Contents

1 Basic conformal relations

1.1 Structure equations

The Maurer-Cartan equations for the conformal group are

$$\begin{aligned} \mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + 2\Delta^{ac}_{db}\omega_c\omega^d \\ \mathbf{d}\omega^a &= \omega^c\omega^a_c + \omega\omega^a \\ \mathbf{d}\omega_a &= \omega^c_a\omega_c + \omega_a\omega \\ \mathbf{d}\omega &= \omega^a\omega_a \end{aligned}$$

where the geometric interpretation of each set of generators is

$$\begin{array}{cl} \omega^a_b & Lorentz transformation \\ \omega^a & Translation \\ \omega_a & Special conformal transformation \\ \omega & Dilatation \end{array}$$

This particular breakdown of the generators is important because each of these four sets of transformations is invariant under the homothetic group. We discuss this below.

If we set the special conformal transformations to zero, $\omega_a = 0$, the remaining subgroup is the inhomogeneous Weyl group, described by the structure equations.

$$\begin{aligned} \mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c \\ \mathbf{d}\omega^a &= \omega^c\omega^a_c + \omega\omega^a \\ \mathbf{d}\omega &= 0 \end{aligned}$$

This has further subgroups. If we set the Weyl vector to zero, $\omega = 0$, so that there are no dilatations, then we have the Maurer-Cartan equations of the Poincaré group

$$\begin{aligned} \mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c \\ \mathbf{d}\omega^a &= \omega^c\omega^a_c \end{aligned}$$

If, instead of eliminating dilatations, we set the translational gauge vector to zero, $\omega^a = 0$, then we have the homogeneous Weyl group,

$$\begin{aligned} \mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c \\ \mathbf{d}\omega &= 0 \end{aligned}$$

comprised of Lorentz transformations and dilatations only. Finally, setting both translatiosn and dilatations to zero, $\omega^a = \omega = 0$, only the Maurer-Cartan structure equations for the Lorentz group remain:

$$\mathbf{d}\omega^a_b = \omega^c_b\omega^a_c$$

1.2 Killing metric

The Killing metric of the conformal group, given by

$$K_{\Sigma\Lambda} = \lambda c_{\Xi\Sigma}{}^{\Delta} c_{\Delta\Lambda}{}^{\Xi}$$

With a suitable choice of the constant λ this becomes

$$K_{\Sigma\Lambda} = \begin{pmatrix} \frac{1}{2}\Delta^{ac}_{db} & 0 & 0 & 0 \\ 0 & 0 & \delta^a_b & 0 \\ 0 & \delta^b_a & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

While the conformal Killing metric is itself non-degenerate, its restriction to the various subgroups may or may not induce a natural, non-degenerate metric.

The restrictions of $K_{\Sigma\Lambda}$ to the inhomogeneous Weyl group and the Poincaré group, respectively, give

$$K_{AB} = \begin{pmatrix} \frac{1}{2}\Delta^{ac}_{db} & 0 & 0 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$K_{AB} = \begin{pmatrix} \frac{1}{2}\Delta^{ac}_{db} & 0 \\ 0 & \mathbf{0} \end{pmatrix}$$

where the bold zero, $\mathbf{0}$, denotes the vanishing $n \times n$ translation sector. When gauged, these groups do not have natural metrics induced from the Killing metric.

The homogeneous Weyl and Lorentz groups fare better, both having non-degenerate Killing metrics

$$K_{AB} = \begin{pmatrix} \frac{1}{2}\Delta^{ac}_{db} & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$K_{AB} = \begin{pmatrix} \frac{1}{2}\Delta^{ac}_{db} \end{pmatrix}$$

respectively.

1.3 Gauge transformations

A Lie group acts on its connection 1-forms according to

$$\tilde{\omega} = g\omega g^{-1} - \mathbf{d}g\, g^{-1}$$

Expanding this infinitesimally for the conformal group using the 6-dim representation,

$$\begin{aligned} g^A_{\ B} &= \begin{pmatrix} \Lambda^a_b & \Lambda^a_4 & \Lambda^a_5 \\ \Lambda^4_b & \Lambda^4_4 & \Lambda^4_5 \\ \Lambda^5_b & \Lambda^5_4 & \Lambda^5_5 \end{pmatrix} \\ &= \begin{pmatrix} \Lambda^a_b & \Lambda^a_4 & \eta^{ac}\Lambda^4_c \\ \Lambda^4_b & \Lambda^4_4 & 0 \\ \eta_{bc}\Lambda^c_4 & 0 & -\Lambda^4_4 \end{pmatrix} \end{aligned}$$

where Λ^4_4 generates a dilatation, Λ^a_b generates a Lorentz transformation, Λ^a_4 generates a translation and Λ^4_a generates a special conformal transformation. Let $\bar{\Lambda}^A_B$ denote the inverse transformations, and in particular, $\bar{\Lambda}^4_4 = \frac{1}{\Lambda^4_4}$. Then we have for the Lorentz piece,

$$\begin{aligned} \tilde{\omega}^a_b &= \Lambda^a_B \omega^B_C \bar{\Lambda}^C_b - \mathbf{d}\Lambda^a_C \bar{\Lambda}^C_b \\ &= \Lambda^a_c \omega^c_d \bar{\Lambda}^d_b + \Lambda^a_c \omega^c_4 \bar{\Lambda}^4_b + \Lambda^a_c \omega^c_5 \bar{\Lambda}^5_b \\ &\quad + \Lambda^a_4 \omega^4_d \bar{\Lambda}^d_b + \Lambda^a_4 \omega^4_4 \bar{\Lambda}^4_b + \Lambda^a_5 \omega^5_d \bar{\Lambda}^d_b + \Lambda^a_5 \omega^5_5 \bar{\Lambda}^5_b \\ &\quad - \mathbf{d}\Lambda^a_c \bar{\Lambda}^c_b - \mathbf{d}\Lambda^a_4 \bar{\Lambda}^4_b - \mathbf{d}\Lambda^a_5 \bar{\Lambda}^5_b \\ &= \Lambda^a_c \omega^c_d \bar{\Lambda}^d_b + \Lambda^a_c \omega^c_4 \bar{\Lambda}^4_b + \Lambda^a_c \eta^{ce} \omega^4_e \eta_{bd} \bar{\Lambda}^d_4 \\ &\quad + \Lambda^a_4 \omega^4_d \bar{\Lambda}^d_b + \Lambda^a_4 \omega^4_4 \bar{\Lambda}^4_b + \eta^{ac} \Lambda^4_c \omega^e_4 \eta_{ed} \bar{\Lambda}^d_b - \eta^{ac} \Lambda^4_c \omega^4_4 \eta_{bd} \bar{\Lambda}^d_4 \\ &\quad - \mathbf{d}\Lambda^a_c \bar{\Lambda}^c_b - \mathbf{d}\Lambda^a_4 \bar{\Lambda}^4_b - \eta^{ac} \mathbf{d}\Lambda^4_c \eta_{bd} \bar{\Lambda}^d_4 \end{aligned}$$

For translations,

$$\begin{aligned}
\tilde{\omega}^a_4 &= \Lambda^a_B\omega^B_C\bar{\Lambda}^C_4 - \mathbf{d}\Lambda^a_C\,\bar{\Lambda}^C_4 \\
&= \Lambda^a_c\omega^c_d\bar{\Lambda}^d_4 + \Lambda^a_c\omega^c_4\bar{\Lambda}^4_4 \\
&\quad +\Lambda^a_4\omega^4_d\bar{\Lambda}^d_4 + \Lambda^a_4\omega^4_4\bar{\Lambda}^4_4 \\
&\quad +\Lambda^a_5\omega^5_d\bar{\Lambda}^d_4 \\
&\quad -\mathbf{d}\Lambda^a_c\,\bar{\Lambda}^c_4 - \mathbf{d}\Lambda^a_4\,\bar{\Lambda}^4_4 \\
&= \Lambda^a_c\omega^c_d\bar{\Lambda}^d_4 + \Lambda^a_c\omega^c_4\bar{\Lambda}^4_4 \\
&\quad +\Lambda^a_4\omega^4_d\bar{\Lambda}^d_4 + \Lambda^a_4\omega^4_4\bar{\Lambda}^4_4 + \eta^{ac}\Lambda^4_c\omega^e_4\eta_{ed}\bar{\Lambda}^d_4 \\
&\quad -\mathbf{d}\Lambda^a_c\,\bar{\Lambda}^c_4 - \mathbf{d}\Lambda^a_4\,\bar{\Lambda}^4_4
\end{aligned}$$

For special conformal transformations,

$$\begin{aligned}
\tilde{\omega}^4_a &= \Lambda^4_B\omega^B_C\bar{\Lambda}^C_a - \mathbf{d}\Lambda^4_C\,\bar{\Lambda}^C_a \\
&= \Lambda^4_c\omega^c_d\bar{\Lambda}^d_a + \Lambda^4_c\omega^c_4\bar{\Lambda}^4_a + \Lambda^4_c\omega^c_5\bar{\Lambda}^5_a \\
&\quad +\Lambda^4_4\omega^4_d\bar{\Lambda}^d_a + \Lambda^4_4\omega^4_4\bar{\Lambda}^4_a \\
&\quad +\Lambda^4_5\omega^5_d\bar{\Lambda}^d_a + \Lambda^4_5\omega^5_5\bar{\Lambda}^5_a \\
&\quad -\mathbf{d}\Lambda^4_c\,\bar{\Lambda}^c_a - \mathbf{d}\Lambda^4_4\,\bar{\Lambda}^4_a \\
&= \Lambda^4_c\omega^c_d\bar{\Lambda}^d_a + \Lambda^4_c\omega^c_4\bar{\Lambda}^4_a + \Lambda^4_c\eta^{cd}\omega^4_d\eta_{ac}\bar{\Lambda}^c_4 \\
&\quad +\Lambda^4_4\omega^4_d\bar{\Lambda}^d_a + \Lambda^4_4\omega^4_4\bar{\Lambda}^4_a \\
&\quad -\mathbf{d}\Lambda^4_c\,\bar{\Lambda}^c_a - \mathbf{d}\Lambda^4_4\,\bar{\Lambda}^4_a
\end{aligned}$$

and dilatations,

$$\begin{aligned}
\tilde{\omega}^4_4 &= \Lambda^4_B\omega^B_C\bar{\Lambda}^C_4 - \mathbf{d}\Lambda^4_C\,\bar{\Lambda}^C_4 \\
&= \Lambda^4_c\omega^c_d\bar{\Lambda}^d_4 + \Lambda^4_c\omega^c_4\bar{\Lambda}^4_4 \\
&\quad +\Lambda^4_4\omega^4_d\bar{\Lambda}^d_4 + \Lambda^4_4\omega^4_4\bar{\Lambda}^4_4 \\
&\quad -\mathbf{d}\Lambda^4_c\,\bar{\Lambda}^c_4 - \mathbf{d}\Lambda^4_4\,\bar{\Lambda}^4_4
\end{aligned}$$

Lie-algebra valued tensors transform in the same way except for the inhomogeneous $-\mathbf{d}\Lambda^A_C\,\bar{\Lambda}^C_B$ terms.

Now that we have the right transformations, we can simplify the notation by dropping all $A = 4$ labels, and setting $\bar{\Lambda} = \frac{1}{\Lambda}$ for dilatations. Collecting the results:

$$\begin{aligned}
\tilde{\omega}^a_b &= \Lambda^a_c\omega^c_d\bar{\Lambda}^d_b + \Lambda^a_c\omega^c\bar{\Lambda}_b + \Lambda^a_c\eta^{ce}\omega_e\eta_{bd}\bar{\Lambda}^d \\
&\quad +\Lambda^a\omega_d\bar{\Lambda}^d_b + \Lambda^a\omega\bar{\Lambda}_b + \eta^{ac}\Lambda_c\omega^e\eta_{ed}\bar{\Lambda}^d_b - \eta^{ac}\Lambda_c\omega\eta_{bd}\bar{\Lambda}^d \\
&\quad -\mathbf{d}\Lambda^a_c\,\bar{\Lambda}^c_b - \mathbf{d}\Lambda^a\,\bar{\Lambda}_b - \eta^{ac}\mathbf{d}\Lambda_c\,\eta_{bd}\bar{\Lambda}^d \\
\tilde{\omega}^a &= \Lambda^a_c\omega^c_d\bar{\Lambda}^d + \Lambda^a_c\omega^c\bar{\Lambda} + \Lambda^a\omega_d\bar{\Lambda}^d \\
&\quad +\Lambda^a\omega\bar{\Lambda} + \eta^{ac}\Lambda_c\omega^e\eta_{ed}\bar{\Lambda}^d - \mathbf{d}\Lambda^a_c\,\bar{\Lambda}^c - \mathbf{d}\Lambda^a\,\bar{\Lambda} \\
\tilde{\omega}_a &= \Lambda_c\omega^c_d\bar{\Lambda}^d_a + \Lambda_c\omega^c\bar{\Lambda}_a + \Lambda_c\eta^{cd}\omega_d\eta_{ac}\bar{\Lambda}^c \\
&\quad +\Lambda\omega_d\bar{\Lambda}^d_a + \Lambda\omega\bar{\Lambda}_a - \mathbf{d}\Lambda_c\,\bar{\Lambda}^c_a - \mathbf{d}\Lambda\,\bar{\Lambda}_a \\
\tilde{\omega} &= \Lambda_c\omega^c_d\bar{\Lambda}^d + \Lambda_c\omega^c\bar{\Lambda} + \Lambda\omega_d\bar{\Lambda}^d \\
&\quad +\Lambda\omega\bar{\Lambda} - \mathbf{d}\Lambda_c\,\bar{\Lambda}^c - \mathbf{d}\Lambda\,\bar{\Lambda}
\end{aligned}$$

These relations specialize immediately to subgroups of the conformal group.

1.4 Lorentz-invariant subgroups

The conformal group acts on its own Lie algebra,

$$\tilde{G}_A = gG_Ag^{-1}$$

and since we expect, at a minimum, homothetic symmetry of our final gauged space, it is useful to have a breakdown of the conformal transformations into homothetically invariant subgroups. It is sufficient to consider an infinitesimal homothetic transformation, given by an arbitrary linear combination of the Lorentz and dilatational generators,

$$\delta = \frac{1}{2} w^a_{\ b} M^b_{\ a} + \lambda D$$

It is easy to see from the Lie algebra,

$$\begin{aligned}
[M^a_{\ \ b}, M^c_{\ \ d}] &= -\frac{1}{2}\left(\delta^c_b \delta^a_f \delta^e_d - \eta_{bd}\eta^{ce}\delta^a_f - \eta^{ac}\eta_{bf}\delta^e_d + \delta^a_d \eta^{ce}\eta_{bf}\right) M^f_{\ \ e} \\
[M^a_{\ \ b}, P_c] &= \frac{1}{2}\left(\delta^a_c \delta^d_b - \eta_{bc}\eta^{ad}\right) P_d \\
[M^a_{\ \ b}, K^c] &= -\frac{1}{2}\left(\delta^c_b \delta^a_d - \eta_{bd}\eta^{ac}\right) K^d \\
\left[P_a, K^b\right] &= -\delta^b_a D - \left(\eta_{ac}\eta^{bd} - \delta^d_a \delta^b_c\right) M^c_{\ \ d} \\
[D, P_a] &= -\delta^b_a P_b \\
[D, K^a] &= \delta^a_b K^b
\end{aligned}$$

that

$$\begin{aligned}
[\delta, M^a_{\ \ b}] &\sim M^a_{\ \ b} \\
[\delta, P_a] &\sim P_a \\
[\delta, K^a] &\sim K^a \\
[\delta, D] &= 0
\end{aligned}$$

Moreover, since

$$\begin{aligned}
[D, P_a] &= -P_a \\
[D, K^a] &= K^a
\end{aligned}$$

we may assign conformal weight -1 to the generator, P_a, of translations and weight $+1$ to the generator, K^a, of special conformal transformations. Combining this with the behavior under Lorentz transformations, we may summarize the irreducible homothetic parts:

$$\begin{array}{lll}
M^a_{\ \ b} & \textit{weight } 0 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \textit{tensor} \\
P_a & \textit{weight } -1 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \textit{tensor} \\
K^a & \textit{weight } +1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \textit{tensor} \\
D & \textit{weight } 0 & \textit{scalar}
\end{array}$$

By combining these in various ways and exponentiating we readily identify all homothetically invariant subgroups of the conformal group by checking whether the commutators close. First, notice that since

$$\left[P_a, K^b\right] = -\delta^b_a D + 2M^b_{\ \ a}$$

we cannot have any proper subgroup with both P_a and K^a among the generators, since it must necessarily also include the D and $M^a_{\ \ b}$ generators. Denoting homothetically invariant subgroups by the subset of generators that generate them, we have the four single-sets,

$$\{M^a_{\ \ b}\}, \{P_a\} \cong \left\{K^b\right\}, \{D\}$$

Since the n-dimensional translation groups generated by $\{P_a\}$ and $\left\{K^b\right\}$ are isomorphic, this gives three groups. There are five closed double-sets,

$$\{M^a_{\ \ b}, P_a\} \cong \{M^a_{\ \ b}, K^a\}, \{M^a_{\ \ b}, D\}, \{P_a, D\} \cong \{K^a, D\}$$

(but not the final pair, $,\left\{P_a,K^b\right\}$!), with isomorphisms reducing these to three independent homothetically invariant subgroups. There are only two triples:

$$\{M^a{}_b,P_a,D\}\cong\{M^a{}_b,K^a,D\}$$

and these generate isomorphic (inhomogeneous, homothetic) groups. Finally the improper subgroup generated by all four sets, $\left\{M^a{}_b,P_a,K^a,D\right\}$. There are therefore seven homothetically invariant proper subgroups of the conformal group.

We note for future reference that only two of these subgroups *contain* the homothetic group – the one generated by $\left\{M^a{}_b,D\right\}$ is the homothetic group itself, while the group generated by $\left\{M^a{}_b,K^a,D\right\}$ is the inhomogeneous homothetic group.

Also, notice that neither of these gaugings contain P_a in the subgroup. This means that the translational symmetry will be broken when we generalize the base manifold geometry. The gauge transformation of the torsion then becomes tensorial, so setting the torsion to zero is a gauge invariant specification. We may therefore consider torsion-free gauge theories.

1.5 Bianchi identities

We find the Bianchi identities in Riemannian, Weyl and conformal geometries.

