf{d}x^a$.

The connection is then

$$\begin{aligned}
\omega_b^a &= -2\Delta_{cb}^{ad} (-y_b + \partial_b \varphi)\, \mathbf{e}^c &(309)\\
\omega^a &= \mathbf{e}^a = \mathbf{d}x^a &(310)\\
\omega_a &= a \left(\mathbf{d}y_a - y_a y_e \mathbf{d}x^e + \frac{1}{2} y^2 \eta_{ae} \mathbf{d}x^e \right) &(311)\\
\omega_0^0 &= W_c \mathbf{e}^c = W_\beta \mathbf{d}x^\beta \\
&= (-y_b + \partial_b \varphi)\, \mathbf{d}x^b &(312)
\end{aligned}$$

again agreeing with the previous solution.

8 Special Case

Now consider the special case when $2\beta - (n-2)\,\alpha = 0$. In this case we find that the spacetime dilatational curvature, Ω_{0bc}^0, is now undetermined.

To find the full solution, we return to the relevant curvature field equations,

$$\alpha \Omega_{bac}^a + \beta \Omega_{0bc}^0 = 0$$

and Bianchi identity:

$$\Omega_{bac}^a - \Omega_{cab}^a = -(n-2)\, \Omega_{0bc}^0$$

The symmetric and antisymmetric parts of the first combine with the Bianchi identities to yield

$$\begin{aligned}
\Omega_{bac}^a + \Omega_{cab}^a &= 0 &(313)\\
\Omega_{bac}^a - \Omega_{cab}^a &= -(n-2)\, \Omega_{0bc}^0 &(314)
\end{aligned}$$

and now give no constraint on the spacetime dilatation.

The cross term of the curvature and dilatation are related as before, except that Λ_b^a and κ take the simpler forms

$$\Lambda_b^a \equiv \frac{n}{2} (\alpha + 2\gamma n)\, \delta_b^a$$

and

$$\kappa = -\frac{1}{(n-1)}\left(1 + \frac{2\gamma n}{\alpha}\right) \tag{315}$$

The remaining field equations and Bianchi identities are manipulated as before with the same results. Collecting results we have:

Curvature:

$$\begin{aligned} \Omega_b^{acd} &= 0 && (316) \\ \Omega_{cd}^{ab} &= -2\kappa\Delta_{dc}^{ab} && (317) \\ \Omega_{bac}^{a} + \Omega_{cab}^{a} &= 0 && (318) \\ \Omega_{bac}^{a} - \Omega_{cab}^{a} &= -(n-2)\,\Omega_{0bc}^{0} && (319) \end{aligned}$$

Torsion:

$$\mathbf{\Omega}^a = 0 \tag{320}$$

Dilatation:

$$\begin{aligned} \Omega_0^{0cd} &= 0 && (321) \\ \Omega_{0b}^{0a} &= \kappa\delta_b^a && (322) \end{aligned}$$

Co-torsion:

$$\begin{aligned} \Omega_a{}^{ab} &= 0 && (323) \\ \alpha\Delta_{eb}^{cf}\left(\Omega_{gc}{}^{b} - \delta_g^b\Omega_{ac}{}^{a}\right) &= 0 && (324) \\ \beta\left(\Omega_{cb}{}^{c} - \Omega_{ba}{}^{a}\right) &= 0 && (325) \\ -(n-2)\,\Omega_{ae}{}^{a} &= \eta_{eb}\eta^{cf}\Omega_{fc}{}^{b} && (326) \end{aligned}$$

where

$$\kappa = -\frac{\alpha + 2n\gamma}{(n-1)\,\alpha}$$

8.1 Differential part of the solution

First, check the dilatation Bianchi. We need to know the momentum piece of the co-torsion before studying the involution of the solder form. We still have $\omega^b = \mathbf{e}^b(x)$ so

$$\mathbf{d}\omega_0^0 = (1-\kappa)\,\mathbf{e}^a\omega_a + \frac{1}{2}\Omega_{0bc}^{0}\mathbf{e}^b\mathbf{e}^c \tag{327}$$

$$
\begin{aligned}
0 &= (1-\kappa)\,\mathbf{e}^a\mathbf{\Omega}_a + \mathbf{d}\left(\frac{1}{2}\Omega^0_{0bc}\omega^b\omega^c\right) && (328)\\
&= (1-\kappa)\,\mathbf{e}^a\mathbf{\Omega}_a + \frac{1}{2}\partial_\alpha\Omega^0_{0bc}e_a{}^{\alpha}\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c && (329)\\
&\quad +\frac{1}{2}\partial^\alpha\Omega^0_{0bc}\mathbf{d}y_\alpha\mathbf{e}^b\mathbf{e}^c && (330)\\
&\quad +\frac{1}{2}\Omega^0_{0bc}\mathbf{d}\mathbf{e}^b\mathbf{e}^c - \frac{1}{2}\Omega^0_{0bc}\mathbf{e}^b\mathbf{d}\mathbf{e}^c && (331)\\
&= (1-\kappa)\,\mathbf{e}^a\mathbf{\Omega}_a + \frac{1}{2}\partial_\alpha\Omega^0_{0bc}e_a{}^{\alpha}\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c && (332)\\
&\quad +\frac{1}{2}\partial^\alpha\Omega^0_{0bc}\mathbf{d}y_\alpha\mathbf{e}^b\mathbf{e}^c + \Omega^0_{0bc}\mathbf{d}\mathbf{e}^b\mathbf{e}^c && (333)
\end{aligned}
$$