1.5.1 Bianchi identities of conformal geometry

The conformal Cartan equations with vanishing torsion are

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c+2\Delta^{ac}_{db}\omega_c\omega^d+\Omega^a_b\\
\mathbf{d}\omega^a &= \omega^c\omega^a_c+\omega\omega^a\\
\mathbf{d}\omega_a &= \omega^c_a\omega_c+\omega_a\omega+\Omega_a\\
\mathbf{d}\omega &= \omega^a\omega_a+\Omega
\end{aligned}$$

Taking the exterior derivative of each equation and substituting for the derivatives leads to

$$\begin{aligned}
0 &= \mathbf{d}^2\omega^a_b\\
&= \mathbf{d}\omega^c_b\omega^a_c-\omega^c_b\mathbf{d}\omega^a_c+2\Delta^{ac}_{db}\mathbf{d}\omega_c\omega^d-2\Delta^{ac}_{db}\omega_c\mathbf{d}\omega^d+\mathbf{d}\Omega^a_b\\
&= \Omega^c_b\omega^a_c-\omega^c_b\Omega^a_c+2\Delta^{ac}_{db}\Omega_c\omega^d+\mathbf{d}\Omega^a_b\\
&= \mathbf{D}\Omega^a_b+2\Delta^{ac}_{db}\Omega_c\omega^d\\
0 &= \mathbf{d}^2\omega^a\\
&= \mathbf{d}\omega^c\omega^a_c-\omega^c\mathbf{d}\omega^a_c+\mathbf{d}\omega\omega^a-\omega\mathbf{d}\omega^a\\
&= -\omega^c\Omega^a_c+\Omega\omega^a\\
0 &= \mathbf{d}^2\omega_a\\
&= \mathbf{d}\omega^c_a\omega_c-\omega^c_a\mathbf{d}\omega_c+\mathbf{d}\omega_a\omega-\omega_a\mathbf{d}\omega+\mathbf{d}\Omega_a\\
&= \Omega^c_a\omega_c-\omega^c_a\Omega_c+\Omega_a\omega-\omega_a\Omega+\mathbf{d}\Omega_a\\
&= \mathbf{D}\Omega_a+\Omega^c_a\omega_c-\omega_a\Omega\\
0 &= \mathbf{d}^2\omega\\
&= \mathbf{d}\omega^a\omega_a-\omega^a\mathbf{d}\omega_a+\mathbf{d}\Omega\\
&= \mathbf{d}\Omega-\omega^a\Omega_a
\end{aligned}$$

Collecting terms, we have

$$\begin{aligned}
\mathbf{D}\Omega^a_b+2\Delta^{ac}_{db}\Omega_c\omega^d &= 0\\
\omega^c\Omega^a_c-\Omega\omega^a &= 0\\
\mathbf{D}\Omega_a+\Omega^c_a\omega_c-\omega_a\Omega &= 0\\
\mathbf{d}\Omega-\omega^a\Omega_a &= 0
\end{aligned}$$

The Bianchi identities for Weyl and Riemannian geometries follow immediately as

$$\begin{aligned} \mathbf{D}\Omega^a_b &= 0 \\ \omega^c \Omega^a_{\ c} - \Omega \omega^a &= 0 \\ \mathbf{d}\Omega &= 0 \end{aligned}$$

in a Weyl geometry and

$$\begin{aligned} \mathbf{D}\mathbf{R}^a_b &= 0 \\ \mathbf{e}^c \mathbf{R}^a_c &= 0 \end{aligned}$$

in Riemannian.

In the scale-invariant geometries, i.e., conformal or Weyl, the torsion-free condition leads to an algebraic constraint relating the dilatational and Lorentz curvatures,

$$\omega^b \Omega^a_{\ b} - \Omega \omega^a = 0$$

Expanding this in components, we have

$$\begin{aligned} 0 &= \Omega^a_{\ [bcd]} - \delta^a_{[b} \Omega_{cd]} \\ &= \frac{1}{3} \left(\Omega^a_{\ bcd} + \Omega^a_{\ cdb} + \Omega^a_{\ dbc} - \delta^a_b \Omega_{cd} - \delta^a_c \Omega_{db} - \delta^a_d \Omega_{bc} \right) \end{aligned}$$

Now contract on the *ac* components,

$$0 = \Omega^c_{\ bcd} + \Omega^c_{\ cdb} + \Omega^c_{\ dbc} - \Omega_{bd} - n\Omega_{db} - \Omega_{bd}$$

so that the antisymmetric part of the trace of the curvature is proportional to the dilatation:

$$\Omega^c_{\ bcd} - \Omega^c_{\ dcb} = -(n-2)\,\Omega_{bd}$$

The trace of the curvature is symmetric if and only if the dilatation vanishes.

1.5.2 Bianchi identities in Riemannian geometry

The first Bianchi identity is

$$\begin{aligned} 0 &= \mathbf{e}^b \mathbf{R}^a_b \\ &= \mathbf{e}^b \left(\mathbf{C}^a_b + 2\Delta^{ac}_{db} \mathscr{R}_c \mathbf{e}^d \right) \\ &= \mathbf{e}^b \mathbf{C}^a_b + \mathscr{R}_b \mathbf{e}^b \mathbf{e}^a \\ &= \mathbf{e}^b \mathbf{C}^a_b \end{aligned}$$

The second curvature Bianchi identity is

$$\begin{aligned} \mathbf{D}\mathbf{R}^a_b &= 0 \\ R^a_{\ b[cd;e]} &= 0 \end{aligned}$$

Substítiting

$$R^a_{\ bcd} = C^a_{\ bcd} - \delta^a_c \mathscr{R}_{bd} + \delta^a_d \mathscr{R}_{bc} + \eta_{bc} \mathscr{R}^a_{\ d} - \eta_{bd} \mathscr{R}^a_{\ c}$$

for the curvature,

$$\begin{aligned} 0 &= \mathbf{D}\mathbf{R}^a_b \\ &= \mathbf{D}\mathbf{C}^a_b + 2\Delta^{ac}_{db} \mathbf{D}\mathscr{R}_c \mathbf{e}^d \end{aligned}$$

$$
\begin{aligned}
0 &= R^a_{bcd;e} + R^a_{bde;c} + R^a_{bec;d} \\
&= C^a_{bcd;e} - \delta^a_c \mathscr{R}_{bd;e} + \delta^a_d \mathscr{R}_{bc;e} + \eta_{bc} \mathscr{R}^a_{\ d;e} - \eta_{bd} \mathscr{R}^a_{\ c;e} \\
&\quad + C^a_{bde;c} - \delta^a_d \mathscr{R}_{be;c} + \delta^a_e \mathscr{R}_{bd;c} + \eta_{bd} \mathscr{R}^a_{\ e;c} - \eta_{be} \mathscr{R}^a_{\ d;c} \\
&\quad + C^a_{bec;d} - \delta^a_e \mathscr{R}_{bc;d} + \delta^a_c \mathscr{R}_{be;d} + \eta_{be} \mathscr{R}^a_{\ c;d} - \eta_{bc} \mathscr{R}^a_{\ e;d}
\end{aligned}
$$

and take the trace on ae,

$$
\begin{aligned}
0 &= C^a_{bcd;a} - \mathscr{R}_{bd;c} + \mathscr{R}_{bc;d} + \eta_{bc} \mathscr{R}^a_{\ d;a} - \eta_{bd} \mathscr{R}^a_{\ c;a} \\
&\quad - \mathscr{R}_{bd;c} + n \mathscr{R}_{bd;c} + \eta_{bd} \mathscr{R}^a_{\ a;c} - \mathscr{R}_{bd;c} \\
&\quad - n \mathscr{R}_{bc;d} + \mathscr{R}_{bc;d} + \mathscr{R}_{bc;d} - \eta_{bc} \mathscr{R}^a_{\ a;d} \\
&= C^a_{bcd;a} - (n-3) \mathscr{R}_{bc;d} + (n-3) \mathscr{R}_{bd;c} + \eta_{bd} \left(\mathscr{R}^a_{\ a;c} - \mathscr{R}^a_{\ c;a} \right) - \eta_{bc} \left(\mathscr{R}^a_{\ a;d} - \mathscr{R}^a_{\ d;a} \right)
\end{aligned}
$$

and contract again on bd,

$$
0 = 2(n-2) \left(\mathscr{R}^a_{\ a;c} - \mathscr{R}^a_{\ c;a} \right)
$$

so,

$$
C^a_{bcd;a} = (n-3) \left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c} \right)
$$

1.5.3 Bianchi identities in Weyl geometry

The torsion-free structure equations are:

$$
\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b \omega^a_c + \mathbf{R}^a_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c \omega^a_c + \omega \mathbf{e}^a \\
\mathbf{d}\omega &= \Omega
\end{aligned}
$$

Taking the exterior derivative of each, we have

$$
\begin{aligned}
\mathbf{D}\mathbf{R}^a_b &= \mathbf{d}\mathbf{R}^a_b + \mathbf{R}^c_b \omega^a_c - \omega^c_b \mathbf{R}^a_c = 0 \\
\mathbf{e}^c \Omega^a_c &= \Omega \mathbf{e}^a \\
\mathbf{d}\Omega &= 0
\end{aligned}
$$

In components the second becomes

$$
\Omega^a_{bcd} + \Omega^a_{cdb} + \Omega^a_{dbc} = \delta^a_b \Omega_{cd} + \delta^a_c \Omega_{db} + \delta^a_d \Omega_{bc}
$$

with trace

$$
\Omega^c_{bcd} - \Omega^c_{dcb} = -(n-2) \Omega_{bd}
$$

In contrast to Riemannian geometry, the trace of the curvature tensor has an antisymmetric part.

1.6 Relationship between the Riemann, Weyl, Ricci and Eisenhart and dilatation tensors

We consider these relationships in Riemannian, Weyl, and conformal geometries. We consider only torsion-free geometries.

1.6.1 Curvature relations in Riemannian geometry

The structure equations are:

$$
\begin{aligned}
\mathbf{d}\omega^a_{\ b} &= \omega^c_{\ b} \omega^a_{\ c} + R^a_b \\
\mathbf{d}\omega^a &= \omega^c \omega^a_{\ c}
\end{aligned}
$$

Riemann, Weyl and Ehrenfest The Weyl curvature is given by

$$C^a_{bcd} = R^a_{bcd} - \frac{1}{n-2}\left(\delta^a_c R_{bd} - \delta^a_d R_{bc} - \eta_{bc} R^a_{\ d} + \eta_{bd} R^a_{\ c}\right) + \frac{1}{(n-1)(n-2)} R\left(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc}\right)$$

Check the trace,

$$\begin{aligned}
C^c_{bcd} &= R^c_{bcd} - \frac{1}{n-2}\left(nR_{bd} - R_{bd} - R_{bd} + \eta_{bd} R\right) + \frac{1}{(n-1)(n-2)} R(n-1)\eta_{bd} \\
&= R_{bd} - R_{bd} - \frac{1}{n-2}\eta_{bd} R + \frac{1}{n-2} R\eta_{bd} \\
&= 0
\end{aligned}$$

The Eisenhart tensor is defined in terms of the Ricci tensor as

$$\mathscr{R}_{ab} = -\frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R\eta_{ab}\right)$$

This is invertible,

$$\begin{aligned}
R_{ab} &= -(n-2)\mathscr{R}_{ab} - \mathscr{R}\eta_{ab} \\
R &= -2(n-1)\mathscr{R}
\end{aligned}$$

Check,

$$\begin{aligned}
\mathscr{R}_{ab} &= -\frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R\eta_{ab}\right) \\
&= -\frac{1}{n-2}\left(-(n-2)\mathscr{R}_{ab} - \mathscr{R}\eta_{ab} - \frac{1}{2(n-1)}\left(-2(n-1)\mathscr{R}\right)\eta_{ab}\right) \\
&= \mathscr{R}_{ab}
\end{aligned}$$

The relationship between the curvatures may be rewritten as

$$\begin{aligned}
C^a_{bcd} &= R^a_{bcd} \\
&\quad - \frac{1}{n-2}\delta^a_c\left(R_{bd} - \frac{1}{2(n-1)} R\eta_{bd}\right) + \frac{1}{n-2}\delta^a_d\left(R_{bc} - \frac{1}{2(n-1)} R\eta_{bc}\right) \\
&\quad + \frac{1}{n-2}\eta_{bc}\left(R^a_{\ d} - \frac{1}{2(n-1)} R\delta^a_d\right) - \frac{1}{n-2}\eta_{bd}\left(R^a_{\ c} - \frac{1}{2(n-1)} R\delta^a_c\right) \\
&\quad - \frac{1}{2(n-1)(n-2)} R\left(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc} - \eta_{bc}\delta^a_d + \eta_{bd}\delta^a_c\right) + \frac{1}{(n-1)(n-2)} R\left(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc}\right) \\
&= R^a_{bcd} + \delta^a_c \mathscr{R}_{bd} - \delta^a_d \mathscr{R}_{bc} - \eta_{bc}\mathscr{R}^a_{\ d} + \eta_{bd}\mathscr{R}^a_{\ c} \\
&= R^a_{bcd} + \delta^a_c \mathscr{R}_{bd} - \delta^a_d \mathscr{R}_{bc} - \eta_{bc}\mathscr{R}^a_{\ d} + \eta_{bd}\mathscr{R}^a_{\ c}
\end{aligned}$$

Then in terms of curvature 2-forms,

$$\begin{aligned}
\mathbf{C}^a_b &= \mathbf{R}^a_b + \frac{1}{2}\left(\delta^a_c \mathscr{R}_{bd} - \delta^a_d \mathscr{R}_{bc} - \eta_{bc}\mathscr{R}^a_{\ d} + \eta_{bd}\mathscr{R}^a_{\ c}\right)\mathbf{e}^c\mathbf{e}^d \\
&= \mathbf{R}^a_b + \frac{1}{2}\left(2\delta^a_c\delta^e_b - 2\eta_{bc}\eta^{ae}\right)\mathscr{R}_{ed}\mathbf{e}^c\mathbf{e}^d \\
&= \mathbf{R}^a_b - 2\Delta^{ad}_{cb}\mathscr{R}_d\mathbf{e}^c
\end{aligned}$$

so that finally,

$$\mathbf{R}^a_b = \mathbf{C}^a_b + 2\Delta^{ac}_{db}\mathscr{R}_c\mathbf{e}^d$$

Divergence of the Weyl curvature We may use this relationship to write the divergence of the Weyl curvature in terms of the curl of the Ehrenfest tensor.

Starting from the algebraic Bianchi identity,

$$0 = R^a_{bcd;e} + R^a_{bde;c} + R^a_{bec;d}$$

we substitute

$$R^a_{bcd} = C^a_{bcd} - \delta^a_c \mathscr{R}_{bd} + \delta^a_d \mathscr{R}_{bc} + \eta_{bc}\mathscr{R}^a_{\ d} - \eta_{bd}\mathscr{R}^a_{\ c}$$

for the curvature to find

$$\begin{aligned} 0 &= R^a_{bcd;e} + R^a_{bde;c} + R^a_{bec;d} \\ &= C^a_{bcd;e} - \delta^a_c \mathscr{R}_{bd;e} + \delta^a_d \mathscr{R}_{bc;e} + \eta_{bc}\mathscr{R}^a_{\ d;e} - \eta_{bd}\mathscr{R}^a_{\ c;e} \\ &\quad + C^a_{bde;c} - \delta^a_d \mathscr{R}_{be;c} + \delta^a_e \mathscr{R}_{bd;c} + \eta_{bd}\mathscr{R}^a_{\ e;c} - \eta_{be}\mathscr{R}^a_{\ d;c} \\ &\quad + C^a_{bec;d} - \delta^a_e \mathscr{R}_{bc;d} + \delta^a_c \mathscr{R}_{be;d} + \eta_{be}\mathscr{R}^a_{\ c;d} - \eta_{bc}\mathscr{R}^a_{\ e;d} \end{aligned}$$

Taking the trace on ae,

$$\begin{aligned} 0 &= C^a_{bcd;a} - \mathscr{R}_{bd;c} + \mathscr{R}_{bc;d} + \eta_{bc}\mathscr{R}^a_{\ d;a} - \eta_{bd}\mathscr{R}^a_{\ c;a} \\ &\quad - \mathscr{R}_{bd;c} + n\mathscr{R}_{bd;c} + \eta_{bd}\mathscr{R}^a_{\ a;c} - \mathscr{R}_{bd;c} \\ &\quad - n\mathscr{R}_{bc;d} + \mathscr{R}_{bc;d} + \mathscr{R}_{bc;d} - \eta_{bc}\mathscr{R}^a_{\ a;d} \\ &= C^a_{bcd;a} - (n-3)\mathscr{R}_{bc;d} + (n-3)\mathscr{R}_{bd;c} + \eta_{bd}\left(\mathscr{R}^a_{\ a;c} - \mathscr{R}^a_{\ c;a}\right) - \eta_{bc}\left(\mathscr{R}^a_{\ a;d} - \mathscr{R}^a_{\ d;a}\right) \end{aligned}$$

and contracting again on bd,

$$0 = 2(n-2)\left(\mathscr{R}^a_{\ a;c} - \mathscr{R}^a_{\ c;a}\right)$$

we find

$$C^a_{bcd;a} = (n-3)\left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c}\right)$$

1.6.2 Curvature relations in Weyl geometry

We find the corresponding results for Weyl geometry.

Riemann, Weyl, Ehrenfest and dilatational curvatures In Weyl geometry, the Bianchi identities show that the trace of the curvature has an antisymmetric part,

$$\Omega^c_{bcd} - \Omega^c_{dcb} = -(n-2)\,\Omega_{bd}$$

so that we no longer have $\Omega^a_b = \mathbf{C}^a_b + 2\Delta^{ac}_{db}\mathscr{R}_c\mathbf{e}^d$. Instead, let

$$\begin{aligned} \Omega^a_{bac} &= R_{bc} - \frac{1}{2}(n-2)\,\Omega_{bc} \\ \Omega^{ac}_{\ \ ac} &= \eta^{ac}R_{ac} = R \end{aligned}$$

where $R_{ab} = R_{ba}$. Then we may still write the traceless part of the curvature as

$$\begin{aligned} C^a_{bcd} &= \Omega^a_{bcd} - \frac{1}{(n-2)}\left(\delta^a_c\Omega^e_{bed} - \delta^a_d\Omega^e_{bec} - \eta_{bc}\Omega^{ea}_{\ \ ed} + \eta_{bd}\Omega^{ea}_{\ \ ec}\right) \\ &\quad + \frac{1}{(n-1)(n-2)}\Omega^{ef}_{\ \ ef}\left(\delta^a_c\eta_{bd} - \delta^a_d\eta_{bc}\right) \\ &= \Omega^a_{bcd} - \frac{1}{(n-2)}\left(\delta^a_c R_{bd} - \delta^a_d R_{bc} - \eta_{bc}R^a_{\ d} + \eta_{bd}R^a_{\ c}\right) \\ &\quad + \frac{1}{2}\left(\delta^a_c\Omega_{bd} - \delta^a_d\Omega_{bc} - \eta_{bc}\Omega^a_{\ d} + \eta_{bd}\Omega^a_{\ c}\right) \\ &\quad + \frac{1}{(n-1)(n-2)}R\left(\delta^a_c\eta_{bd} - \delta^a_d\eta_{bc}\right) \\ &= \Omega^a_{bcd} + \left(\delta^a_c\mathscr{R}_{bd} - \delta^a_d\mathscr{R}_{bc} - \eta_{bc}\mathscr{R}^a_{\ d} + \eta_{bd}\mathscr{R}^a_{\ c}\right) \\ &\quad + \frac{1}{2}\left(\delta^a_c\Omega_{bd} - \delta^a_d\Omega_{bc} - \eta_{bc}\Omega^a_{\ d} + \eta_{bd}\Omega^a_{\ c}\right) \end{aligned}$$

or, using differential forms, we may decompose the full curvature 2-form as

$$\Omega^a_b = \mathbf{C}^a_b + 2\Delta^{ac}_{db}\left(\mathscr{R}_c + \frac{1}{2}\Omega_{ce}\omega^e\right)\omega^d$$

that is, the usual decomposition in terms of the Weyl and Eisenhart tensors, together with a dilatation term. The Eisenhart tensor is still defined in terms of the (symmetric) Ricci tensor as

$$\mathscr{R}_{ab} = -\frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}R\eta_{ab}\right)$$

and the Ricci scalar is unchanged.

Symmetries of the Weyl curvature We find the symmetries of the Weyl curvature in a Weyl geometry. From

$$\mathbf{C}^a_b = \Omega^a_b - 2\Delta^{ac}_{db}\left(\mathscr{R}_c + \frac{1}{2}\Omega_{ce}\omega^e\right)\omega^d$$

we have the decomposition of the full curvature,

$$\begin{aligned}\Omega^a_b &= \mathbf{C}^a_b + 2\Delta^{ad}_{cb}\left(\mathscr{R}_d + \frac{1}{2}\Omega_{de}\omega^e\right)\omega^c \\ &= \mathbf{R}^a_b + \Delta^{ad}_{cb}\Omega_d\omega^c\end{aligned}$$

where $\mathbf{R}^a_b$ has the usual symmetries of the Riemann tensor. Therefore, we have antisymmetry on both the first and second pairs of indices. The triple antisymmetry usually follows from the Bianchi identity. Here we have:

$$\begin{aligned}\Omega^a_{bcd} &= R^a_{bcd} + \Delta^{ae}_{cb}\Omega_{ed} - \Delta^{ae}_{db}\Omega_{ec} \\ \Omega^a_{[bcd]} &= R^a_{[bcd]} + \frac{1}{3}\left(\Delta^{ae}_{cb}\Omega_{ed} - \Delta^{ae}_{db}\Omega_{ec}\right) + \frac{1}{3}\left(\Delta^{ae}_{dc}\Omega_{eb} - \Delta^{ae}_{bc}\Omega_{ed}\right) + \frac{1}{3}\left(\Delta^{ae}_{bd}\Omega_{ec} - \Delta^{ae}_{cd}\Omega_{eb}\right) \\ &= \frac{1}{3}\left(\Delta^{ae}_{cb}\Omega_{ed} - \Delta^{ae}_{db}\Omega_{ec} + \Delta^{ae}_{dc}\Omega_{eb} - \Delta^{ae}_{bc}\Omega_{ed} + \Delta^{ae}_{bd}\Omega_{ec} - \Delta^{ae}_{cd}\Omega_{eb}\right) \\ &= \frac{1}{6}\left(\delta^a_c\delta^e_b\Omega_{ed} - \delta^a_d\delta^e_b\Omega_{ec} + \delta^a_d\delta^e_c\Omega_{eb} - \delta^a_b\delta^e_c\Omega_{ed} + \delta^a_b\delta^e_d\Omega_{ec} - \delta^a_c\delta^e_d\Omega_{eb}\right) \\ &\quad -\frac{1}{6}\left(\eta^{ae}\eta_{bc}\Omega_{ed} - \eta^{ae}\eta_{bd}\Omega_{ec} + \eta^{ae}\eta_{cd}\Omega_{eb} - \eta^{ae}\eta_{bc}\Omega_{ed} + \eta^{ae}\eta_{bd}\Omega_{ec} - \eta^{ae}\eta_{cd}\Omega_{eb}\right) \\ &= -\frac{1}{3}\left(\delta^a_d\Omega_{bc} + \delta^a_b\Omega_{cd} + \delta^a_c\Omega_{db}\right)\end{aligned}$$

Then

$$\Omega^a_{[bcd]} = -\delta^a_{[b}\Omega_{cd]}$$

Writing the curvature in covariant form,

$$\Omega^a_{bcd} = R^a_{bcd} + \Delta^{ae}_{db}\Omega_{ec} - \Delta^{ae}_{cb}\Omega_{ed}$$

$$\begin{aligned}\Omega^a_{bcd} &= R^a_{bcd} + \frac{1}{2}\delta^a_d\Omega_{bc} - \frac{1}{2}\eta^{ae}\eta_{bd}\Omega_{ec} - \frac{1}{2}\delta^a_c\Omega_{bd} + \frac{1}{2}\eta^{ae}\eta_{bc}\Omega_{ed} \\ \Omega_{abcd} &= R_{abcd} + \frac{1}{2}\eta_{ad}\Omega_{bc} - \frac{1}{2}\eta_{bd}\Omega_{ac} - \frac{1}{2}\eta_{ac}\Omega_{bd} + \frac{1}{2}\eta_{bc}\Omega_{ad}\end{aligned}$$

we also have

$$\begin{aligned}\Omega_{abcd} - \Omega_{cdab} &= R_{abcd} - R_{cdab} + \frac{1}{2}\eta_{ad}\Omega_{bc} - \frac{1}{2}\eta_{bd}\Omega_{ac} - \frac{1}{2}\eta_{ac}\Omega_{bd} + \frac{1}{2}\eta_{bc}\Omega_{ad} \\ &\quad -\frac{1}{2}\eta_{cb}\Omega_{da} + \frac{1}{2}\eta_{db}\Omega_{ca} + \frac{1}{2}\eta_{ac}\Omega_{db} - \frac{1}{2}\eta_{da}\Omega_{cb} \\ &= \frac{1}{2}\eta_{ad}\Omega_{bc} - \frac{1}{2}\eta_{da}\Omega_{cb} - \frac{1}{2}\eta_{bd}\Omega_{ac} + \frac{1}{2}\eta_{db}\Omega_{ca} - \frac{1}{2}\eta_{ac}\Omega_{bd} + \frac{1}{2}\eta_{ac}\Omega_{db} + \frac{1}{2}\eta_{bc}\Omega_{ad} - \frac{1}{2}\eta_{cb}\Omega_{da} \\ &= \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac} - \eta_{ac}\Omega_{bd} + \eta_{bc}\Omega_{ad}\end{aligned}$$

Therefore, the new symmetries are:

$$
\begin{aligned}
\Omega_{abcd} &= -\Omega_{bacd} = -\Omega_{abdc} \\
\Omega^a_{[bcd]} &= \delta^a_{[b}\Omega_{cd]} \\
\Omega_{abcd} - \Omega_{cdab} &= \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac} - \eta_{ac}\Omega_{bd} + \eta_{bc}\Omega_{ad}
\end{aligned}
$$

Divergence of the Weyl curvature As before, we start with the Bianch identity

$$
\mathbf{D}\Omega^a_b = \mathbf{d}\Omega^a_b + \Omega^c_b\omega^a_c - \omega^c_b\Omega^a_c = 0
$$

where

$$
\begin{aligned}
\Omega^a_{bcd} &= C^a_{bcd} - \left(\delta^a_c\mathscr{R}_{bd} - \delta^a_d\mathscr{R}_{bc} - \eta_{bc}\mathscr{R}^a_{\ d} + \eta_{bd}\mathscr{R}^a_{\ c}\right) \\
&\quad -\frac{1}{2}\left(\delta^a_c\Omega_{bd} - \delta^a_d\Omega_{bc} - \eta_{bc}\Omega^a_{\ d} + \eta_{bd}\Omega^a_{\ c}\right)
\end{aligned}
$$

Combining these, we have

$$
\begin{aligned}
0 &= \Omega^a_{bcd;e} + \Omega^a_{bde;c} + \Omega^a_{bec;d} \\
&= C^a_{bcd;e} + C^a_{bde;c} + C^a_{bec;d} \\
&\quad - \left(\delta^a_c\mathscr{R}_{bd;e} - \delta^a_d\mathscr{R}_{bc;e} - \eta_{bc}\mathscr{R}^a_{\ d;e} + \eta_{bd}\mathscr{R}^a_{\ c;e}\right) - \frac{1}{2}\left(\delta^a_c\Omega_{bd;e} - \delta^a_d\Omega_{bc;e} - \eta_{bc}\Omega^a_{\ d;e} + \eta_{bd}\Omega^a_{\ c;e}\right) \\
&\quad - \left(\delta^a_d\mathscr{R}_{be;c} - \delta^a_e\mathscr{R}_{bd;c} - \eta_{bd}\mathscr{R}^a_{\ e;c} + \eta_{be}\mathscr{R}^a_{\ d;c}\right) - \frac{1}{2}\left(\delta^a_d\Omega_{be;c} - \delta^a_e\Omega_{bd;c} - \eta_{bd}\Omega^a_{\ e;c} + \eta_{be}\Omega^a_{\ d;c}\right) \\
&\quad - \left(\delta^a_e\mathscr{R}_{bc;d} - \delta^a_c\mathscr{R}_{be;d} - \eta_{be}\mathscr{R}^a_{\ c;d} + \eta_{bc}\mathscr{R}^a_{\ e;d}\right) - \frac{1}{2}\left(\delta^a_e\Omega_{bc;d} - \delta^a_c\Omega_{be;d} - \eta_{be}\Omega^a_{\ c;d} + \eta_{bc}\Omega^a_{\ e;d}\right)
\end{aligned}
$$

Now contract ae,

$$
\begin{aligned}
0 &= C^a_{bcd;a} \\
&\quad - \left(\mathscr{R}_{bd;c} - \mathscr{R}_{bc;d} - \eta_{bc}\mathscr{R}^a_{\ d;a} + \eta_{bd}\mathscr{R}^a_{\ c;a}\right) - \frac{1}{2}\left(\Omega_{bd;c} - \Omega_{bc;d} - \eta_{bc}\Omega^a_{\ d;a} + \eta_{bd}\Omega^a_{\ c;a}\right) \\
&\quad - \left(\mathscr{R}_{bd;c} - n\mathscr{R}_{bd;c} - \eta_{bd}\mathscr{R}^a_{\ a;c} + \mathscr{R}_{bd;c}\right) - \frac{1}{2}\left(\Omega_{bd;c} - n\Omega_{bd;c} + \Omega_{bd;c}\right) \\
&\quad - \left(n\mathscr{R}_{bc;d} - \mathscr{R}_{bc;d} - \mathscr{R}_{bc;d} + \eta_{bc}\mathscr{R}^a_{\ a;d}\right) - \frac{1}{2}\left(n\Omega_{bc;d} - \Omega_{bc;d} - \Omega_{bc;d}\right)
\end{aligned}
$$

Simplify,

$$
\begin{aligned}
C^a_{bcd;a} &= (n-3)\mathscr{R}_{bc;d} - (n-3)\mathscr{R}_{bd;c} + \eta_{bc}\mathscr{R}^a_{\ a;d} - \eta_{bc}\mathscr{R}^a_{\ d;a} + \eta_{bd}\mathscr{R}^a_{\ c;a} - \eta_{bd}\mathscr{R}^a_{\ a;c} \\
&\quad + \frac{1}{2}\left((n-3)\Omega_{bc;d} - (n-3)\Omega_{bd;c} + \eta_{bd}\Omega^a_{\ c;a} - \eta_{bc}\Omega^a_{\ d;a}\right)
\end{aligned}
$$

One further trace, on bd, gives

$$
0 = 2(n-2)\left(\mathscr{R}^a_{\ c;a} - \mathscr{R}_{;c} + \frac{1}{2}\Omega^a_{\ c;a}\right)
$$

so that $\mathscr{R}^a_{\ c;a} = \mathscr{R}_{;c} - \frac{1}{2}\Omega^a_{\ c;a}$

$$
\begin{aligned}
C^a_{bcd;a} &= (n-3)\mathscr{R}_{bc;d} - (n-3)\mathscr{R}_{bd;c} + \eta_{bc}\mathscr{R}_{;d} - \eta_{bc}\mathscr{R}_{;d} + \frac{1}{2}\eta_{bc}\Omega^a_{\ d;a} + \eta_{bd}\mathscr{R}_{;c} - \frac{1}{2}\eta_{bd}\Omega^a_{\ c;a} - \eta_{bd}\mathscr{R}_{;c} \\
&\quad + \frac{1}{2}\left((n-3)\Omega_{bc;d} - (n-3)\Omega_{bd;c} + \eta_{bd}\Omega^a_{\ c;a} - \eta_{bc}\Omega^a_{\ d;a}\right) \\
&= (n-3)\left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c} + \frac{1}{2}\Omega_{bc;d} - \frac{1}{2}\Omega_{bd;c}\right)
\end{aligned}
$$

Now cycle and combine:

$$
\begin{aligned}
C^a_{bcd;a} &= (n-3)\left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c} + \frac{1}{2}\Omega_{bc;d} - \frac{1}{2}\Omega_{bd;c}\right) \\
C^a_{cdb;a} &= (n-3)\left(\mathscr{R}_{cd;b} - \mathscr{R}_{cb;d} + \frac{1}{2}\Omega_{cd;b} - \frac{1}{2}\Omega_{cb;d}\right) \\
-C^a_{dbc;a} &= (n-3)\left(-\mathscr{R}_{db;c} + \mathscr{R}_{dc;b} - \frac{1}{2}\Omega_{db;c} + \frac{1}{2}\Omega_{dc;b}\right)
\end{aligned}
$$

Sum:

$$
\begin{aligned}
C^a_{bcd;a} + C^a_{cdb;a} - C^a_{dbc;a} &= (n-3)\left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c} + \mathscr{R}_{cd;b} - \mathscr{R}_{cb;d} - \mathscr{R}_{db;c} + \mathscr{R}_{dc;b} + \frac{1}{2}\Omega_{bc;d} - \frac{1}{2}\Omega_{bd;c} + \frac{1}{2}\Omega_{cd;b} - \frac{1}{2}\Omega_{cb;d} - \frac{1}{2}\Omega_{db;c} - \right. \\
&= (n-3)\left(2\mathscr{R}_{cd;b} - 2\mathscr{R}_{bd;c} + \Omega_{bc;d}\right)
\end{aligned}
$$

Now use

$$
C^a_{bcd} + C^a_{cdb} + C^a_{dbc} = 0
$$

to write this as

$$
C^a_{bcd;a} = (n-3)\left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c} - \frac{1}{2}\Omega_{cd;b}\right)
$$

1.7 Transformation of the Weyl and Eisenhart tensors under conformal transformation

The structure equations for a Riemannian geometry are

$$
\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + \mathbf{R}^a_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\omega^a_c
\end{aligned}
$$

Changing the solder form $\mathbf{e}^a$ by a conformal factor, $\mathbf{e}^a \to e^{\varphi}\mathbf{e}^a$ changes connection to, say, α^a_b, and the second equation gives

$$
e^{\varphi}\mathbf{d}\mathbf{e}^a + e^{\varphi}\mathbf{d}\varphi\mathbf{e}^a = e^{\varphi}\mathbf{e}^b\alpha^a_b
$$

This is solved by setting $\alpha^a_b = \omega^a_b + 2\Delta^{ac}_{db}\varphi_{,c}\mathbf{e}^d$, where ω^a_b is the original spin connection, since then

$$
\begin{aligned}
e^{\varphi}\mathbf{d}\mathbf{e}^a + e^{\varphi}\mathbf{d}\varphi\mathbf{e}^a &= e^{\varphi}\mathbf{e}^b\alpha^a_b \\
&= e^{\varphi}\mathbf{e}^b\left(\omega^a_b + 2\Delta^{ac}_{db}\varphi_{,c}\mathbf{e}^d\right) \\
&= e^{\varphi}\mathbf{e}^b\omega^a_b + 2\Delta^{ac}_{db}\varphi_{,c}e^{\varphi}\mathbf{e}^b\mathbf{e}^d \\
&= e^{\varphi}\mathbf{e}^b\omega^a_b + e^{\varphi}\varphi_{,b}\mathbf{e}^b\mathbf{e}^a - \eta^{ac}\eta_{bd}\varphi_{,c}e^{\varphi}\mathbf{e}^b\mathbf{e}^d \\
e^{\varphi}\mathbf{d}\mathbf{e}^a + e^{\varphi}\mathbf{d}\varphi\mathbf{e}^a &= e^{\varphi}\mathbf{e}^b\omega^a_b + e^{\varphi}\mathbf{d}\varphi\mathbf{e}^a \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega^a_b
\end{aligned}
$$

Then the first equation gives the change in curvature,

$$
\begin{aligned}
\tilde{\mathbf{R}}^a_b &= \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c \\
&= \mathbf{d}\left(\omega^a_b + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\varphi_{,c}\mathbf{e}^d\right) \\
&\quad - \left(\omega^c_b + \left(\delta^c_d\delta^e_b - \eta^{ce}\eta_{db}\right)\varphi_{,e}\mathbf{e}^d\right)\left(\omega^a_c + \left(\delta^a_g\delta^f_c - \eta^{af}\eta_{cg}\right)\varphi_{,f}\mathbf{e}^g\right) \\
&= \mathbf{d}\omega^a_b + \mathbf{d}\varphi_{,b}\mathbf{e}^a - \eta^{ac}\eta_{db}\mathbf{d}\varphi_{,c}\mathbf{e}^d + \varphi_{,b}\mathbf{d}\mathbf{e}^a - \eta^{ac}\eta_{db}\varphi_{,c}\mathbf{d}\mathbf{e}^d \\
&\quad - \omega^c_b\omega^a_c - \varphi_{,b}\mathbf{e}^c\omega^a_c + \eta^{ce}\eta_{db}\varphi_{,e}\mathbf{e}^d\omega^a_c - \omega^c_b\varphi_{,c}\mathbf{e}^a + \omega^c_b\eta^{af}\eta_{cg}\varphi_{,f}\mathbf{e}^g \\
&\quad - \varphi_{,b}\mathbf{e}^c\varphi_{,c}\mathbf{e}^a + \varphi_{,b}\mathbf{e}^c\eta^{af}\eta_{cg}\varphi_{,f}\mathbf{e}^g + \eta^{ce}\eta_{db}\varphi_{,e}\mathbf{e}^d\varphi_{,c}\mathbf{e}^a - \eta^{ce}\eta_{db}\varphi_{,e}\mathbf{e}^d\eta^{af}\eta_{cg}\varphi_{,f}\mathbf{e}^g
\end{aligned}
$$