Since

$$\mathbf{d}y_\alpha = A_{\alpha a}\mathbf{e}^a + B^\alpha_\alpha\omega_a$$

for some $A_{\alpha a}$ and B^α_α, and $\mathbf{d}\mathbf{e}^b$ is bilinear in the solder form, the only term with two co-solder forms is

$$\frac{1}{2}(1-\kappa)\,\mathbf{e}^a\Omega_a{}^{bc}\omega_b\omega_c = 0$$

which implies

$$(1-\kappa)\,\Omega_a{}^{bc} = 0$$

Unless $\kappa = 1$, the involution of the solder form goes through as for the generic case, giving

$$
\begin{aligned}
\omega^a_b &= \omega^a_{bc}\mathbf{e}^c && (334)\\
\omega^a &= \mathbf{e}^a && (335)\\
&= e_\alpha{}^a(x,y)\,\mathbf{d}x^\alpha && (336)\\
\omega_a &= \mathbf{f}_a + b_{ab}\mathbf{e}^b && (337)\\
&= \partial^\alpha\theta_a(x,y)\,\mathbf{d}y_\alpha + b_{a\beta}(x,y)\,\mathbf{d}x^\beta && (338)\\
\omega^0_0 &= W_c\mathbf{e}^c = W_\beta\mathbf{d}x^\beta && (339)
\end{aligned}
$$

Both $e_\alpha{}^a$ and $f_a{}^\alpha$ must be invertible.

Again the structure equation for the solder form restricts the functional dependence of the solder form to x only, $\mathbf{e}^a = \mathbf{e}^a(x)$.

We can simplify the form of the co-solder form as well. First, observe that there is total freedom in the choice of the coordinate functions θ_a. Extracting an inverse solder form $e_a^{\ \ \mu}(x)$ and a constant, a, from both θ_a and $b_{a\beta}$,

$$\begin{aligned}
\omega_a &= \partial^\alpha \theta_a (x,y)\, \mathbf{d}y_\alpha + b_{a\beta}(x,y)\,\mathbf{d}x^\beta \\
&= a e_a^{\ \ \mu} \left(\partial^\alpha \theta_\mu \mathbf{d}y_\alpha + b_{\mu\beta}(x,y)\,\mathbf{d}x^\beta\right)
\end{aligned}$$

we rewrite the y_α derivative of θ_μ as an exterior derivative on the full biconformal space,

$$\begin{aligned}
\omega_a &= a e_a^{\ \ \mu} \left(\partial^\alpha \theta_\mu \mathbf{d}y_\alpha + b_{\mu\beta}(x,y)\,\mathbf{d}x^\beta\right) && (340)\\
&= a e_a^{\ \ \mu} \left(\mathbf{d}\theta_\mu - \partial_\beta \theta_\mu \mathbf{d}x^\beta + b_{\mu\beta}\mathbf{d}x^\beta\right) && (341)\\
&= a e_a^{\ \ \mu} \left(\mathbf{d}\theta_\mu + (b_{\mu\beta} - \partial_\beta\theta_\mu)\,\mathbf{d}x^\beta\right) && (342)
\end{aligned}$$

we see that coordinate freedom on the $\mathbf{e}^a = 0$ submanifold leads to a change in $b_{\alpha\beta}$. We use this freedom below to simplify the form of $b_{\alpha\beta}$.

8.1.1 Back to the structure equations

In the following subsections, we work systematically through the structure equations to arrive at a final form for the connection.

The solder form structure equation We begin with eq.(123) for the solder form,

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \omega_b^a + \omega_0^0 \mathbf{e}^a \tag{343}$$

This is solved for the spin connection as before, leading to

$$\omega_b^a = \alpha_b^a - 2\Delta_{db}^{ac} W_c \mathbf{e}^d \tag{344}$$

The dilatation equation Next, consider eq.(125) for the dilatation,

$$\mathbf{d}\omega_0^0 = (1-\kappa)\,\omega^a\omega_a + \frac{1}{2}\Omega_{0bc}^0 \mathbf{e}^b\mathbf{e}^c \tag{345}$$

Expanding the right side,

$$\begin{aligned}
\mathbf{d}\omega_0^0 &= (1-\kappa)\,\omega^a\omega_a + \frac{1}{2}\Omega_{0bc}^0 \mathbf{e}^b\mathbf{e}^c && (346)\\
\mathbf{d}\left(W_\beta \mathbf{d}x^\beta\right) &= (1-\kappa)\left(\mathbf{e}_\alpha^{\ \ a}\mathbf{d}x^\alpha\right) a e_a^{\ \ \mu}\left(\mathbf{d}\theta_\mu + (b_{\mu\beta}-\partial_\beta\theta_\mu)\,\mathbf{d}x^\beta\right) + \frac{1}{2}\Omega_{0\mu\beta}^0 \mathbf{d}x^\mu\mathbf{d}x^\beta && (347)\\
&= (1-\kappa)\,a\left(\mathbf{d}x^\mu\mathbf{d}\theta_\mu + (b_{\mu\beta}-\partial_\beta\theta_\mu)\,\mathbf{d}x^\mu\mathbf{d}x^\beta\right) + \frac{1}{2}\Omega_{0\mu\beta}^0 \mathbf{d}x^\mu\mathbf{d}x^\beta && (348)
\end{aligned}$$