Simplify,

$$\begin{aligned}
\tilde{\mathbf{R}}^a_b &= \mathbf{R}^a_b + \left(\mathbf{d}\varphi_{,b} - \varphi_{,c}\omega^c_b\right)\mathbf{e}^a + \varphi_{,b}\left(\mathbf{d}\mathbf{e}^a - \mathbf{e}^c\omega^a_c\right) \\
&\quad -\eta^{ac}\eta_{bd}\left(\mathbf{d}\varphi_{,c} - \varphi_{,e}\omega^e_c\right)\mathbf{e}^d - \eta^{ac}\varphi_{,c}\eta_{bd}\left(\mathbf{d}\mathbf{e}^d - \mathbf{e}^e\omega^d_e\right) \\
&\quad +\varphi_{,b}\mathbf{e}^a\mathbf{d}\varphi - \eta^{af}\eta_{bd}\varphi_{,f}\mathbf{e}^d\mathbf{d}\varphi + \left(\eta^{ce}\varphi_{,c}\varphi_{,e}\right)\eta_{bd}\mathbf{e}^d\mathbf{e}^a \\
&= \mathbf{R}^a_b + \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd}\right)\left(\mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e\right)\mathbf{e}^d
\end{aligned}$$

Then using

$$\begin{aligned}
\tilde{\mathbf{R}}^a_b &= \tilde{\mathbf{C}}^a_b + 2\Delta^{ac}_{db}\tilde{\mathscr{R}}_c\tilde{\mathbf{e}}^d \\
&= \tilde{\mathbf{C}}^a_b + 2\Delta^{ac}_{db}\tilde{\mathscr{R}}_c e^{\phi}\mathbf{e}^d \\
&= \mathbf{C}^a_b + 2\Delta^{ac}_{db}\left(\mathscr{R}_c + \mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e\right)\mathbf{e}^d
\end{aligned}$$

we have

$$\begin{aligned}
\tilde{\mathbf{C}}^a_b &= \mathbf{C}^a_b \\
\tilde{\mathscr{R}}_c &= e^{-\phi}\left(\mathscr{R}_c + \mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e\right)
\end{aligned}$$

1.8 Symmetries of the Weyl curvature with nonzero dilatation

In scale-invariant geometries, there may exist a nonvanishing dilational curvature. We find the symmetries of the Weyl curvature when this dilatational curvature is non-zero. From

$$\mathbf{C}^a_b = \Omega^a_b - 2\Delta^{ac}_{db}\left(\mathscr{R}_c + \frac{1}{2}\Omega_{ce}\omega^e\right)\omega^d$$

we have the decomposition of the full curvature,

$$\begin{aligned}
\Omega^a_b &= \mathbf{C}^a_b + 2\Delta^{ad}_{cb}\left(\mathscr{R}_d + \frac{1}{2}\Omega_{de}\omega^e\right)\omega^c \\
&= \mathbf{R}^a_b + \Delta^{ad}_{cb}\Omega_d\omega^c
\end{aligned}$$

where $\mathbf{R}^a_b$ has the usual symmetries of the Riemann tensor. Therefore, we have antisymmetry on both the first and second pairs of indices. The triple antisymmetry usually follows from the Bianchi identity. Here we have:

$$\begin{aligned}
\Omega^a_{bcd} &= R^a_{bcd} + \Delta^{ae}_{cb}\Omega_{ed} - \Delta^{ae}_{db}\Omega_{ec} \\
\Omega^a_{[bcd]} &= R^a_{[bcd]} + \frac{1}{3}\left(\Delta^{ae}_{cb}\Omega_{ed} - \Delta^{ae}_{db}\Omega_{ec}\right) + \frac{1}{3}\left(\Delta^{ae}_{dc}\Omega_{eb} - \Delta^{ae}_{bc}\Omega_{ed}\right) + \frac{1}{3}\left(\Delta^{ae}_{bd}\Omega_{ec} - \Delta^{ae}_{cd}\Omega_{eb}\right) \\
&= \frac{1}{3}\left(\Delta^{ae}_{cb}\Omega_{ed} - \Delta^{ae}_{db}\Omega_{ec} + \Delta^{ae}_{dc}\Omega_{eb} - \Delta^{ae}_{bc}\Omega_{ed} + \Delta^{ae}_{bd}\Omega_{ec} - \Delta^{ae}_{cd}\Omega_{eb}\right) \\
&= \frac{1}{6}\left(\delta^a_c\delta^e_b\Omega_{ed} - \delta^a_d\delta^e_b\Omega_{ec} + \delta^a_d\delta^e_c\Omega_{eb} - \delta^a_b\delta^e_c\Omega_{ed} + \delta^a_b\delta^e_d\Omega_{ec} - \delta^a_c\delta^e_d\Omega_{eb}\right) \\
&\quad -\frac{1}{6}\left(\eta^{ae}\eta_{bc}\Omega_{ed} - \eta^{ae}\eta_{bd}\Omega_{ec} + \eta^{ae}\eta_{cd}\Omega_{eb} - \eta^{ae}\eta_{bc}\Omega_{ed} + \eta^{ae}\eta_{bd}\Omega_{ec} - \eta^{ae}\eta_{cd}\Omega_{eb}\right) \\
&= -\frac{1}{3}\left(\delta^a_d\Omega_{bc} + \delta^a_b\Omega_{cd} + \delta^a_c\Omega_{db}\right)
\end{aligned}$$

Then

$$\Omega^a_{[bcd]} = -\delta^a_{[b}\Omega_{cd]}$$

Writing the curvature in covariant form,

$$\Omega^a_{bcd} = R^a_{bcd} + \Delta^{ae}_{db}\Omega_{ec} - \Delta^{ae}_{cb}\Omega_{ed}$$

$$
\begin{aligned}
\Omega^a_{bcd} &= R^a_{bcd} + \frac{1}{2}\delta^a_d \Omega_{bc} - \frac{1}{2}\eta^{ae}\eta_{bd}\Omega_{ec} - \frac{1}{2}\delta^a_c \Omega_{bd} + \frac{1}{2}\eta^{ae}\eta_{bc}\Omega_{ed} \\
\Omega_{abcd} &= R_{abcd} + \frac{1}{2}\eta_{ad}\Omega_{bc} - \frac{1}{2}\eta_{bd}\Omega_{ac} - \frac{1}{2}\eta_{ac}\Omega_{bd} + \frac{1}{2}\eta_{bc}\Omega_{ad}
\end{aligned}
$$

we also have

$$
\begin{aligned}
\Omega_{abcd} - \Omega_{cdab} &= R_{abcd} - R_{cdab} + \frac{1}{2}\eta_{ad}\Omega_{bc} - \frac{1}{2}\eta_{bd}\Omega_{ac} - \frac{1}{2}\eta_{ac}\Omega_{bd} + \frac{1}{2}\eta_{bc}\Omega_{ad} \\
&\quad - \frac{1}{2}\eta_{cb}\Omega_{da} + \frac{1}{2}\eta_{db}\Omega_{ca} + \frac{1}{2}\eta_{ac}\Omega_{db} - \frac{1}{2}\eta_{da}\Omega_{cb} \\
&= \frac{1}{2}\eta_{ad}\Omega_{bc} - \frac{1}{2}\eta_{da}\Omega_{cb} - \frac{1}{2}\eta_{bd}\Omega_{ac} + \frac{1}{2}\eta_{db}\Omega_{ca} - \frac{1}{2}\eta_{ac}\Omega_{bd} + \frac{1}{2}\eta_{ac}\Omega_{db} + \frac{1}{2}\eta_{bc}\Omega_{ad} - \frac{1}{2}\eta_{cb}\Omega_{da} \\
&= \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac} - \eta_{ac}\Omega_{bd} + \eta_{bc}\Omega_{ad}
\end{aligned}
$$

Therefore, the new symmetries are:

$$
\begin{aligned}
\Omega_{abcd} &= -\Omega_{bacd} = -\Omega_{abdc} \\
\Omega^a_{[bcd]} &= \delta^a_{[b}\Omega_{cd]} \\
\Omega_{abcd} - \Omega_{cdab} &= \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac} - \eta_{ac}\Omega_{bd} + \eta_{bc}\Omega_{ad}
\end{aligned}
$$

2 Poincaré gauge theory

We begin our set of gauge theories of general relativity with the gauging of the Poincaré group, taking the quotient of the Pioncaré group by its Lorentz subgroup. The base manifold is interpreted as spacetime and the quotient bundle gives local Lorentz symmetry.

2.1 The structure equations and Bianchi identities

The Maurer-Cartan structure equations, extended to the Cartan equations with the addition of horizontal curvatures, are

$$
\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b \omega^a_c + \mathbf{R}^a_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c \omega^a_c + \mathbf{T}^a
\end{aligned}
$$

with Bianchi identities

$$
\begin{aligned}
0 &= \mathbf{D}\mathbf{R}^a_b = \mathbf{d}\mathbf{R}^a_b + \mathbf{R}^c_b \omega^a_c + \omega^c_b \mathbf{R}^a_c \\
0 &= \mathbf{e}^c \mathbf{R}^a_c + \mathbf{D}\mathbf{T}^a = \mathbf{e}^c \mathbf{R}^a_c + \mathbf{d}\mathbf{T}^a + \mathbf{T}^c \omega^a_c
\end{aligned}
$$

or, in components,

$$
\begin{aligned}
R^a_{b[cd;e]} &= 0 \\
R^a_{[bcd]} + T^a_{[bc;d]} &= 0
\end{aligned}
$$

2.2 The Einstein-Hilbert action

After eliminating the translational gauge transformations, the gauge transformations of the connection reduce to

$$
\begin{aligned}
\tilde{\omega}^a_b &= \Lambda^a_c \omega^c_d \bar{\Lambda}^d_b - \mathbf{d}\Lambda^a_c \bar{\Lambda}^c_b \\
\tilde{\mathbf{e}}^a &= \Lambda^a_c \mathbf{e}^c
\end{aligned}
$$

so the solder form now transforms as a tensor. For the curvatures we have

$$
\begin{aligned}
\tilde{\mathbf{R}}^a_b &= \Lambda^a_c \mathbf{R}^c_d \bar{\Lambda}^d_b \\
\tilde{\mathbf{T}}^a &= \Lambda^a_c \mathbf{T}^c
\end{aligned}
$$

so the torsion and Lorentz components of the Poincaré curvature do not mix. The full set of available tensors is therefore $\left\{\eta_{ab}, e_{abcd}, \mathbf{e}^a, \mathbf{T}^a, \mathbf{R}^a_b\right\}$. The most general action we can build from these, up to terms linear in the curvature, is

$$S = \int \left(\mathbf{R}^{ab}\mathbf{e}^c \dots \mathbf{e}^d + \Lambda \mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \dots \mathbf{e}^d\right) e_{abc\dots d}$$

where Λ is constant. Torsion terms can enter only at second order.

2.3 The field equations

Varying the connection, we have

$$\begin{aligned}
0 &= \delta S \\
&= \int \left(\delta\mathbf{R}^{ab}\mathbf{e}^c \dots \mathbf{e}^d + (n-2)\,\mathbf{R}^{ab}\mathbf{e}^c \dots \delta\mathbf{e}^d + n\Lambda\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \dots \delta\mathbf{e}^d\right) e_{abc\dots d} \\
&= \int \left(\mathbf{D}\delta\omega^{ab}\mathbf{e}^c \dots \mathbf{e}^d\mathbf{e}^e + (n-2)\,\mathbf{R}^{ab}\mathbf{e}^c \dots \mathbf{e}^d\,\delta\mathbf{e}^e + n\Lambda\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \dots \mathbf{e}^d\,\delta\mathbf{e}^e\right) e_{abc\dots de} \\
&= \int \left((n-2)\,\delta\omega^{ab}\mathbf{D}\mathbf{e}^c \dots \mathbf{e}^d\mathbf{e}^e + (n-2)\,\mathbf{R}^{ab}\mathbf{e}^c \dots \mathbf{e}^d\,\delta\mathbf{e}^e + n\Lambda\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \dots \mathbf{e}^d\,\delta\mathbf{e}^e\right) e_{abc\dots de}
\end{aligned}$$

Since each component of the connection is varied independently we have

$$\begin{aligned}
0 &= (n-2)\,\mathbf{D}\mathbf{e}^c\mathbf{e}^d \dots \mathbf{e}^e e_{abcd\dots e} \\
0 &= \left((n-2)\,\mathbf{R}^{ab}\mathbf{e}^c \dots \mathbf{e}^d + n\Lambda\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \dots \mathbf{e}^d\right) e_{abc\dots de}
\end{aligned}$$

We consider these in turn.

In the first field equation, we recognize the covariant exterior derivative of the solder form, $\mathbf{D}\mathbf{e}^c$, as the torsion, so (for $n > 2$)

$$\begin{aligned}
0 &= \mathbf{T}^c\mathbf{e}^d \dots \mathbf{e}^e e_{abcd\dots e} \\
&= \frac{1}{2}T^c_{fg}\mathbf{e}^f\mathbf{e}^g\mathbf{e}^d \dots \mathbf{e}^e e_{abcd\dots e}
\end{aligned}$$

Take the wedge product of this equation with one further arbitrary solder form, $\mathbf{e}^h$, then define the convenient volume form

$$\Phi = \frac{1}{n!}e_{a\dots b}\mathbf{e}^a \dots \mathbf{e}^b$$

so that

$$\mathbf{e}^a \dots \mathbf{e}^b = -e^{a\dots b}\Phi$$

We may then write

$$0 = -\frac{1}{2}T^c_{fg}e^{hfgd\dots e}e_{abcd\dots e}\Phi$$

Taking the Hodge dual and discarding the overall constant this becomes

$$\begin{aligned}
0 &= T^c_{fg}e^{hfgd\dots e}e_{abcd\dots e} \\
&= T^c_{fg}\delta^{hfg}_{abc}
\end{aligned}$$

where $\delta^{hfg}_{abc} = \delta^{[h}_a\delta^f_b\delta^{g]}_c$. Expanding, while relying on the antisymmetry of the fg indices on the torsion,

$$\begin{aligned}
0 &= T^c_{fg}\delta^{hfg}_{abc} \\
&= \frac{1}{3}T^c_{fg}\left(\delta^h_a\delta^f_b\delta^g_c + \delta^f_a\delta^g_b\delta^h_c + \delta^g_a\delta^h_b\delta^f_c\right) \\
&= \frac{1}{3}\left(T^c_{bc}\delta^h_a + T^c_{ab}\delta^h_c + T^c_{ca}\delta^h_b\right)
\end{aligned}$$

Contract on the hb indices,

$$\begin{aligned}0 &= \frac{1}{3}\left(T^c_{ac}+T^c_{ac}+nT^c_{ca}\right)\\0 &= -\frac{1}{3}\left(n-2\right)T^c_{ac}\end{aligned}$$

so the trace of the torsion must be zero and the full field equation reduces to vanish torsion, $T^c_{ab}=0$.

The second field equation is

$$\begin{aligned}0 &= \left(\left(n-2\right)\mathbf{R}^{ab}\mathbf{e}^c\ldots\mathbf{e}^d+n\Lambda\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c\ldots\mathbf{e}^d\right)e_{abc\ldots de}\\0 &= \left(\frac{1}{2}\left(n-2\right)R^{ab}{}_{fg}+n\Lambda\delta^a_f\delta^b_g\right)\mathbf{e}^f\mathbf{e}^g\mathbf{e}^c\ldots\mathbf{e}^d e_{abc\ldots de}\end{aligned}$$

Taking the wedge product with one further solder form and using the volume form as before leads to

$$\begin{aligned}0 &= \left(\frac{1}{2}\left(n-2\right)R^{ab}{}_{fg}+n\Lambda\delta^a_f\delta^b_g\right)\mathbf{e}^h\mathbf{e}^f\mathbf{e}^g\mathbf{e}^c\ldots\mathbf{e}^d e_{abc\ldots de}\\0 &= -\left(-1\right)^{n-1}\left(\frac{1}{2}\left(n-2\right)R^{ab}{}_{fg}+n\Lambda\delta^a_f\delta^b_g\right)e^{hfgc\ldots d}e_{eabc\ldots d}\Phi\end{aligned}$$

and therefore, taking the dual,

$$\begin{aligned}0 &= \left(\frac{1}{2}\left(n-2\right)R^{ab}{}_{fg}+\frac{1}{2}n\Lambda\left(\delta^a_f\delta^b_g-\delta^a_g\delta^b_f\right)\right)\delta^{hfg}_{eab}\\&= \frac{1}{6}\left(\left(n-2\right)R^{ab}{}_{fg}+n\Lambda\left(\delta^a_f\delta^b_g-\delta^a_g\delta^b_f\right)\right)\left(\delta^h_e\delta^f_a\delta^g_b+\delta^f_e\delta^g_a\delta^h_b+\delta^g_e\delta^h_a\delta^f_b\right)\\&= \frac{1}{6}\left(\left(n-2\right)\left(R^{ab}{}_{ab}\delta^h_e+R^{ah}{}_{ea}+R^{hb}{}_{be}\right)+n\Lambda\left(n^2\delta^h_e-n\delta^h_e+\delta^h_e-n\delta^h_e+\delta^h_e-n\delta^h_e\right)\right)\\&= \frac{1}{6}\left(\left(n-2\right)\left(R\delta^h_e-2R^h{}_e\right)+n\left(n-1\right)\left(n-2\right)\Lambda\delta^h_e\right)\end{aligned}$$

Lowering the upper index this becomes the Einstein equation with cosmological constant, $\tilde{\Lambda}=\frac{1}{2}n\left(n-1\right)\Lambda$,

$$R_{ab}-\frac{1}{2}R\eta_{ab}-\tilde{\Lambda}\eta_{ab}=0$$

2.4 Conformal Ricci flatness in Riemannian geometry

We find the necessary and sufficient conditions for a Riemannian geometry to be conformally related to a Ricci-flat Riemannian geometry.