Choosing $a = (1-\kappa)^{-1} = const.$, setting

$$\tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \partial_\beta \theta_\alpha$$

and expanding the exterior derivative,

$$\partial^\alpha W_\beta \mathbf{d}\theta_\alpha \mathbf{d}x^\beta + \partial_\alpha W_\beta \mathbf{d}x^\alpha \mathbf{d}x^\beta = \mathbf{d}x^\beta \mathbf{d}\theta_\beta + \tilde{b}_{\alpha\beta} \mathbf{d}x^\alpha \mathbf{d}x^\beta + \frac{1}{2}\Omega^0_{0\alpha\beta} \mathbf{d}x^\alpha \mathbf{d}x^\beta \quad (349)$$

we equate the coefficients of like terms

$$\begin{aligned} \partial_\alpha W_\beta - \partial_\beta W_\alpha &= \tilde{b}_{\alpha\beta} - \tilde{b}_{\beta\alpha} + \Omega^0_{0\alpha\beta} && (350) \\ \partial^\alpha W_\beta &= -\delta^\beta_\alpha && (351) \end{aligned}$$

The second of these, eq.(351), may be integrated immediately to give

$$W_\beta = -\theta_\beta + g_\beta(x) \quad (352)$$

This form is independent of the choice for the θ_β coordinates. Substituting this solution into eq.(350),

$$\partial_\alpha g_\beta - \partial_\beta g_\alpha = b_{\alpha\beta} - \partial_\beta \theta_\alpha - b_{\beta\alpha} + \partial_\alpha \theta_\beta + \Omega^0_{0\alpha\beta} \quad (353)$$

Rearranging, we have

$$b_{\alpha\beta} - b_{\beta\alpha} + \Omega^0_{0\alpha\beta} = \partial_\beta (\theta_\alpha - g_\alpha) - \partial_\alpha (\theta_\beta - g_\beta) \quad (354)$$

The best we can do now with our choice of θ_α is make the right side vanish. With this choice we have

$$b_{\alpha\beta} - b_{\beta\alpha} + \Omega^0_{0\alpha\beta} = 0 \quad (355)$$

The result is still accomplished by setting the coordinate on the $\mathbf{e}^a = 0$ submanifolds to be

$$y_\alpha = \theta_\alpha - g_\alpha$$

With this choice the Weyl vector still takes the simple form

$$W_\beta = -y_\beta$$

The only difference is that b_{ab} now has an antisymmetric part.

The curvature equation Now consider the curvature equation

$$\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d - 2\kappa\Delta^{ac}_{db}\omega_c\mathbf{e}^d = \mathbf{d}\omega^a_b - \omega^c_b\omega^a_c - \Delta^{ad}_{cb}\ \omega_d\mathbf{e}^c \tag{356}$$

The first part of the calculation proceeds as for the generic case. Define the curvature of $\alpha^a_b\left(e\left(x\right)\right)$ in the usual way,

$$\mathbf{R}^a_b = \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c \tag{357}$$

In particular, $\mathbf{R}^a_b$ depends only on x. Expanding,

$$\begin{aligned}\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d - 2\kappa\Delta^{ac}_{db}\omega_c\mathbf{e}^d &= \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c - 2\Delta^{ac}_{db}\omega_c\mathbf{e}^d \\ &\quad +\mathbf{d}\beta^a_b - \alpha^c_b\beta^a_c - \beta^c_b\alpha^a_c - \beta^c_b\beta^a_c &(358)\\ &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\omega_c\mathbf{e}^d \\ &\quad +\mathbf{d}\beta^a_b - \alpha^c_b\beta^a_c - \beta^c_b\alpha^a_c - \beta^c_b\beta^a_c &(359)\end{aligned}$$

or

$$\begin{aligned}\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\left(1-\kappa\right)\Delta^{ac}_{db}\omega_c\mathbf{e}^d \\ &\quad +\mathbf{d}\beta^a_b - \alpha^c_b\beta^a_c - \beta^c_b\alpha^a_c - \beta^c_b\beta^a_c\end{aligned}$$

The exterior derivative is

$$\begin{aligned}\mathbf{d}\beta^a_b &= \mathbf{d}\left(-2\Delta^{ac}_{db}e_c\ ^{\mu}W_\mu\mathbf{e}^d\right) \\ &= \mathbf{d}\left(2\Delta^{ac}_{db}e_c\ ^{\mu}y_\mu\mathbf{e}^d\right) \\ &= 2\Delta^{ac}_{db}\mathbf{d}e_c\ ^{\mu}y_\mu\mathbf{e}^d + 2\Delta^{ac}_{db}e_c\ ^{\mu}\mathbf{d}y_\mu\mathbf{e}^d + 2\Delta^{ac}_{db}e_c\ ^{\mu}y_\mu\mathbf{d}\mathbf{e}^d\end{aligned}$$

Then since $\omega_a = \left(1-\kappa\right)^{-1}\left(\mathbf{f}_a + \mathbf{b}_a\right)$,

$$\begin{aligned}\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\left(e_c\ ^{\mu}\mathbf{d}y_\mu + \mathbf{b}_c\right)\mathbf{e}^d \\ &\quad +2\Delta^{ac}_{db}\mathbf{d}e_c\ ^{\mu}y_\mu\mathbf{e}^d + 2\Delta^{ac}_{db}e_c\ ^{\mu}\mathbf{d}y_\mu\mathbf{e}^d \\ &\quad +2\Delta^{ac}_{db}e_c\ ^{\mu}y_\mu\mathbf{d}\mathbf{e}^d - \alpha^c_b\left(-2\Delta^{ae}_{dc}e_e\ ^{\mu}W_\mu\mathbf{e}^d\right) \\ &\quad -\left(-2\Delta^{ce}_{db}e_e\ ^{\mu}W_\mu\mathbf{e}^d\right)\alpha^a_c \\ &\quad -\left(-2\Delta^{ce}_{db}e_e\ ^{\mu}W_\mu\mathbf{e}^d\right)\left(-2\Delta^{af}_{gc}e_f\ ^{\mu}W_\mu\mathbf{e}^g\right) &(360)\end{aligned}$$

The $\mathbf{d}y_\mu$ dependent terms,

$$0 = -2\Delta^{ac}_{db}e_c\ ^{\mu}\mathbf{d}y_\mu\mathbf{e}^d + 2\Delta^{ac}_{db}e_c\ ^{\mu}\mathbf{d}y_\mu\mathbf{e}^d \tag{361}$$

cancel identically. Expanding the remainder, and using the antisymmetry of the spin connection, $\eta_{dc}\alpha_b^c = -\eta_{bc}\alpha_d^c$,

$$
\begin{aligned}
\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathbf{b}_c\mathbf{e}^d \\
&\quad +2\Delta^{ac}_{db}\mathbf{d}e_c{}^{\mu}y_\mu\mathbf{e}^d + 2\Delta^{ac}_{db}e_c{}^{\mu}y_\mu\mathbf{d}\mathbf{e}^d \\
&\quad -2\alpha^c_b\Delta^{ae}_{dc}e_e{}^{\mu}y_\mu\mathbf{e}^d - 2\Delta^{ce}_{db}e_e{}^{\mu}y_\mu\mathbf{e}^d\alpha^a_c \\
&\quad -4\Delta^{ce}_{db}e_e{}^{\mu}y_\mu\mathbf{e}^d\Delta^{af}_{gc}e_f{}^{\nu}y_\nu\mathbf{e}^g \\
&= \mathbf{R}^a_b - \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{b}_c\mathbf{e}^d \\
&\quad + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{d}e_c{}^{\mu}y_\mu\mathbf{e}^d \\
&\quad + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)e_c{}^{\mu}y_\mu\mathbf{d}\mathbf{e}^d \qquad (362) \\
&\quad -\alpha^e_b e_e{}^{\mu}y_\mu\mathbf{e}^a - \eta^{ae}\eta_{bc}e_e{}^{\mu}y_\mu\alpha^c_d\mathbf{e}^d \\
&\quad -e_b{}^{\mu}y_\mu\mathbf{e}^c\alpha^a_c - \eta^{ca}\eta_{db}e_e{}^{\mu}y_\mu\mathbf{e}^d\alpha^e_c \\
&\quad -e_b{}^{\mu}y_\mu\mathbf{e}^d e_d{}^{\nu}y_\nu\mathbf{e}^a + \eta^{ef}\eta_{db}e_e{}^{\mu}y_\mu\mathbf{e}^d e_f{}^{\nu}y_\nu\mathbf{e}^a \\
&\quad -\eta_{db}\eta^{af}e_e{}^{\mu}y_\mu\mathbf{e}^d e_f{}^{\nu}y_\nu\mathbf{e}^e \qquad (363)
\end{aligned}
$$

Setting $y_a = e_a{}^{\mu}y_\mu$, this becomes

$$
\begin{aligned}
\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{R}^a_b - \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{b}_c\mathbf{e}^d \qquad (364) \\
&\quad + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\left(\mathbf{d}e_c{}^{\mu} - \alpha^e_c e_e{}^{\mu}\right)y_\mu\mathbf{e}^d \qquad (365) \\
&\quad + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)y_c\left(\mathbf{d}\mathbf{e}^d - \mathbf{e}^e\alpha^d_e\right) \qquad (366) \\
&\quad - \left(\delta^c_b\delta^a_d - \eta_{db}\eta^{ac}\right)y_c y_e\mathbf{e}^e\mathbf{e}^d \qquad (367) \\
&\quad + \left(\delta^c_b\delta^a_d - \eta_{db}\eta^{ca}\right)\frac{1}{2}\eta_{ce}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^e\mathbf{e}^d \qquad (368) \\
&= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\Big(\mathbf{b}_c\mathbf{e}^d + \left(e_c{}^{\nu}y_\mu\mathbf{\Gamma}^\mu_\nu\right)\mathbf{e}^d \qquad (369) \\
&\quad +y_c y_e\mathbf{e}^e\mathbf{e}^d - \frac{1}{2}\eta_{ce}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^e\mathbf{e}^d\Big) \qquad (370)
\end{aligned}
$$