2.4.1 Necessary condition: Exterior derivative of the Eisenhart tensor

We compute the covariant exterior derivative of the Eisenhart tensor when the metric is conformal to a Ricci flat metric.

Suppose we have

$$\begin{aligned}\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c+\mathbf{R}^a_b\\\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\omega^a_c\end{aligned}$$

where the Ricci and Eisenhart tensors of $\mathbf{R}^a_b$ vanish, $R_{ab}=0$; $\mathscr{R}_a=0$, and let

$$\tilde{\mathbf{e}}^a=e^{\varphi}\mathbf{e}^a$$

The corresponding spin connection is given by

$$\begin{aligned}\mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^c\tilde{\omega}^a_c\\\mathbf{d}\varphi e^{\varphi}\mathbf{e}^a+e^{\varphi}\mathbf{d}\mathbf{e}^a &= e^{\varphi}\mathbf{e}^c\tilde{\omega}^a_c\\\mathbf{d}\varphi\mathbf{e}^a+\mathbf{e}^c\omega^a_c &= \mathbf{e}^c\tilde{\omega}^a_c\end{aligned}$$

Let

$$\tilde{\omega}^a_c = \omega^a_c + 2\Delta^{ae}_{dc}\varphi_{,e}\mathbf{e}^d$$

Then

$$\begin{aligned}
\mathbf{e}^c\tilde{\omega}^a_c &= \mathbf{e}^c\left(\omega^a_c + 2\Delta^{ae}_{dc}\varphi_{,e}\mathbf{e}^d\right) \\
&= \mathbf{e}^c\omega^a_c + \mathbf{e}^c\left(\delta^a_d\delta^e_c - \eta^{ae}\eta_{dc}\right)\varphi_{,e}\mathbf{e}^d \\
&= \mathbf{e}^c\omega^a_c + \mathbf{d}\varphi\mathbf{e}^a
\end{aligned}$$

as desired.

We wish to compute $\tilde{\mathbf{D}}\tilde{\mathscr{R}}_a$,

$$\begin{aligned}
\tilde{\mathbf{D}}\tilde{\mathscr{R}}_a &= \mathbf{d}\tilde{\mathscr{R}}_a - \tilde{\omega}^c_a\tilde{\mathscr{R}}_c \\
&= \mathbf{d}\tilde{\mathscr{R}}_a - \left(\omega^c_a + 2\Delta^{ce}_{da}\varphi_{,e}\mathbf{e}^d\right)\tilde{\mathscr{R}}_c \\
&= \mathbf{D}\tilde{\mathscr{R}}_a - 2\Delta^{ce}_{da}\phi_{,e}\mathbf{e}^d\tilde{\mathscr{R}}_c \\
&= \mathbf{D}\left(e^{-\varphi}\left(\mathbf{D}\varphi_{,a} - \varphi_{,a}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ae}\mathbf{e}^e\right)\right) \\
&\quad -2\Delta^{ce}_{da}\varphi_{,e}\mathbf{e}^d\left(e^{-\varphi}\left(\mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{cb}\mathbf{e}^b\right)\right) \\
&= -e^{-\varphi}\mathbf{D}\varphi\left(\mathbf{D}\varphi_{,a} - \varphi_{,a}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ae}\mathbf{e}^e\right) \\
&\quad +e^{-\varphi}\left(\mathbf{D}\mathbf{D}\varphi_{,a} - \mathbf{D}\left(\varphi_{,a}\mathbf{d}\varphi\right) + \frac{1}{2}\mathbf{D}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ae}\mathbf{e}^e\right) \\
&\quad -2\Delta^{ce}_{da}\varphi_{,e}\mathbf{e}^d\left(e^{-\phi}\left(\mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{cb}\mathbf{e}^b\right)\right)
\end{aligned}$$

Then

$$\begin{aligned}
e^{\varphi}\tilde{\mathbf{D}}\tilde{\mathscr{R}}_a &= -\mathbf{d}\varphi\mathbf{D}\varphi_{,a} - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\mathbf{d}\varphi\eta_{ae}\mathbf{e}^e \\
&\quad +\mathbf{D}\mathbf{D}\varphi_{,a} - \mathbf{D}\left(\varphi_{,a}\mathbf{d}\varphi\right) + \frac{1}{2}\mathbf{D}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ae}\mathbf{e}^e \\
&\quad -\varphi_{,a}\mathbf{e}^c\mathbf{D}\varphi_{,c} + \phi_{,a}\mathbf{e}^c\varphi_{,c}\mathbf{d}\varphi - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\varphi_{,a}\mathbf{e}^c\eta_{cb}\mathbf{e}^b \\
&\quad +\eta^{ce}\eta_{da}\varphi_{,e}\mathbf{e}^d\mathbf{D}\varphi_{,c} - \left(\eta^{ce}\phi_{,e}\varphi_{,c}\right)\eta_{da}\mathbf{e}^d\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{da}\mathbf{e}^d\mathbf{d}\varphi
\end{aligned}$$

and so

$$\begin{aligned}
e^{\varphi}\tilde{\mathbf{D}}\tilde{\mathscr{R}}_a &= \mathbf{D}\mathbf{D}\varphi_{,a} + \eta^{fg}\varphi_{,f}\mathbf{D}\varphi_{,g}\eta_{ae}\mathbf{e}^e - \varphi_{,a}\mathbf{D}\left(\mathbf{d}\varphi\right) + \eta^{ce}\eta_{da}\varphi_{,e}\mathbf{e}^d\mathbf{D}\varphi_{,c} \\
&\quad +\frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\left(\eta_{da}\mathbf{e}^d\mathbf{d}\varphi + \eta_{ae}\mathbf{e}^e\mathbf{d}\varphi - 2\eta_{da}\mathbf{e}^d\mathbf{d}\varphi\right) \\
&= \mathbf{D}\mathbf{D}\varphi_{,a} + \left(\eta^{fg}\varphi_{,f}\mathbf{D}\varphi_{,g} - \eta^{ce}\varphi_{,e}\mathbf{D}\varphi_{,c}\right)\left(\eta_{ad}\mathbf{e}^d\right) \\
&= \mathbf{D}\mathbf{D}\varphi_{,a}
\end{aligned}$$

Finally, the Ricci identity gives

$$\begin{aligned}
\mathbf{D}\mathbf{D}\varphi_{,a} &= \mathbf{D}\left(\mathbf{d}\varphi_{,a} - \varphi_b\omega^b_a\right) \\
&= \mathbf{d}\left(\mathbf{d}\varphi_{,a} - \varphi_b\omega^b_a\right) + \left(\mathbf{d}\varphi_{,c} - \varphi_b\omega^b_c\right)\omega^c_a \\
&= -\varphi_b\left(\mathbf{d}\omega^b_a - \omega^c_a\omega^b_c\right) \\
&= -\varphi_b\mathbf{R}^b_a \\
&= -\varphi_b\mathbf{C}^b_a \\
&= -\varphi_b\tilde{\mathbf{C}}^b_a
\end{aligned}$$

Thus,

$$\begin{aligned} e^{\varphi}\tilde{\mathbf{D}}\tilde{\mathscr{R}}_a + \varphi_b \tilde{\mathbf{C}}^b_a &= 0 \\ \tilde{\mathbf{D}}\tilde{\mathscr{R}}_a - \left(e^{-\varphi}\right)_{,b} \tilde{\mathbf{C}}^b_a &= 0 \end{aligned}$$

This condition must hold in any geometry related to a Ricci flat geometry by a conformal transformation.

However, notice that our original choice of φ was fully arbitrary. Therefore, this equation holds *regardless of* φ. We can choose e^{φ} to be *any* gauge factor, and an equation of this form holds. What this means is that there is one gauge in which $\varphi = 0$; this is the Ricci-flat gauge. In general, when this equation is satisfied, the Ricci and Eisenhart tensors are constructed from derivatives of φ. The curl of the Eisenhart tensor doesn't vanish, but ends up being $\left(e^{-\varphi}\right)_{,b} \tilde{\mathbf{C}}^b_a$.

2.4.2 Sufficient condition: Integrability for conformal Ricci flatness

We have shown that if a spacetime is conformal to a Ricci flat spacetime, then we necessarily have

$$\tilde{\mathbf{D}}\tilde{\mathscr{R}}_a - \left(e^{-\varphi}\right)_{,b} \tilde{\mathbf{C}}^b_a = 0$$

We now seek a sufficient conditon.

Start with a spacetime with curvatures $\mathbf{C}^a_b, \mathscr{R}_c$, and perform a conformal transformation. The new curvatures are given by

$$\begin{aligned} \tilde{\mathbf{C}}^a_b &= \mathbf{C}^a_b \\ \tilde{\mathscr{R}}_c &= e^{-\varphi}\left(\mathscr{R}_c + \mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e\right) \end{aligned}$$

and the new spacetime will be Ricci flat if and only if $\tilde{\mathscr{R}}_c = 0$. We therefore require the existence of a scalar φ satisfying

$$\begin{aligned} 0 &= \mathscr{R}_c + \mathbf{D}\varphi_{,c} - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e \\ &= \mathscr{R}_c + \mathbf{d}\varphi_{,c} - \varphi_{,a}\omega^a_c - \varphi_{,c}\mathbf{d}\varphi + \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e \end{aligned}$$

Write this as

$$\mathbf{d}\varphi_{,c} = \varphi_{,a}\omega^a_c + \varphi_{,c}\mathbf{d}\varphi - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e - \mathscr{R}_c$$

Then the integrability condition is given by the Poincaré lemma, $\mathbf{d}^2\varphi_{,c} = 0$. We therefore require

$$\begin{aligned} 0 &= \mathbf{d}\left(\varphi_{,a}\omega^a_c + \varphi_{,c}\mathbf{d}\varphi - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e - \mathscr{R}_c\right) \\ &= \mathbf{d}\varphi_{,a}\omega^a_c + \varphi_{,a}\mathbf{d}\omega^a_c + \mathbf{d}\varphi_{,c}\mathbf{d}\varphi - \left(\eta^{fg}\varphi_{,f}\mathbf{d}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{d}\mathbf{e}^e - \mathbf{d}\mathscr{R}_c \end{aligned}$$

Now substitute for $\mathbf{d}\varphi_{,c}$,

$$\begin{aligned} 0 = &\left(\varphi_{,b}\omega^b_a + \varphi_{,a}\mathbf{d}\varphi - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ae}\mathbf{e}^e - \mathscr{R}_a\right)\omega^a_c + \varphi_{,a}\mathbf{d}\omega^a_c \\ &+ \left(\varphi_{,a}\omega^a_c + \varphi_{,c}\mathbf{d}\varphi - \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{e}^e - \mathscr{R}_c\right)\mathbf{d}\varphi \\ &- \left(\eta^{fg}\varphi_{,f}\left(\varphi_{,a}\omega^a_g + \varphi_{,g}\mathbf{d}\varphi - \frac{1}{2}\left(\eta^{hk}\varphi_{,h}\varphi_{,k}\right)\eta_{gd}\mathbf{e}^d - \mathscr{R}_g\right)\right)\eta_{ce}\mathbf{e}^e \\ &- \frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\eta_{ce}\mathbf{d}\mathbf{e}^e - \mathbf{d}\mathscr{R}_c \end{aligned}$$

Simplify,

$$
\begin{aligned}
0 \;=\;& \varphi_{,a}\left(\mathbf{d}\omega_c^a - \omega_c^b\omega_b^a\right) + \varphi_{,a}\mathbf{d}\varphi\omega_c^a - \mathscr{R}_c\mathbf{d}\varphi - \left(\mathbf{d}\mathscr{R}_c - \omega_c^a\mathscr{R}_a\right) \\
& +\varphi_{,a}\omega_c^a\mathbf{d}\varphi - \left(\varphi_{,f}\varphi_{,a}\omega^{af}\right)\eta_{ce}\mathbf{e}^e + \eta^{fg}\varphi_{,f}\mathscr{R}_g\eta_{ce}\mathbf{e}^e \\
& -\frac{1}{2}\left(\eta^{fg}\varphi_{,f}\varphi_{,g}\right)\left(\eta_{ce}\mathbf{e}^e\mathbf{d}\varphi + 2\mathbf{d}\varphi\eta_{ce}\mathbf{e}^e - \mathbf{d}\varphi\eta_{ce}\mathbf{e}^e + \eta_{ac}\left(\mathbf{d}\mathbf{e}^a - \mathbf{e}^e\omega_e^a\right)\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
0 \;=\;& \varphi_{,a}\left(\mathbf{d}\omega_c^a - \omega_c^b\omega_b^a\right) \\
& -\left(\mathbf{d}\mathscr{R}_c - \omega_c^a\mathscr{R}_a + \mathscr{R}_b\left(\delta_c^b\delta_a^d - \eta^{bd}\eta_{ca}\right)\varphi_{,d}\mathbf{e}^a\right) \\
=\;& \varphi_{,a}\mathbf{R}_c^a - \left(\mathbf{d}\mathscr{R}_c - \left(\omega_c^b + 2\Delta_{ca}^{bd}\varphi_{,d}\mathbf{e}^a\right)\mathscr{R}_b\right) \\
=\;& \varphi_{,a}\mathbf{R}_c^a - \left(\mathbf{d}\mathscr{R}_c - \tilde{\omega}_c^b\mathscr{R}_b\right) \\
=\;& \varphi_{,a}\mathbf{R}_c^a - \tilde{\mathbf{D}}\mathscr{R}_c
\end{aligned}
$$

Try to separate out the φ-dependence,

$$0 = \varphi_{,a}\mathbf{R}_c^a - \mathbf{D}\mathscr{R}_c + 2\Delta_{ca}^{bd}\varphi_{,d}\mathbf{e}^a\mathscr{R}_b$$

Expand the curvature,

$$\mathbf{R}_b^a = \mathbf{C}_b^a + 2\Delta_{db}^{ac}\mathscr{R}_c\mathbf{e}^d$$

so that

$$
\begin{aligned}
0 \;=\;& \varphi_{,a}\mathbf{C}_c^a + \varphi_{,a}2\Delta_{dc}^{ae}\mathscr{R}_e\mathbf{e}^d - \mathbf{D}\mathscr{R}_c + 2\Delta_{ca}^{bd}\varphi_{,d}\mathbf{e}^a\mathscr{R}_b \\
=\;& \varphi_{,a}\mathbf{C}_c^a - \mathbf{D}\mathscr{R}_c + 2\Delta_{ca}^{bd}\varphi_{,d}\left(\mathscr{R}_b\mathbf{e}^a + \mathbf{e}^a\mathscr{R}_b\right) \\
=\;& \varphi_{,a}\mathbf{C}_c^a - \mathbf{D}\mathscr{R}_c
\end{aligned}
$$

Therefore, a sufficient condition for the space to be conformally Ricci flat is the existence of a scalar field such that

$$\varphi_{,a}\mathbf{C}_c^a - \mathbf{D}\mathscr{R}_c = 0$$

This is the usual form of the condition. Unfortunately, the condition still depends on the scalar field.

This condition is therefore necessary and sufficient.

How do we use this condition to solve for the geometry? Here's the program: Having this condition tells us that there exists a gauge in which the Ricci tensor vanishes. Choose that gauge and solve for the metric. Now, gauge again by an arbitrary conformal factor. This gives the conformal equivalence class of metrics which solve the problem. Each choice of gauge will satisify $\varphi_{,a}\mathbf{C}_c^a - \mathbf{D}\mathscr{R}_c = 0$ with its appropriate conformal factor, Weyl curvature and Eisenhart tensor.

The usefulness of the condition is that it is the conformal equivalent of the Einstein equation. It is not particularly useful for *solving* for the metric – the whole point is that we can still use the Einstein equation for that, but then have the freedom to choose a gauge. But knowing that $\varphi_{,a}\mathbf{C}_c^a - \mathbf{D}\mathscr{R}_c = 0$ is the conformal Einstein equation lets us interpret that expression when it occurs in other gravity theories. Any gravity theory that leads to this condition is on experimentally sound footing – all data that support general relativity support such a theory as well, and the freedom to choose a local scale makes it epistemologically more sound.

3 Weyl (homothetic) gauge theory

We ask corresponding questions in a Weyl geometry. The overall symmetry is that of the Poincaré group together with dilatations. We take the quotient of the inhomogeneous Weyl group by the homogeneous Weyl group to get an n-dim base manifold with local homothetic (Lorentz plus dilatations) symmetry.

The structure equations are now

$$\begin{aligned}\mathbf{d}\omega_b^a &= \omega_b^c\omega_c^a + \Omega_b^a \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\omega_c^a + \omega\mathbf{e}^a \\ \mathbf{d}\omega &= \Omega\end{aligned}$$

and setting $\omega = W_a\mathbf{e}^a$, the connection may be written in the form

$$\omega_c^a = \alpha_c^a - 2\Delta_{bc}^{ad}W_d\mathbf{e}^b$$

where α_c^a is the usual spin connection for $\mathbf{e}^a$, since then

$$\begin{aligned}\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\omega_c^a + \omega\mathbf{e}^a \\ &= \mathbf{e}^c\alpha_c^a - 2\Delta_{bc}^{ad}W_d\mathbf{e}^c\mathbf{e}^b + \omega\mathbf{e}^a \\ &= \mathbf{e}^c\alpha_c^a - W_c\mathbf{e}^c\mathbf{e}^a + \omega\mathbf{e}^a \\ &= \mathbf{e}^c\alpha_c^a\end{aligned}$$

Notice that under conformal transformation, the connection changes by

$$\begin{aligned}\tilde{\mathbf{e}}^a &= e^{\varphi}\mathbf{e}^a \\ \tilde{\omega} &= \omega + \mathbf{d}\varphi \\ \tilde{\omega}_b^a &= \omega_b^a\end{aligned}$$

Notice that the spin connection is unchanged (this happens because dilatations and Lorentz transformations commute) and therefore the curvature is unchanged:

$$\begin{aligned}\tilde{\Omega}_b^a &= \Omega_b^a \\ \tilde{\mathbf{C}}_b^a &= \mathbf{C}_b^a \\ \tilde{\mathscr{R}}_a &= \mathscr{R}_a\end{aligned}$$

As a result, it does not make sense to ask for a conformally related metric with vanishing Ricci tensor – the Ricci tensor is conformally invariant. We now consider the homothetic equivalent of the vanishing of the Ricci tensor, namely,

$$\Omega_{bac}^a = 0$$

We now show that this is the necessary and sufficient condition for the existence of a gauge in which the Riemannian part of the curvature has vanishing Ricci tensor.

3.1 Conformal Ricci flatness

We now consider the necessary and sufficient conditions under which the Ehrenfest tensor, $\mathscr{R}_a$, vanishes in a Weyl geometry. Since the vanishing of the Ehrenfest tensor is equivalent to vanishing Ricci tensor, this condition solves the vacuum Einstein equation.