where we use

$$
\begin{aligned}
\mathbf{d}e_c{}^{\mu} - \alpha^e_c e_e{}^{\mu} + e_c{}^{\nu}\Gamma^\mu_{\nu\alpha}\mathbf{d}x^\alpha &= 0 \\
\Gamma^\mu_{\nu\alpha}\mathbf{d}x^\alpha &= \mathbf{\Gamma}^\mu_\nu
\end{aligned}
$$

in the last step. Thus, $\Gamma^\mu_{\nu\alpha}$ is the Christoffel connection corresponding to solder form $\mathbf{e}^a$ and spin connection α^e_c. For simplicity, define

$$
\mathbf{c}_a = c_{ac}\mathbf{e}^c = \mathbf{b}_a + e_a{}^{\nu}y_\mu\mathbf{\Gamma}^\mu_\nu + y_a y_c\mathbf{e}^c - \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c \qquad (371)
$$

Then we have simply

$$\frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d = \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathbf{c}_c\mathbf{e}^d \tag{372}$$

or, in components,

$$\begin{aligned}
\Omega^a_{bcd} &= R^a_{bcd} - 2\Delta^{ae}_{db}c_{ec} + 2\Delta^{ae}_{cb}c_{ed} && (373)\\
&= R^a_{bcd} - \left(\delta^a_d\delta^e_b - \eta_{db}\eta^{ae}\right)c_{ec} + \left(\delta^a_c\delta^e_b - \eta_{cb}\eta^{ae}\right)c_{ed} && (374)\\
&= R^a_{bcd} - \delta^a_d c_{bc} + \eta_{db}c^a{}_c + \delta^a_c c_{bd} - \eta_{cb}c^a{}_d && (375)
\end{aligned}$$

$$\Omega^a_{bcd} = R^a_{bcd} - \delta^a_d c_{bc} + \eta_{db}c^a{}_c + \delta^a_c c_{bd} - \eta_{cb}c^a{}_d \tag{376}$$

where

$$\mathbf{c}_a = c_{ac}\mathbf{e}^c = \mathbf{b}_a + e_a{}^\nu y_\mu\boldsymbol{\Gamma}^\mu_\nu + y_a y_c\mathbf{e}^c - \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^c \tag{377}$$

Here the calculation starts to differ, because the field equation still contains the dilatational curvature. It is convenient to find the symmetric and antisymmetric parts of c_{ab},

$$\begin{aligned}
c_{(ab)} &= b_{(ab)} + y_\mu\boldsymbol{\Gamma}^\mu_{ab} + y_a y_b - \frac{1}{2}\eta_{ab}\left(\eta^{gh}y_g y_h\right)\\
c_{[ab]} &= b_{[ab]}
\end{aligned}$$

The field equation is

$$\begin{aligned}
\Omega^a_{bac} + \Omega^a_{cab} &= 0 && (378)\\
\Omega^a_{bac} - \Omega^a_{cab} &= -(n-2)\,\Omega^0_{0bc} && (379)
\end{aligned}$$

Inserting the expression for the trace of the curvature,

$$\Omega^c_{bcd} = R^c_{bcd} - c_{bd} + \eta_{db}c^c{}_c + nc_{bd} - c_{bd} \tag{380}$$

we find

$$\begin{aligned}
0 &= \Omega^c_{bcd} + \Omega^c_{dcb} = 2R^c_{bcd} + 2\eta_{db}c^c{}_c + (n-2)\left(c_{bd} + c_{db}\right) && (381)\\
\Omega^a_{bac} - \Omega^a_{cab} &= -(n-2)\,\Omega^0_{0bc} = 2(n-2)\,b_{[bd]} && (382)
\end{aligned}$$

By eq.(**??**), the second equation is exactly satisfied, while the first gives the Eisenhart tensor as before.

$$
\begin{aligned}
c_{(bd)} &= -\frac{1}{n-2}\left(R_{bd} - \frac{1}{2(n-1)}\eta_{bd}R\right) \equiv \mathcal{R}_{bd} \\
c_{[ab]} &= -\frac{1}{2}\Omega^{0}_{0bc}
\end{aligned}
$$