We have seen that the Lorentz curvature is given by

$$\Omega_b^a = \mathbf{R}_b^a - 2\Delta_{db}^{ac}\left(\mathbf{D}W_c + W_c\omega - \frac{1}{2}\eta_{ce}\left(\eta^{fg}W_fW_g\right)\mathbf{e}^e\right)\mathbf{e}^d$$

Suppose the trace of this curvature vanishes,

$$\Omega_{bac}^a = 0$$

Then, writing Ω_b^a in components,

$$\Omega_{bcd}^a = R_{bcd}^a - 2\Delta_{db}^{ae}\left(W_{e;c} + W_eW_c - \frac{1}{2}\eta_{ce}\left(\eta^{fg}W_fW_g\right)\right)\mathbf{e}^c\mathbf{e}^d$$

we must have

$$\begin{aligned} 0 &= \Omega^c_{bcd} \\ &= R_{bd} + (n-2)\, W_{b;d} + \eta^{ce}\eta_{bd} W_{e;c} \\ &\quad + (n-2)\, W_b W_d + \eta^{ce}\eta_{bd} W_e W_c \\ &\quad - \eta_{bd}\,(n-1)\left(\eta^{fg} W_f W_g\right) \end{aligned}$$

The trace gives

$$\begin{aligned} 0 &= R + 2\,(n-1)\,\eta^{bd} W_{b;d} \\ &\quad -(n-1)\,(n-2)\left(\eta^{fg} W_f W_g\right) \\ \eta^{bd} W_{b;d} &= \frac{1}{2\,(n-1)}\left(-R + (n-1)\,(n-2)\left(\eta^{fg} W_f W_g\right)\right) \end{aligned}$$

Substituting,

$$\begin{aligned} 0 &= R_{bd} - \frac{1}{2\,(n-1)}\eta_{bd} R \\ &\quad + (n-2)\, W_{b;d} + (n-2)\, W_b W_d - \frac{1}{2}(n-2)\,\eta_{bd}\left(\eta^{fg} W_f W_g\right) \end{aligned}$$

or,

$$\begin{aligned} \mathscr{R}_{bd} &= -\frac{1}{n-2}\left(R_{bd} - \frac{1}{2\,(n-1)}\eta_{bd} R\right) \\ &= W_{b;d} + W_b W_d - \frac{1}{2}\eta_{bd}\left(\eta^{fg} W_f W_g\right) \end{aligned}$$

This has symmetric and antisymmetric parts,

$$\begin{aligned} \mathscr{R}_{bd} &= W_{(b;d)} + W_b W_d - \frac{1}{2}\eta_{bd}\left(\eta^{fg} W_f W_g\right) \\ 0 &= W_{[b;d]} \end{aligned}$$

The antisymmetric part is just half the dilatation. Therefore, $\Omega^c_{bcd} = 0$ if and only if

$$\begin{aligned} \mathscr{R}_{bd} &= W_{b;d} + W_b W_d - \frac{1}{2}\eta_{bd}\left(\eta^{fg} W_f W_g\right) \\ \Omega_{bd} &= 0 \end{aligned}$$

Solution for the Weyl vector We treat this as a differential equation for the Weyl vector. Contract with $\mathbf{e}^d$ to write it as a 1-form equation,

$$\begin{aligned} 0 &= \mathbf{D} W_b + W_b \omega - \frac{1}{2}\eta_{bd} W^2 \mathbf{e}^d - \mathscr{R}_b \\ \mathbf{d} W_b &= W_c \alpha^c_b - W_b \omega + \frac{1}{2}\eta_{bd} W^2 \mathbf{e}^d + \mathscr{R}_b \end{aligned}$$

The integrability condition is then

$$\begin{aligned} 0 &= \mathbf{d}^2 W_b \\ &= \mathbf{d} W_c \alpha^c_b + W_c \mathbf{d}\alpha^c_b - \mathbf{d} W_b \omega - W_b \mathbf{d}\omega + \eta_{bd} W^c \mathbf{d} W_c \mathbf{e}^d + \frac{1}{2}\eta_{bd} W^2 \mathbf{d}\mathbf{e}^d + \mathbf{d}\mathscr{R}_b \\ &= \mathbf{d}\mathscr{R}_b + \left(W_e \alpha^e_c - W_c \omega + \frac{1}{2}\eta_{cd} W^2 \mathbf{e}^d + \mathscr{R}_c\right)\alpha^c_b + W_c \mathbf{d}\alpha^c_b \\ &\quad - \left(W_c \alpha^c_b - W_b \omega + \frac{1}{2}\eta_{bd} W^2 \mathbf{e}^d + \mathscr{R}_b\right)\omega - W_b \mathbf{d}\omega \\ &\quad + \eta_{bd} W^c \left(W_e \alpha^e_c - W_c \omega + \frac{1}{2}\eta_{cd} W^2 \mathbf{e}^d + \mathscr{R}_c\right)\mathbf{e}^d + \frac{1}{2}\eta_{bd} W^2 \mathbf{d}\mathbf{e}^d \end{aligned}$$

Now collect terms,

$$\begin{aligned} 0 &= \mathbf{d}\mathscr{R}_b - \alpha^c_b \mathscr{R}_c - \mathscr{R}_b \omega + W_c \mathbf{d}\alpha^c_b - W_e \alpha^c_b \alpha^e_c + \eta_{bd} W^c \mathscr{R}_c \mathbf{e}^d \\ &\quad + \frac{1}{2}\eta_{bd} W^2 \mathbf{d}\mathbf{e}^d - \frac{1}{2}\eta_{cb} W^2 \mathbf{e}^d \alpha^c_d - \frac{1}{2}\eta_{bd} W^2 \mathbf{e}^d \omega - \eta_{bd} W^2 \omega \mathbf{e}^d + \frac{1}{2}\eta_{bd} W^2 W_e \mathbf{e}^e \mathbf{e}^d \\ &\quad - W_b \mathbf{d}\omega \end{aligned}$$

and therefore,

$$0 = \mathbf{D}_{(\alpha)} \mathscr{R}_b - W_b \Omega + W_c \mathbf{R}^c_b - \mathscr{R}_b \omega + \eta_{bd} W^c \mathscr{R}_c \mathbf{e}^d$$

Now write

$$\begin{aligned} -\mathscr{R}_b \omega + \eta_{bd} W^c \mathscr{R}_c \mathbf{e}^d &= W_e \left(-\delta^e_d \delta^c_b + \eta_{bd} \eta^{ec}\right) \mathscr{R}_c \mathbf{e}^d \\ &= -2 W_e \Delta^{ec}_{db} \mathscr{R}_c \mathbf{e}^d \end{aligned}$$

so we have

$$\begin{aligned} 0 &= \mathbf{D}_{(\alpha)} \mathscr{R}_b - W_b \Omega + W_c \left(\mathbf{R}^c_b - 2\Delta^{ce}_{db} \mathscr{R}_e \mathbf{e}^d\right) \\ &= \mathbf{D}_{(\alpha)} \mathscr{R}_b - W_b \Omega + W_c C^c_b \end{aligned}$$

together with

$$\Omega = 0$$

Since the dilatation vanishes, the Weyl vector is a gradient, so there exists a function φ such that

$$\begin{aligned} 0 &= \mathbf{D}_{(\alpha)} \mathscr{R}_b - W_b \Omega + W_c C^c_b \\ &= \mathbf{D}_{(\alpha)} \mathscr{R}_b + \varphi_{,c} C^c_b \end{aligned}$$

Of course, this is the condition in the underlying Riemannian geometry for conformal Ricci flatness.

Conversely, suppose the underlying Riemannian geometry is conformally related to a Ricci flat geometry. Then the condition above holds and we can run the calculation backwards to show that the integrability condition hold. This, in turn, means that we can solve

$$0 = \mathbf{D} W_b + W_b \omega - \frac{1}{2}\eta_{bd} W^2 \mathbf{e}^d - \mathscr{R}_b$$

Such a solution gives

$$\Omega^c_{bcd} = 0$$

Therefore, the Riemannian geometry underlying a Weyl geometry is conformal to a Ricci flat geometry if and only if $\Omega^c_{bcd} = 0$.

3.2 An action for General Relativity

Usually, scale invariant actions for gravity theories are taken to be quadratic in the curvature. There is a simple reason for this: while the curvatures, Ω^a_b, $\mathscr{R}_a$, and Ω are conformally invariant, it is not obvious how to build a scale invariant action because the volume element has conformal dimension $+4$. Thus, although the functional

$$S_1 = \int \Omega^{ab} \mathbf{e}^c \mathbf{e}^d e_{abcd}$$

will lead to the field equation we desire, the integrand has conformal weight $+2$. The usual solution is to write an action quadratic in the curvature, such as

$$S_2 = \int \Omega^a_b {}^*\Omega^b_a e_{abcd}$$

This leads field equations that contain second derivatives of the connection, and are therefore of third order for the metric. Such higher derivatives are often found to lead to negative norm states – ghosts – in the quantum theory. However, the situation is not actually as bad as that, for if we use the quadratic action the field equation,

$$\mathbf{D}^* \Omega^a_b = 0$$

may be related to the integrability condition for the conformal Einstein equation. But there is a still better solution.

An often overlooked fact of scale-invariant geometry is that dimensionful "constants" may have nontrivial evolution. To see this, note that the homothetically covariant derivative of a tensor with dimensions of $(length)^n$ is given by

$$D_a T^{b\ldots c} = \partial_a T^{b\ldots c} + T^{e\ldots c}\Gamma^b_{ea} + \ldots + T^{b\ldots e}\Gamma^c_{ea} + nW_a T^{b\ldots c}$$

and tensors are labeled by both Lorentz type, $\begin{pmatrix} p \\ q \end{pmatrix}$, and by conformal weight, n. We may therefore have tensors, κ, of Lorentz type $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but conformal weight n, and these will have covariant derivative

$$D_a \kappa = \partial_a \kappa + nW_a \kappa$$

even though in a Riemannian spacetime κ would be scalar or even constant. The physical meaning of this is clear. Suppose κ represents the length of a meter stick. Then, if we use global units with $W_a = 0$, that length is unchanging,

$$\partial_a \kappa_0 = 0$$

and the length is constant. But suppose we choose a local length standard given by a set of springs. The length of those springs, l_s, relative the the meter stick, changes from place to place and time to time. If the meter stick were our standard, then the length of the spring at spacetime position (t, x^i) is a function,

$$l_s\left(t, x^i\right) = l_{s0} e^{\varphi}$$

However, if we take the length of the springs as our length standard, it is the length of the meter stick which changes, $\kappa = \frac{\kappa_0}{l_s} = \kappa\left(t, x^i\right)$: it is the dimensionless ratio,

$$\frac{\kappa}{l_{s0}} = \frac{\kappa_0}{l_s} = e^{-\varphi}$$

which is the measured quantity. With the springs as standards, the Weyl vector is given by $W_a = \partial_a \varphi$. We still regard κ as constant because it satisfies the condition

$$\begin{aligned} D_a \kappa &= \partial_a \kappa + W_a \kappa \\ &= \partial_a \kappa + \partial_a \varphi \kappa \\ &= \partial_a \kappa - \kappa \partial_a \ln\left(\kappa / l_{s0}\right) \\ &= \partial_a \kappa - \frac{\partial_a \left(\kappa / l_{s0}\right)}{\left(\kappa / l_{s0}\right)} \kappa \\ &= 0 \end{aligned}$$

Notice that the existence of global constants of this type, that is, one or more quantities satisfying $\mathbf{D}\kappa = 0$, we must satisfy the integrability condition

$$\begin{aligned} 0 &= \mathbf{d}^2 \kappa \\ &= \mathbf{d}\omega\kappa - \omega\mathbf{d}\kappa \\ &= \mathbf{d}\omega\kappa - \omega\omega\kappa \\ &= \mathbf{d}\omega\kappa \end{aligned}$$

If the constant κ is not to vanish, then the dilatation must be zero, $\Omega = \mathbf{d}\omega = 0$. A weaker condition is possible: we can demand the existence of a congruence of curves along which κ remains constant. In simple applications, these curves turn out to be the classical paths of motion. See *Gauging Newton's law*.

Now, if we insert a constant of the appropriate conformal weight into the functional S_1, we have a suitable action,

$$S = \int \kappa \Omega^{ab} \mathbf{e}^c \mathbf{e}^d e_{abcd}$$

We require κ to have units of $(length)^{-2}$ in order for the integrand to be of weight zero. If we use the gravitational constant,

$$[G] = \frac{m^3}{kg \cdot sec^2}$$

together with the speed of light and Planck's constant,

$$\begin{aligned} \left[\frac{G\hbar}{c^3}\right] &= \frac{m^3}{kg \cdot sec^2} \cdot \frac{sec^3}{m^3} \cdot \frac{kg \cdot m^2}{sec} \\ &= m^2 \end{aligned}$$

then we may set

$$\kappa = \frac{c^3}{G\hbar}$$

Dirac, in his large numbers paper, takes this action a step further by adding a kinetic term for κ. By allowing solutions with slowly changing κ, he accounts for the relative sizes of certain fundamental quantities. This approach may be taken in any dimension, n, with

$$S = \int \kappa^{(n-2)/2} \Omega^{ab} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc \cdots d} + Kinetic\, term$$

Notice that the form of the action depends on the dimension of the space.

4 Auxiliary conformal gauging

The auxiliary conformal gauging has the structure equations

$$\mathbf{d}\omega_b^a = \omega_b^c \omega_c^a + 2\Delta_{cb}^{ad} \omega_d \omega^c + \Omega_b^a \quad (1)$$

$$\mathbf{d}\omega^a = \omega^b \omega_b^a + \omega \omega^a + \Omega^a \quad (2)$$

$$\mathbf{d}\omega_a = \omega_a^b \omega_b + \omega_a \omega + \Omega_a \quad (3)$$

$$\mathbf{d}\omega = \omega^a \omega_a + \Omega \quad (4)$$

where the horizontal directions are spanned by the gauge fields of translations, ω^a. The quotient is $\left\{M_b^a, P_a, K^a, D\right\} / \left\{M_b^a, K^a, D\right\}$, resulting in an n-dimensional manifold which we identify with spacetime. The solder forms, ω^a, span the base manifold, and all forms and curvatures are horizontal, for example,

$$\Omega_b^a = \frac{1}{2} \Omega_{bcd}^a \omega^c \omega^d$$

Each of the connection 1-forms is a function of n coordinates, x^α, and is a linear combination of $\mathbf{d}x^\alpha$,

$$\omega^A = \omega_\alpha^A \left(x^\beta\right) \mathbf{d}x^\alpha$$

4.1 Bianchi identities

In the next sub-Section, we show that it is consistent to set the torsion, Ω^a, to zero. We have found the Bianchi identities for any torsion-free conformal gauging to be

$$\begin{aligned} \mathbf{D}\Omega_b^a + 2\Delta_{db}^{ac} \Omega_c \omega^d &= 0 \\ \omega^c \Omega_c^a - \Omega \omega^a &= 0 \\ \mathbf{D}\Omega_a + \Omega_a^c \omega_c - \omega_a \Omega &= 0 \\ \mathbf{d}\Omega - \omega^a \Omega_a &= 0 \end{aligned}$$

Because the torsion vanishes, the second of these is algebraic:

$$0 = \delta_b^a \Omega_{cd} + \delta_c^a \Omega_{db} + \delta_d^a \Omega_{bc} - \Omega_{bcd}^a - \Omega_{cdb}^a - \Omega_{dbc}^a$$

The trace on ac,

$$\begin{aligned} 0 &= \Omega_{bd} + n\Omega_{db} + \Omega_{bd} - \Omega^{c}_{bcd} - \Omega^{c}_{dbc} \\ &= -(n-2)\,\Omega_{bd} - \Omega^{c}_{bcd} + \Omega^{c}_{dcb} \end{aligned}$$

so that

$$\Omega^{c}_{bcd} - \Omega^{c}_{dcb} = -(n-2)\,\Omega_{bd}$$

then shows that, just as in the Weyl case, there is an antisymmetric part to the trace of the curvature, which is proportional to the dilatation.

The first Bianchi expands to give

$$\begin{aligned} 0 &= \mathbf{D}\Omega^{a}_{b} + 2\Delta^{af}_{eb}\Omega_{f}\omega^{e} \\ &= \frac{1}{2}\left(\Omega^{a}_{bcd;e} + 2\Delta^{af}_{eb}\Omega_{fcd}\right)\omega^{c}\omega^{d}\omega^{e} \end{aligned}$$

Take the trace on ac,

$$\begin{aligned} 0 &= \Omega^{c}_{bcd;e} + \Omega^{c}_{bde;c} + \Omega^{c}_{bec;d} \\ &\quad + 2\Delta^{cf}_{eb}\Omega_{fcd} + 2\Delta^{cf}_{cb}\Omega_{fde} + 2\Delta^{cf}_{db}\Omega_{fec} \\ &= \Omega^{c}_{bcd;e} + \Omega^{c}_{bde;c} + \Omega^{c}_{bec;d} \\ &\quad + (n-3)\,\Omega_{bde} + \eta_{db}\eta^{cf}\Omega_{fce} - \eta_{eb}\eta^{cf}\Omega_{fcd} \end{aligned}$$

Take an additional trace, on bd,

$$\begin{aligned} 0 &= \Omega^{cd}{}_{cd;e} + \Omega^{cd}{}_{ec;d} - \Omega^{cd}{}_{ed;c} \\ &\quad + 2\,(n-2)\,\Omega^{d}_{de} \end{aligned}$$

so we have the trace of the co-torsion,

$$\begin{aligned} \Omega^{d}_{de} &= \frac{1}{2\,(n-2)}\left(\Omega^{cd}{}_{ed;c} - \Omega^{cd}{}_{ec;d} - \Omega^{cd}{}_{cd;e}\right) \\ &= \frac{1}{2\,(n-2)}\left(\Omega^{dc}{}_{de;c} + \Omega^{dc}{}_{de;c} - \Omega^{cd}{}_{cd;e}\right) \\ &= \frac{1}{2\,(n-2)}\left(2\Omega^{dc}{}_{de;c} - \Omega^{dc}{}_{dc;e}\right) \end{aligned}$$

Substitute back to find

$$\begin{aligned} 0 &= \Omega^{c}_{bcd;e} + \Omega^{c}_{bde;c} + \Omega^{c}_{bec;d} \\ &\quad + (n-3)\,\Omega_{bde} + \frac{1}{2\,(n-2)}\eta_{db}\left(2\Omega^{ac}{}_{ae;c} - \Omega^{ac}{}_{ac;e}\right) - \frac{1}{2\,(n-2)}\eta_{eb}\left(2\Omega^{ac}{}_{ad;c} - \Omega^{ac}{}_{ac;d}\right) \end{aligned}$$

and we may solve for the entire co-torsion,

$$\begin{aligned} \Omega_{bde} &= -\frac{1}{n-3}\left(\Omega^{c}_{bcd;e} + \Omega^{c}_{bde;c} - \Omega^{c}_{bce;d}\right) \\ &\quad - \frac{1}{2\,(n-2)\,(n-3)}\eta_{db}\left(2\Omega^{ac}{}_{ae;c} - \Omega^{ac}{}_{ac;e}\right) + \frac{1}{2\,(n-2)\,(n-3)}\eta_{eb}\left(2\Omega^{ac}{}_{ad;c} - \Omega^{ac}{}_{ac;d}\right) \end{aligned}$$

Skipping the third, the final identity is

$$\mathbf{d}\Omega - \omega^{a}\Omega_{a} = 0$$

Expanding,

$$\Omega_{[ab;c]} - \Omega_{[cab]} = 0$$

or

$$\Omega_{ab;c} + \Omega_{bc;a} + \Omega_{ca;b} - \Omega_{cab} - \Omega_{abc} - \Omega_{bca} = 0$$

When combined with the result from the first identity, this relates the Lorentz curvature and the derivative of the dilatation.

4.2 Inconsistency of the linear action

4.2.1 An invariant action

To write an action for the auxiliary conformal gauging, we first find the available tensors. Setting the translational gauge transformations to zero, $\Lambda_4^a = 0$, the gauge transformations of the connection forms become

$$\begin{aligned}
\tilde{\omega}_b^a &= \Lambda_c^a \omega_d^c \bar{\Lambda}_b^d + \Lambda_c^a \omega^c \bar{\Lambda}_b \\
&\quad + \eta^{ac} \Lambda_c \omega^e \eta_{ed} \bar{\Lambda}_b^d - \mathbf{d}\Lambda_c^a \bar{\Lambda}_b^c \\
\tilde{\omega}^a &= \Lambda_c^a \omega^c \bar{\Lambda} \\
\tilde{\omega}_a &= \Lambda_c \omega_d^c \bar{\Lambda}_a^d + \Lambda_c \omega^c \bar{\Lambda}_a \\
&\quad + \Lambda \omega_d \bar{\Lambda}_a^d + \Lambda \omega \bar{\Lambda}_a - \mathbf{d}\Lambda_c \bar{\Lambda}_a^c - \mathbf{d}\Lambda \bar{\Lambda}_a \\
\tilde{\omega} &= \omega + \Lambda_c \omega^c \bar{\Lambda} - \mathbf{d}\Lambda \bar{\Lambda}
\end{aligned}$$

so the solder form, ω^a, becomes a tensor. Notice that a special conformal transformation, Λ_a, can be chosen such that the Weyl vector vanishes, $\tilde{\omega} = 0$. The curvatures gauge in the same way except for the derivative terms,

$$\begin{aligned}
\tilde{\Omega}_b^a &= \Lambda_c^a \Omega_d^c \bar{\Lambda}_b^d + \Lambda_c^a \Omega^c \bar{\Lambda}_b \\
&\quad + \eta^{ac} \Lambda_c \Omega^e \eta_{ed} \bar{\Lambda}_b^d \\
\tilde{\Omega}^a &= \Lambda_c^a \Omega^c \bar{\Lambda} \\
\tilde{\Omega}_a &= \Lambda_c \Omega_d^c \bar{\Lambda}_a^d + \Lambda_c \Omega^c \bar{\Lambda}_a \\
&\quad + \Lambda \Omega_d \bar{\Lambda}_a^d + \Lambda \Omega \bar{\Lambda}_a \\
\tilde{\Omega} &= \Omega + \Lambda_c \Omega^c \bar{\Lambda}
\end{aligned}$$

Only one of the irreducible curvature components is a tensor – the Lorentz curvature, special conformal curvature and dilatational curvature mix with one another. This makes it difficult to write an invariant action. However, the situation is considerably improved if we set the torsion to zero,

$$\Omega^a = 0$$

This is a gauge invariant constraint because the torsion, at least, is a tensor. The constraint is also consistent with our expectation that ω^a is the spacetime solder form, so that to recover general relativity we will want vanishing torsion.

Dropping all torsion terms, the curvatures transform as

$$\begin{aligned}
\tilde{\Omega}_b^a &= \Lambda_c^a \Omega_d^c \bar{\Lambda}_b^d \\
\tilde{\Omega}^a &= \Lambda_c^a \Omega^c \bar{\Lambda} = 0 \\
\tilde{\Omega}_a &= \Lambda_c \Omega_d^c \bar{\Lambda}_a^d + \Lambda \Omega_d \bar{\Lambda}_a^d + \Lambda \Omega \bar{\Lambda}_a \\
\tilde{\Omega} &= \Omega
\end{aligned}$$

so now the Lorentz and dilatational curvatures transform as independent tensors, both of conformal weight zero.

For the action, the simplest choice is similar to that for Weyl geometry, by introducing a dimensionful "constant" or field,

$$S = \int \kappa^2 \left(\Omega^{ab} + \Lambda \mathbf{e}^a \mathbf{e}^b \right) \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} + \mathbf{D}\kappa^* \mathbf{D}\kappa + m^2 \kappa^* \kappa$$

Look at the kinetic term

$$\begin{aligned}
\mathbf{D}\kappa^* \mathbf{D}\kappa + m^2 \kappa^* \kappa &= \frac{1}{(n-1)!} D_a \kappa D_b \kappa g^{bc} e_{cd\cdots e} \mathbf{e}^a \mathbf{e}^d \cdots \mathbf{e}^e + \frac{1}{n!} m^2 \kappa^2 e_{cd\cdots e} \mathbf{e}^c \mathbf{e}^d \cdots \mathbf{e}^e \\
&= \frac{1}{(n-1)!} D_a \kappa D_b \kappa g^{bc} e_{cd\cdots e} e^{ad\cdots e} \Phi + \frac{1}{n!} m^2 \kappa^2 e_{cd\cdots e} e^{cd\cdots e} \Phi \\
&= -\left(g^{ab} D_a \kappa D_b \kappa + m^2 \kappa^2 \right) \Phi
\end{aligned}$$

Since the conformal weight of Φ is n, and both m^2 and g^{ab} have conformal weight -2, we require κ to have conformal weight $-\frac{n-2}{2}$, and the covariant derivative is given by

$$D_a\kappa = \partial_a\kappa - \left(\frac{n-2}{2}\right) W_a\kappa$$

Notice that we must also have

$$D_a m = \partial_a m - mW_a = 0$$

if the mass is to be considered constant.

The cosmological constant Λ, written as we have with a factor of κ^2, has conformal weight -2.

It quickly becomes clear that we must modify the linear action, because the varying the special conformal gauge field leads to a contradiction,

$$\begin{aligned} \delta S &= \delta \int \kappa^2 \Omega^{ab} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\ &= \int \kappa^2 \left(-2\eta^{bh}\Delta^{ae}_{fh}\delta\omega_e \mathbf{e}^f\right) \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \end{aligned}$$

Then

$$\begin{aligned} 0 &= 2\eta^{bh}\Delta^{ae}_{fh}\mathbf{e}^g\mathbf{e}^f\mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\ 0 &= 2\eta^{bh}\Delta^{ae}_{fh}\left(\delta^g_a\delta^f_b - \delta^g_b\delta^f_a\right) \\ &= \left(\delta^a_f\delta^e_h - \eta^{ae}\eta_{fh}\right)\left(\eta^{bh}\delta^g_a\delta^f_b - \eta^{bh}\delta^g_b\delta^f_a\right) \\ &= \eta^{ge} - n\eta^{ge} - n\eta^{ge} + \eta^{ge} \\ &= -2\,(n-1)\,\eta^{ge} \end{aligned}$$

and since we have $n > 2$ and $\kappa \neq 0$, this is incorrect. We must therefore modify the action by adding one or more additional terms.

We may try to fix this problem by adding additional terms. Since the torsion vanishes, the only possibilities at quadratic order are the square of the dilatation or the square of the Lorentz curvature.

The simplest term to add is one quadratic in the dilatation, $\Omega^*\Omega$. Then we have

$$S = \int \kappa^2 \left(\Omega^{ab} + \Lambda \mathbf{e}^a\mathbf{e}^b\right) \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} + \kappa^2\beta\Omega^*\Omega + \mathbf{D}\kappa^*\mathbf{D}\kappa + m^2\kappa^*\kappa$$

The conformal weight of β is given by expanding the dilatation term,

$$\beta\Omega^*\Omega = \beta\Omega_{ab}\Omega_{cd}g^{ce}g^{df}e_{efg\cdots h}\mathbf{e}^a\mathbf{e}^b\mathbf{e}^g\cdots\mathbf{e}^h$$

The solder forms give a weight of n, the inverse metrics -4, the factor κ^2 a weight of $-n+2$, and the components of the dilatation -4. Therefore, the conformal weight of β must be -6.

However, when we vary the special conformal gauge field, we again find a contradiction. We have

$$\begin{aligned} \delta S &= \int \kappa^2 \delta\Omega^{ab}\mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} + 2\kappa^2\beta\,\delta\Omega^*\Omega \\ &= \int \kappa^2\left(-2\eta^{be}\Delta^{ac}_{de}\delta\omega_c\mathbf{e}^d\right)\mathbf{e}^f\cdots\mathbf{e}^g e_{abf\cdots g} + 2\kappa^2\beta\left(-\mathbf{e}^c\delta\omega_c\right)^*\Omega \\ &= \int 2\delta\omega_c\kappa^2\left(-\eta^{be}\Delta^{ac}_{de}\mathbf{e}^d\mathbf{e}^f\cdots\mathbf{e}^g e_{abf\cdots g} + \beta\mathbf{e}^{c*}\Omega\right) \end{aligned}$$

and therefore,

$$\begin{aligned} 0 &= -\eta^{be}\Delta^{ac}_{de}\mathbf{e}^d\mathbf{e}^f\cdots\mathbf{e}^g e_{abf\cdots g} + \frac{1}{(n-2)!}\beta\mathbf{e}^c\Omega_{ab}\eta^{ad}\eta^{be}e_{def\cdots g}\mathbf{e}^f\cdots\mathbf{e}^g \\ 0 &= -\eta^{be}\Delta^{ac}_{de}\mathbf{e}^h\mathbf{e}^d\mathbf{e}^f\cdots\mathbf{e}^g e_{abf\cdots g} + \frac{1}{(n-2)!}\beta\Omega_{ab}\eta^{ad}\eta^{be}e_{def\cdots g}\mathbf{e}^h\mathbf{e}^c\mathbf{e}^f\cdots\mathbf{e}^g \end{aligned}$$

$$
\begin{aligned}
0 &= -\eta^{be}\Delta^{ac}_{de}e^{hdf\cdots g}e_{abf\cdots g} + \frac{1}{(n-2)!}\beta\Omega_{ab}\eta^{ad}\eta^{be}e_{def\cdots g}e^{hcf\cdots g} \\
0 &= (n-2)!\eta^{be}\Delta^{ac}_{de}\left(\delta^h_a\delta^d_b - \delta^d_a\delta^h_b\right) - \beta\Omega_{ab}\eta^{ad}\eta^{be}\left(\delta^h_d\delta^c_e - \delta^c_d\delta^h_e\right) \\
&= \frac{1}{2}(n-2)!\left(\eta^{ch} - n\eta^{ch} - n\eta^{hc} + \eta^{ch}\right) - \beta\left(\Omega^{hc} - \Omega^{ch}\right) \\
&= -(n-1)!\eta^{ch} - 2\beta\Omega^{hc}
\end{aligned}
$$

so that this generalization does not solve the problem: the antisymmetric part forces the dilatation to vanish, while the symmetric part is inconsistent.

Exercise: Find the remaining variational field equations from this action

Answer: See Appendix A Returning to the central problem, we consider a fully quadratic action.

4.3 Quadratic action theory

Consider the quadratic action:

$$S = \int \alpha\Omega^a_b{}^*\Omega^b_a + \beta\Omega^*\Omega$$

We find the resulting field equations.

4.3.1 Varying the Weyl vector

Varying the action with respect to the Weyl vector,

$$
\begin{aligned}
0 &= \delta_\omega S \\
&= \delta_\omega \int \alpha\Omega^a_b{}^*\Omega^b_a + \beta\Omega^*\Omega \\
&= \int 2\beta\delta_\omega\Omega^*\Omega
\end{aligned}
$$

since the Lorentz curvature is independent of ω. The structure equation for dilatations now gives

$$
\begin{aligned}
\Omega &= \mathbf{d}\omega - \omega^a\omega_a \\
\delta_\omega\Omega &= \mathbf{d}(\delta\omega)
\end{aligned}
$$

Therefore, integrating by parts,

$$
\begin{aligned}
0 &= \int_V 2\beta\mathbf{d}(\delta\omega)\,^*\Omega \\
&= \int_V 2\beta\left(\mathbf{d}((\delta\omega)\,^*\Omega) + \delta\omega\,\mathbf{d}^*\Omega\right) \\
&= 2\beta\int_{\delta V}(\delta\omega)\,^*\Omega + \int_V 2\beta\delta\omega\,\mathbf{d}^*\Omega \\
&= \int_V 2\beta\delta\omega\,\mathbf{d}^*\Omega
\end{aligned}
$$

since the variation vanishes on the boundary, δV. Since the variation is arbitrary, the field equation is

$$\beta\mathbf{d}^*\Omega = 0$$

4.3.2 Varying the spin connection

Next, varying the spin connection, we have a similar calculation,

$$\begin{aligned} 0 &= \delta_{\omega^a_b} S \\ &= \delta_{\omega^a_b} \int \alpha \Omega^b_{\ a} {}^*\Omega^a_{\ b} + \beta \Omega^* \Omega \\ &= 2\alpha \int \delta_{\omega^a_b} \Omega^b_{\ a} {}^*\Omega^a_{\ b} \end{aligned}$$

since the dilatation is independent of ω^a_b. From the Lorentz structure equation

$$\begin{aligned} \Omega^a_{\ b} &= \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b}\omega^a_{\ c} - 2\Delta^{ac}_{db}\omega_c\omega^d \\ \delta_{\omega_c}\Omega^b_{\ a} &= \mathbf{d}\delta\omega^b_{\ a} - \delta\omega^c_{\ a}\omega^b_{\ c} - \omega^c_{\ a}\delta\omega^b_{\ c} \\ &= \mathbf{D}\delta\omega^b_{\ a} \end{aligned}$$

where $\mathbf{D}$ is the covariant exterior derivative. Substituting and integrating by parts,

$$\begin{aligned} 0 &= 2\alpha \int \delta_{\omega^b_a} \Omega^b_{\ a} {}^*\Omega^a_{\ b} \\ &= \int_V 2\beta \mathbf{D}\delta\omega^b_{\ a} {}^*\Omega^a_{\ b} \\ &= \int_V 2\beta \mathbf{D}\left(\delta\omega^b_{\ a} {}^*\Omega^a_{\ b}\right) + 2\beta\delta\omega^b_{\ a}\mathbf{D}^*\Omega^a_{\ b} \\ &= \int_V 2\beta \mathbf{d}\left(\delta\omega^b_{\ a} {}^*\Omega^a_{\ b}\right) + 2\beta\delta\omega^b_{\ a}\mathbf{D}^*\Omega^a_{\ b} \\ &= \int_{\delta V} 2\beta\delta\omega^b_{\ a} {}^*\Omega^a_{\ b} + 2\beta\int_V \delta\omega^b_{\ a}\mathbf{D}^*\Omega^a_{\ b} \\ &= 2\beta\int_V \delta\omega^b_{\ a}\mathbf{D}^*\Omega^a_{\ b} \end{aligned}$$

Notice that $\mathbf{D}\left(\delta\omega^b_{\ a} {}^*\Omega^a_{\ b}\right) = \mathbf{d}\left(\delta\omega^b_{\ a} {}^*\Omega^a_{\ b}\right)$ since the expression in parentheses is a scalar. The variation vanishes on the boundary and is arbitrary inside the volume of interest, V, so the field equation is

$$\alpha\mathbf{D}^*\Omega^a_{\ b} = 0$$

4.3.3 Varying the special conformal transformations

The variation with respect to the special conformal gauge vector is simpler. We have

$$\begin{aligned} 0 &= \delta_{\omega_c} S \\ &= \delta_{\omega_c} \int \alpha \Omega^b_{\ a} {}^*\Omega^a_{\ b} + \beta \Omega^* \Omega \\ &= 2\int \alpha\delta_{\omega_c} \Omega^b_{\ a} {}^*\Omega^a_{\ b} + \beta\delta_{\omega_c}\Omega^*\Omega \end{aligned}$$

From the expressions for the curvature, the variations are:

$$\begin{aligned} \Omega^a_{\ b} &= \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b}\omega^a_{\ c} - 2\Delta^{ac}_{db}\omega_c\omega^d \\ \delta_{\omega_c}\Omega^b_{\ a} &= -2\Delta^{bc}_{da}\delta\omega_c\omega^d \\ \Omega &= \mathbf{d}\omega - \omega^a\omega_a \\ \delta_{\omega_c}\Omega &= -\omega^a\delta\omega_a \end{aligned}$$

so substituting,

$$\begin{aligned} 0 &= 2\int \alpha\delta_{\omega_c}\Omega^b_{\ a} {}^*\Omega^a_{\ b} + \beta\delta_{\omega_c}\Omega^*\Omega \\ &= 2\int \alpha\left(-2\Delta^{bc}_{da}\delta\omega_c\omega^d\right) {}^*\Omega^a_{\ b} + \beta\left(-\omega^a\delta\omega_a\right) {}^*\Omega \\ &= -2\int \delta\omega_c\left(2\alpha\Delta^{bc}_{da}\omega^d {}^*\Omega^a_{\ b} - \beta\omega^c {}^*\Omega\right) \end{aligned}$$

Using the antisymmetry of the Lorentz curvature, the final field equation is

$$2\alpha\omega^{d\,*}\Omega^{c}_{\ d}-\beta\omega^{c\,*}\Omega=0$$

4.3.4 Varying the solder form

The final variation, with respect to the solder form, ω^a, is complicated by the dependence of the volume element on the solder form. To make these terms explicit, we expand the dual form in the action,

$$\begin{aligned} S &= \int\alpha\Omega^{b}_{\ a}{}^{*}\Omega^{a}_{\ b}+\beta\Omega^{*}\Omega \\ &= \frac{1}{8}\int\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu}\end{aligned}$$