With this, the full spacetime curvature 2-form is the Weyl (conformal) curvature 2-form of the spacetime submanifold plus the spacetime dilatational curvature,

$$
\begin{aligned}
\mathbf{\Omega}^{a}_{b} &= \mathbf{R}^{a}_{b} - 2\Delta^{ac}_{db}\mathcal{R}_{c}\mathbf{e}^{d} - 2\Delta^{ac}_{db}\left(-\frac{1}{2}\Omega^{0}_{0ce}\right)\mathbf{e}^{e}\mathbf{e}^{d} && (383) \\
&= \mathbf{C}^{a}_{b} + \Delta^{ac}_{db}\Omega^{0}_{0ce}\mathbf{e}^{e}\mathbf{e}^{d} && (384) \\
\frac{1}{2}\Omega^{a}_{bcd}\mathbf{e}^{c}\mathbf{e}^{d} &= \frac{1}{2}C^{a}_{bcd}\mathbf{e}^{c}\mathbf{e}^{d} + \Delta^{ac}_{db}\Omega^{0}_{0ce}\mathbf{e}^{e}\mathbf{e}^{d} && (385)
\end{aligned}
$$

We also have the form of $\mathbf{b}_a$,

$$
\mathbf{b}_a = \mathcal{R}_a - e_a^{\ \ \nu} y_\mu \mathbf{\Gamma}^{\mu}_{\nu} - y_a y_c \mathbf{e}^{c} + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^{c} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c} \qquad (386)
$$

The symmetry of the components, $b_{ab} = b_{ba}$ is as before, but now the antisymmetric part gives the spacetime dilatation.

We have now solved for the entire connection:

$$
\begin{aligned}
\omega^{a}_{b} &= \alpha^{a}_{b} - 2\Delta^{ac}_{db}W_c\mathbf{e}^{d} && (387) \\
\omega^{a} &= \mathbf{e}^{a}(x) && (388) \\
\omega_{a} &= a\left(\mathbf{f}_a + \mathbf{b}_a\right) && (389) \\
\omega^{0}_{0} &= W_c\mathbf{e}^{c} = -y_\beta \mathbf{d}x^{\beta} && (390)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= (1-\kappa)^{-1} && (391) \\
\mathbf{f}_a &= e_a^{\ \ \beta}(x)\ \mathbf{d}y_\beta && (392) \\
\mathbf{b}_a &= \mathcal{R}_a - e_a^{\ \ \nu} y_\mu \mathbf{\Gamma}^{\mu}_{\nu} - y_a y_c \mathbf{e}^{c} + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_g y_h\right)\mathbf{e}^{c} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c} && (393) \\
&= \mathbf{b}^{0}_{a} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c} && (394)
\end{aligned}
$$

where $\mathbf{b}^{0}_{a}$ denotes the generic form of $\mathbf{b}_a$. All that is left is to solve the co-torsion field and structure equations.

The co-torsion We must still satisfy the following equations:

$$\alpha\Delta^{cf}_{eb}\left(\Omega_{gc}{}^{b} - \delta^{b}_{g}\Omega_{ac}{}^{a}\right) = 0 \quad (395)$$

$$-\left(n-2\right)\Omega_{ae}{}^{a} = \eta_{ea}\eta^{bc}\Omega_{bc}{}^{a} \quad (396)$$

where

$$\Omega_{[abc]} = 0 \quad (397)$$

$$\Omega_{[ac]}^{\;\;b} = 0 \quad (398)$$

$$\Omega_{a}{}^{bc} = 0 \quad (399)$$

In addition we have the structure equation for the co-torsion, eq.(124):

$$\frac{1}{2}\Omega_{abc}\mathbf{e}^{b}\mathbf{e}^{c} + \Omega_{ac}^{\;\;b}\omega_{b}\mathbf{e}^{c} = \mathbf{d}\omega_{a} - \omega^{b}_{a}\omega_{b} - \omega_{a}\omega^{0}_{0} \quad (400)$$

Working in the orthonormal basis, we will need certain exterior derivatives.

$$\begin{aligned}\mathbf{d}\mathbf{f}_{a} &= \mathbf{d}e_{a}{}^{\beta}\mathbf{d}y_{\beta}\\ &= \left(\alpha^{\beta}_{a} - \boldsymbol{\Gamma}^{\beta}_{a}\right)\mathbf{d}y_{\beta}\end{aligned}$$

and

$$\mathbf{d}\mathbf{f}_{a} - \alpha^{c}_{a}\mathbf{f}_{c} + \boldsymbol{\Gamma}^{c}_{a}\mathbf{f}_{c} = 0$$

where

$$\begin{aligned}\mathbf{f}_{a} &= e_{a}{}^{\beta}\left(x\right)\;\mathbf{d}y_{\beta}\\ \mathbf{b}_{a} &= \mathcal{R}_{a} - e_{a}{}^{\nu}y_{\mu}\boldsymbol{\Gamma}^{\mu}_{\nu} - y_{a}y_{c}\mathbf{e}^{c} + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_{g}y_{h}\right)\mathbf{e}^{c} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c}\\ &= \mathbf{b}^{0}_{a} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c}\end{aligned}$$

Using these relations, we compute the terms on the right side:

$$\begin{aligned}\frac{1}{a}\boldsymbol{\Omega}_{a} &= \mathbf{d}\mathbf{f}_{a} + \mathbf{d}\mathbf{b}_{a} - \omega^{b}_{a}\left(\mathbf{f}_{b} + \mathbf{b}_{b}\right) - \left(\mathbf{f}_{a} + \mathbf{b}_{a}\right)\omega^{0}_{0}\\ &= \alpha^{c}_{a}\mathbf{f}_{c} - \boldsymbol{\Gamma}^{c}_{a}\mathbf{f}_{c} - \alpha^{b}_{a}\mathbf{f}_{b} + 2\Delta^{bc}_{da}W_{c}\mathbf{e}^{d}\mathbf{f}_{b} - \mathbf{f}_{a}\omega^{0}_{0}\\ &\quad + \mathbf{d}\mathbf{b}_{a} - \alpha^{b}_{a}\mathbf{b}_{b} + 2\Delta^{bc}_{da}W_{c}\mathbf{e}^{d}\mathbf{b}_{b} - \mathbf{b}_{a}\omega^{0}_{0}\end{aligned}$$

$$
\begin{aligned}
= \quad & -\boldsymbol{\Gamma}^c_a \mathbf{f}_c + W_a \mathbf{e}^b \mathbf{f}_b - \eta^{bc}\eta_{da} W_c \mathbf{e}^d \mathbf{f}_b - \mathbf{f}_a W_c \mathbf{e}^c \\
& + \mathbf{d}\left(\mathbf{b}^0_a - \frac{1}{2}\Omega^0_{0ac}\mathbf{e}^c\right) - \alpha^b_a\left(\mathbf{b}^0_b - \frac{1}{2}\Omega^0_{0bc}\mathbf{e}^c\right) \\
& + 2\Delta^{bc}_{da} W_c \mathbf{e}^d\left(\mathbf{b}^0_b - \frac{1}{2}\Omega^0_{0bc}\mathbf{e}^c\right) - \left(\mathbf{b}^0_a - \frac{1}{2}\Omega^0_{0ac}\mathbf{e}^c\right)\omega^0_0
\end{aligned}
$$

As for the generic case, when we expand $\mathbf{d}\mathbf{b}^0_a$ the terms in $\mathbf{f}_a$ cancel identically, while the remaining $\mathbf{b}^0_a$ terms give the generic result,

$$
\begin{aligned}
\frac{1}{2a}\Omega_{abc}\mathbf{e}^b\mathbf{e}^c + \frac{1}{a}\Omega_{ac}^{\ \ \ b}\omega_b\mathbf{e}^c \quad = \quad & \mathbf{d}\mathcal{R}_a - \mathbf{d}e_a^{\ \ \nu} y_\mu \boldsymbol{\Gamma}^\mu_\nu - e_a^{\ \ \nu} y_\mu \mathbf{d}\boldsymbol{\Gamma}^\mu_\nu - y_a y_c \mathbf{d}\mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{d}\mathbf{e}^c \\
& - \frac{1}{2}\Omega^{0\ \ \ ,\mu}_{0ac}\mathbf{d}y_\mu \mathbf{e}^c - \frac{1}{2}\mathbf{d}_x \Omega^0_{0ac}\mathbf{e}^c - \alpha^b_a\left(\mathbf{b}^0_b - \frac{1}{2}\Omega^0_{0bc}\mathbf{e}^c\right) \\
& + 2\Delta^{bc}_{da} W_c \mathbf{e}^d\left(\mathbf{b}^0_b - \frac{1}{2}\Omega^0_{0bc}\mathbf{e}^c\right) - \left(\mathbf{b}^0_a - \frac{1}{2}\Omega^0_{0ac}\mathbf{e}^c\right)\omega^0_0
\end{aligned}
$$

We therefore have

$$
\frac{1}{2}\Omega_{abc}\mathbf{e}^b\mathbf{e}^c + \Omega_{ac}^{\ \ \ b}\omega_b\mathbf{e}^c = \mathbf{d}\omega_a - \omega^b_a\omega_b - \omega_a\omega^0_0
$$

so that

$$
\begin{aligned}
\frac{1}{2a}\Omega_{abc}\mathbf{e}^b\mathbf{e}^c + \frac{1}{a}\Omega_{ac}^{\ \ \ b}\mathbf{b}^0_b\mathbf{e}^c \quad &= \quad \mathbf{d}\mathcal{R}_a - \alpha^b_a\mathcal{R}_b - \eta^{bc}\eta_{da}W_c\mathbf{e}^d\mathcal{R}_b - \mathcal{R}_a W_b \mathbf{e}^b \\
& \qquad - e_a^{\ \ \beta} y_\mu\left(\mathbf{d}\boldsymbol{\Gamma}^\mu_\beta - \boldsymbol{\Gamma}^\nu_\beta\boldsymbol{\Gamma}^\mu_\nu\right) \\
&= \quad \mathbf{d}\mathcal{R}_a - \alpha^b_a\mathcal{R}_b + W_b\left(\mathbf{R}^b_a - 2\Delta^{bc}_{da}\mathcal{R}_c\mathbf{e}^d\right) \\
&= \quad \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + W_b\mathbf{C}^b_a - \mathbf{D}_{(x,\alpha,W)}\boldsymbol{\Omega}^0_{0a} \\
\frac{1}{a}\Omega_{ac}^{\ \ \ b}\mathbf{f}_b\mathbf{e}^c \quad &= \quad -\frac{1}{2}\Omega^{0\ \ \ ,\mu}_{0ac} e_\mu^{\ \ b}\mathbf{f}_b\mathbf{e}^c
\end{aligned}
$$

where

$$
\frac{1}{2}\Omega^0_{0ac}\mathbf{e}^c = \boldsymbol{\Omega}^0_{0a}
$$

and

$$
\begin{aligned}
\mathbf{D}\boldsymbol{\Omega}^0_{0a} \quad &= \quad \mathbf{d}\boldsymbol{\Omega}^0_{0a} - \omega^b_a\boldsymbol{\Omega}^0_{0b} - \boldsymbol{\Omega}^0_{0a}\omega^0_0 \\
&= \quad \mathbf{d}\boldsymbol{\Omega}^0_{0a} - \omega^b_a\boldsymbol{\Omega}^0_{0b} - \boldsymbol{\Omega}^0_{0a}\omega^0_0 \\
&= \quad \mathbf{d}\boldsymbol{\Omega}^0_{0a} - \left(\alpha^b_a - \left(\delta^b_d W_a + \delta^b_a W_d - \eta_{ad}W^b\right)\mathbf{e}^d\right)\boldsymbol{\Omega}^0_{0b} \\
&= \quad \mathbf{D}_{(x,\alpha,W)}\boldsymbol{\Omega}^0_{0a}
\end{aligned}
$$

Imposing the field equation for the cross term,

$$\begin{aligned}
0 &= \Omega_{ac}^{\ \ b} - \Omega_{ca}^{\ \ b} \\
&= -\frac{a}{2}\Omega^{0}_{0ac}{}^{,\mu} e_{\mu}^{\ \ b} + \frac{a}{2}\Omega^{0}_{0ca}{}^{,\mu} e_{\mu}^{\ \ b} \\
&= a\Omega^{0}_{0ca}{}^{,\mu} e_{\mu}^{\ \ b}
\end{aligned}$$

so that

$$\Omega^{0}_{0ca} = \Omega^{0}_{0ca}(x)$$

and

$$\Omega_{ac}^{\ \ b} = 0$$

Therefore,

$$\frac{1}{2a}\Omega_{abc}\mathbf{e}^{b}\mathbf{e}^{c} = \mathbf{D}_{(x,\alpha)}\mathcal{R}_{a} + W_{b}\mathbf{C}^{b}_{a} - \mathbf{D}_{x,\alpha,W}\left(\frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c}\right)$$

9 Summary

The connection takes the final form

$$\begin{aligned}
\omega^{a}_{b} &= \alpha^{a}_{b} - 2\Delta^{ac}_{db}W_{c}\mathbf{e}^{d} && (401)\\
\omega^{a} &= \mathbf{e}^{a}(x) && (402)\\
\omega_{a} &= a\left(\mathbf{f}_{a}+\mathbf{b}_{a}\right) && (403)\\
\omega^{0}_{0} &= W_{c}\mathbf{e}^{c} = -y_{\beta}\mathbf{d}x^{\beta} && (404)
\end{aligned}$$

where

$$\begin{aligned}
a &= (1-\kappa)^{-1} && (405)\\
\mathbf{f}_{a} &= e_{a}^{\ \ \beta}(x)\ \mathbf{d}y_{\beta} && (406)\\
\mathbf{b}_{a} &= \mathcal{R}_{a} - e_{a}^{\ \ \nu}y_{\mu}\mathbf{\Gamma}^{\mu}_{\nu} - y_{a}y_{c}\mathbf{e}^{c} + \frac{1}{2}\eta_{ac}\left(\eta^{gh}y_{g}y_{h}\right)\mathbf{e}^{c} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c} && (407)\\
&= \mathbf{b}^{0}_{a} - \frac{1}{2}\Omega^{0}_{0ac}\mathbf{e}^{c} && (408)
\end{aligned}$$

The curvatures are

$$
\begin{aligned}
\mathbf{\Omega}^a_b &= \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c\mathbf{e}^d - 2\kappa\Delta^{ac}_{db}\omega_c\mathbf{e}^d && (409)\\
&= \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathcal{R}_c\mathbf{e}^d - 2a\kappa\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d && (410)\\
&= \mathbf{C}^a_b - 2a\kappa\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d && (411)\\
\mathbf{\Omega}^a &= 0 && (412)\\
\mathbf{\Omega}_a &= \mathbf{D}_{(x,\alpha)}\mathcal{R}_a + W_b\mathbf{C}^b_a - \mathbf{D}_{x,\alpha,W}\left(\frac{1}{2}\Omega^0_{0ac}\mathbf{e}^c\right) && (413)\\
\mathbf{\Omega}^0_0 &= -\kappa\mathbf{e}^a\omega_a = -a\kappa\mathbf{e}^a\mathbf{f}_a = -a\kappa\mathbf{d}x^\beta\mathbf{d}y_\beta && (414)
\end{aligned}
$$

10 Final special sub-case (in progress)

There is one final subcase which occurs when

$$\kappa = 1$$