Now, variation of the solder form leads to

$$\begin{aligned} 0 &= \delta_{\omega^k}S \\ &= \frac{1}{4}\int\left(\alpha\delta_{\omega^k}\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\delta_{\omega^k}\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu} \\ &\quad+\frac{1}{8}\int\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)\left(2\delta_{\omega^k}g^{\lambda\rho}g^{\tau\sigma}-\frac{1}{2}g^{\lambda\rho}g^{\tau\sigma}g_{\theta\varphi}\delta_{\omega^k}g^{\theta\varphi}\right)\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu}\end{aligned}$$

Next, we need the curvature and metric variations:

$$\begin{aligned} \frac{1}{2}\delta\Omega^{a}_{\ b\alpha\beta}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} &= \delta\left(\omega^{a}_{\ b\beta,\alpha}-\omega^{c}_{\ b\alpha}\omega^{a}_{\ c\beta}-2\Delta^{ac}_{db}\omega_{c\alpha}\omega^{d}_{\beta}\right)\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} \\ &= -2\Delta^{ac}_{db}\omega_{c\alpha}\delta\omega^{d}_{\beta}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} \\ \delta\Omega^{a}_{\ b\alpha\beta} &= 2\Delta^{ac}_{db}\omega_{c\beta}\delta\omega^{d}_{\alpha}-2\Delta^{ac}_{db}\omega_{c\alpha}\delta\omega^{d}_{\beta} \\ \frac{1}{2}\delta\Omega_{\alpha\beta}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} &= \delta\left(\omega_{\beta,\alpha}-\omega^{a}_{\alpha}\omega_{a\beta}\right)\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} \\ \frac{1}{2}\delta\Omega_{\alpha\beta}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} &= -\delta\omega^{a}_{\alpha}\omega_{a\beta}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta} \\ \delta\Omega_{\alpha\beta} &= \delta\omega^{a}_{\beta}\omega_{a\alpha}-\delta\omega^{a}_{\alpha}\omega_{a\beta}\end{aligned}$$

and (this will be clearer if we denote the solder form by $\omega^{a}_{\beta}=e^{a}_{\beta}$ from here on):

$$\begin{aligned} \delta g^{\alpha\beta} &= \delta\left(g^{\alpha\rho}g^{\beta\sigma}g_{\rho\sigma}\right) \\ &= 2\delta g^{\alpha\rho}g^{\beta\sigma}g_{\rho\sigma}+g^{\alpha\rho}g^{\beta\sigma}\delta g_{\rho\sigma} \\ &= 2\delta g^{\alpha\beta}+g^{\alpha\rho}g^{\beta\sigma}\delta g_{\rho\sigma} \\ \delta g^{\alpha\beta} &= -g^{\alpha\rho}g^{\beta\sigma}\delta g_{\rho\sigma} \\ &= -g^{\alpha\rho}g^{\beta\sigma}\delta\left(\eta_{ab}e^{a}_{\rho}e^{b}_{\sigma}\right) \\ &= -g^{\alpha\rho}g^{\beta\sigma}\eta_{ab}\left(\delta e^{a}_{\rho}e^{b}_{\sigma}+e^{a}_{\rho}\delta e^{b}_{\sigma}\right) \\ &= -g^{\alpha\rho}\left(\eta^{cd}e^{\beta}_{c}e^{\sigma}_{d}\right)\eta_{ab}\delta e^{a}_{\rho}e^{b}_{\sigma}-\left(\eta^{cd}e^{\alpha}_{c}e^{\rho}_{d}\right)g^{\beta\sigma}\eta_{ab}e^{a}_{\rho}\delta e^{b}_{\sigma} \\ &= -g^{\alpha\rho}\left(\eta^{cd}e^{\beta}_{c}\right)\eta_{ad}\delta e^{a}_{\rho}-e^{\alpha}_{c}g^{\beta\sigma}\delta e^{c}_{\sigma} \\ &= -g^{\alpha\rho}e^{\beta}_{a}\delta e^{a}_{\rho}-e^{\alpha}_{c}g^{\beta\sigma}\delta e^{c}_{\sigma} \\ &= -\left(g^{\alpha\sigma}e^{\beta}_{c}+e^{\alpha}_{c}g^{\beta\sigma}\right)\delta e^{c}_{\sigma}\end{aligned}$$

The variation becomes

$$0 = \frac{1}{4}\int\left(\alpha\delta_{\omega^k}\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\delta_{\omega^k}\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu}$$

$$
\begin{aligned}
& +\frac{1}{8}\int\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)\left(2\delta_{\omega^k}g^{\lambda\rho}g^{\tau\sigma}-\frac{1}{2}g^{\lambda\rho}g^{\tau\sigma}g_{\theta\varphi}\delta_{\omega^k}g^{\theta\varphi}\right)\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu} \\
= & \frac{1}{4}\int\left(\alpha\left(4\Delta^{bc}_{da}\omega_{c\beta}\delta\omega_{\alpha}^{\ d}\right)\Omega^{a}_{\ b\lambda\tau}+2\beta\delta\omega_{\beta}^{\ a}\omega_{a\alpha}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu} \\
& -\frac{1}{8}\int\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)\left(2\delta^{\lambda}_{\theta}\delta^{\rho}_{\varphi}g^{\tau\sigma}-\frac{1}{2}g^{\lambda\rho}g^{\tau\sigma}g_{\theta\varphi}\right)\left(\left(g^{\xi\theta}e_{c}^{\ \varphi}+e_{c}^{\ \theta}g^{\varphi\xi}\right)\delta e_{\xi}^{c}\right)\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu} \\
= & \frac{1}{4}\int\left(\alpha\left(4\Delta^{bd}_{ca}\omega_{d\beta}\delta^{\xi}_{\alpha}\right)\Omega^{a}_{\ b\lambda\tau}+2\beta\delta^{\xi}_{\beta}\omega_{c\alpha}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu}\delta\omega_{\xi}^{c} \\
& -\frac{1}{8}\int\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)\left(2\delta^{\lambda}_{\theta}\delta^{\rho}_{\varphi}g^{\tau\sigma}-\frac{1}{2}g^{\lambda\rho}g^{\tau\sigma}g_{\theta\varphi}\right)\left(g^{\xi\theta}e_{c}^{\ \varphi}+e_{c}^{\ \theta}g^{\varphi\xi}\right)\sqrt{-g}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu}\delta e_{\xi}^{c}
\end{aligned}
$$

Then the field equation is

$$
\begin{aligned}
0 = & \frac{1}{4}\left(\alpha\left(4\Delta^{bd}_{ca}\omega_{d\beta}\delta^{\xi}_{\alpha}\right)\Omega^{a}_{\ b\lambda\tau}+2\beta\delta^{\xi}_{\beta}\omega_{c\alpha}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu} \\
& -\frac{1}{8}\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)\left(2\delta^{\lambda}_{\theta}\delta^{\rho}_{\varphi}g^{\tau\sigma}-\frac{1}{2}g^{\lambda\rho}g^{\tau\sigma}g_{\theta\varphi}\right)\left(g^{\xi\theta}e_{c}^{\ \varphi}+e_{c}^{\ \theta}g^{\varphi\xi}\right)\varepsilon_{\rho\sigma\mu\nu}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu}
\end{aligned}
$$

Taking the dual,

$$
\begin{aligned}
0 = & \frac{1}{4}\left(\alpha\left(4\Delta^{bd}_{ca}\omega_{d\beta}\delta^{\xi}_{\alpha}\right)\Omega^{a}_{\ b\lambda\tau}+2\beta\delta^{\xi}_{\beta}\omega_{c\alpha}\Omega_{\lambda\tau}\right)g^{\lambda\rho}g^{\tau\sigma}\delta^{\alpha\beta}_{\rho\sigma} \\
& -\frac{1}{8}\left(\alpha\Omega^{b}_{\ a\alpha\beta}\Omega^{a}_{\ b\lambda\tau}+\beta\Omega_{\alpha\beta}\Omega_{\lambda\tau}\right)\left(2\delta^{\lambda}_{\theta}\delta^{\rho}_{\varphi}g^{\tau\sigma}-\frac{1}{2}g^{\lambda\rho}g^{\tau\sigma}g_{\theta\varphi}\right)\left(g^{\xi\theta}e_{c}^{\ \varphi}+e_{c}^{\ \theta}g^{\varphi\xi}\right)\delta^{\alpha\beta}_{\rho\sigma} \\
0 = & 4\alpha\Delta^{bd}_{ca}\omega_{d\beta}\Omega^{a\ \ \xi\beta}_{\ b}+2\beta\omega_{c\alpha}\Omega^{\alpha\xi} \\
& -2\alpha\left(\Omega^{b}_{\ a\rho\sigma}\Omega^{b\ \ \xi\sigma}_{\ a}e_{c}^{\rho}-\frac{1}{4}\Omega^{b}_{\ a\rho\sigma}\Omega^{a\ \ \rho\sigma}_{\ b}e_{c}^{\xi}\right) \\
& -2\beta\left(\Omega_{\rho\sigma}\Omega^{\xi\sigma}e_{c}^{\rho}-\frac{1}{4}\Omega_{\rho\sigma}\Omega^{\rho\sigma}e_{c}^{\xi}\right)
\end{aligned}
$$

Finally, we rewrite the final equation in the orthonormal basis by multiplying by the solder form, e_{ξ}^{e} and replacing coordinate (Greek) contractions with orthonormal (Latin) contractions:

$$
\begin{aligned}
0 = & 2\alpha\Delta^{bd}_{ca}\omega_{df}\Omega^{a\ \ ef}_{\ b}+\beta\omega_{ca}\Omega^{ae} \\
& -\alpha\left(\Omega^{b}_{\ acf}\Omega^{b\ \ ef}_{\ a}-\frac{1}{4}\Omega^{b}_{\ afg}\Omega^{a\ \ fg}_{\ b}\delta^{e}_{c}\right) \\
& -\beta\left(\Omega_{cd}\Omega^{ed}-\frac{1}{4}\Omega_{ab}\Omega^{ab}\delta^{e}_{c}\right)
\end{aligned}
$$

Now lower the e index, and use $\Delta^{bd}_{ca}\Omega^{a\ \ f}_{\ be}=\Omega^{d\ \ f}_{\ ce}$,

$$
\begin{aligned}
0 = & 2\alpha\omega_{df}\Omega^{d\ \ f}_{\ ce}+\beta\omega_{ca}\Omega^{a}_{e} \\
& -\alpha\left(\Omega^{b}_{\ acf}\Omega^{b\ \ f}_{\ ae}-\frac{1}{4}\eta_{ce}\Omega^{b}_{\ afg}\Omega^{a\ \ fg}_{\ b}\right) \\
& -\beta\left(\Omega_{cd}\Omega_{e}^{\ d}-\frac{1}{4}\eta_{ce}\Omega_{ab}\Omega^{ab}\right)
\end{aligned}
$$

4.3.5 Summary of the field equations

Collecting the results, the conformal field equations are:

$$
\begin{aligned}
\alpha\mathbf{D}^{*}\Omega^{a}_{b} &= 0 \\
\beta\mathbf{d}^{*}\Omega &= 0 \\
2\alpha\omega^{d\,*}\Omega^{c}_{d}-\beta\omega^{c\,*}\Omega &= 0 \\
2\alpha\omega_{df}\Omega^{d\ \ f}_{\ ce}+\beta\omega_{ca}\Omega^{a}_{e} &= \alpha\Theta_{ce}+\beta T_{ce}
\end{aligned}
$$

where

$$\begin{aligned}
\Theta_{ab} &= \Omega^{d}{}_{cae}\Omega^{c}{}_{db}{}^{e} - \frac{1}{4}\Omega^{d}{}_{cef}\Omega^{c}{}_{d}{}^{ef}\eta_{ab} \\
T_{ab} &= \Omega_{ac}\Omega_{b}{}^{c} - \frac{1}{4}\eta_{ab}\Omega_{cd}\Omega^{cd}
\end{aligned}$$

This time, all four equations form a sensible set. In particular, the equation from the variation of the special conformal gauge field may now be consistently solved to eliminate that field.

The two fields, Θ_{ab} and T_{ab} have the form we typically expect for the energy-momentum tensor of a Yang-Mills field. Here, however, they depend on the curvature and dilatation, and may not have such a straightforward physical meaning.

4.4 The auxiliary field

Consider the field equation from the variation of the special conformal variation. Because the co-torsion, Ω_a, does not appear in the action, this equation is algebraic in the curvatures. This means that we can use algebraic techniques to solve for the curvatures, rather than having to integrate.

We begin by choosing a special conformal gauge. Recall that the gauge transformation for the Weyl vector is given by

$$\tilde{\omega} = \omega + \Lambda_c \omega^c \bar{\Lambda} - \mathbf{d}\Lambda\bar{\Lambda}$$

and we may choose the arbitrary gauge, Λ_c, so that $\omega + \Lambda_c \omega^c \bar{\Lambda} = 0$. Then the Weyl vector is pure (conformal) gauge,

$$\begin{aligned}
\tilde{\omega} &= -\mathbf{d}\Lambda\bar{\Lambda} \\
&= -\mathbf{d}\varphi
\end{aligned}$$

For the remainder of the calculation we choose this gauge, while leaving the conformal factor undetermined. In this gauge, let $\omega_a = W_{ab}\omega^b$.

Now consider the special conformal field equation. To write this in components first expand the duals:

$$\begin{aligned}
0 &= 2\alpha\omega^{d}\,{}^{*}\Omega^{c}_{d} - \beta\omega^{c}\,{}^{*}\Omega \\
&= \frac{1}{(n-2)!}\left(2\alpha\Omega^{c}{}_{d}{}^{ef}e_{efg\ldots h} - \beta\delta^{c}_{d}\Omega^{ef}e_{efg\ldots h}\right)\omega^{d}\omega^{g}\ldots\omega^{h}
\end{aligned}$$

Now wedge this with one additional solder form and take the dual:

$$\begin{aligned}
0 &= \frac{1}{(n-2)!}\left(2\alpha\Omega^{c}{}_{d}{}^{ef}e_{efgh} - \beta\delta^{c}_{d}\Omega^{ef}e_{efg\ldots h}\right)\omega^{b}\omega^{d}\omega^{g}\ldots\omega^{h} \\
0 &= \frac{1}{(n-2)!}\left(2\alpha\Omega^{c}{}_{d}{}^{ef} - \beta\delta^{c}_{d}\Omega^{ef}\right)e_{efg\ldots h}e^{bdg\ldots h} \\
&= \left(2\alpha\Omega^{c}{}_{d}{}^{ef} - \beta\delta^{c}_{d}\Omega^{ef}\right)\delta^{bd}_{ef} \\
&= 2\alpha\Omega^{c}{}_{d}{}^{bd} - \beta\Omega^{bc}
\end{aligned}$$

Lowering the bc indices and rearranging gives,

$$2\alpha\Omega^{a}{}_{bac} = -\beta\Omega_{bc}$$

However, the Bianchi identity for the solder form for torsion-free conformal geometry also relates these two curvatures,

$$\omega^{c}\Omega^{a}_{c} - \Omega\omega^{a} = 0$$

which, as we have shown, gives

$$\Omega^{a}_{bad} - \Omega^{a}_{dab} = -(n-2)\,\Omega_{bd}$$

Now compare the antisymmetric part of the field equation to the Bianchi identity:

$$\begin{aligned}\alpha\left(\Omega^{c}{}_{acb}-\Omega^{c}{}_{bca}\right) &= -\beta\Omega_{ab}\\ \Omega^{c}_{acb}-\Omega^{c}_{bca} &= -\left(n-2\right)\Omega_{ab}\end{aligned}$$

The difference between the first, and α times the second gives

$$0 = \left(\beta-\left(n-2\right)\alpha\right)\Omega_{ab}$$

so that generically (i.e., unless $\beta=\left(n-2\right)\alpha$), the dilatational curvature must vanish. Rather than tracking two cases, we set $\beta=\left(n-2\right)\alpha$ in the field equation. Then, whether the dilatational curvature vanishes or not, we must have

$$\Omega^{c}{}_{acb}=-\frac{1}{2}\left(n-2\right)\Omega_{ab}$$

and we simply remember that except for a special choice of α,β in the original action, the dilatation vanishes. We will see below that the dilatation vanishes in the special case as well.

Now define

$$\begin{aligned}\mathbf{R}^{a}_{b} &= \mathbf{d}\omega^{a}_{b}-\omega^{c}_{b}\omega^{a}_{c}\\ \omega_{a} &= W_{ab}\omega^{b}\end{aligned}$$

where ω^{c}_{b} is the Weyl spin connection, satisfying

$$\mathbf{d}\omega^{a}=\omega^{b}\omega^{a}_{b}+\omega\omega^{a}$$

We may now express the special conformal gauge field in terms of the trace, R^{c}_{bcd}, of $\mathbf{R}^{a}_{b}$. Notice that R^{a}_{bcd} is the curvature of a Weyl connection, so that its trace satisfies

$$R^{c}_{bcd}-R^{c}_{dcb}=-\left(n-2\right)\Omega_{bd}$$

Let R_{ab} be the symmetric part of the trace of R^{a}_{bcd}, so that

$$R^{c}_{bcd}+R^{c}_{dcb}=2R_{bd}$$

Then adding,

$$R^{c}_{bcd} = R_{bd}-\frac{1}{2}\left(n-2\right)\Omega_{bd}$$

Substituting these definitions into the structure equation for the Lorentz curvature:

$$\begin{aligned}\Omega^{a}_{b} &= \mathbf{R}^{a}_{b}-2\Delta^{ad}_{cb}\omega_{d}\omega^{c}\\ \Omega^{a}_{bcd} &= R^{a}_{bcd}+2\Delta^{ae}_{cb}W_{ed}-2\Delta^{ae}_{db}W_{ec}\\ &= R^{a}_{bcd}+\delta^{a}_{c}W_{bd}-\eta^{ae}\eta_{cb}W_{ed}-\delta^{a}_{d}W_{bc}+\eta_{db}\eta^{ae}W_{ec}\end{aligned}$$

we impose the field equation:

$$\begin{aligned}-\frac{1}{2}\left(n-2\right)\Omega_{bd} &= \Omega^{c}_{bcd}\\ &= R_{bd}-\frac{1}{2}\left(n-2\right)\Omega_{bd}+nW_{bd}-W_{bd}-W_{bd}+\eta_{bd}\eta^{ec}W_{ec}\\ &= R_{bd}-\frac{1}{2}\left(n-2\right)\Omega_{bd}+\left(n-2\right)W_{bd}+\eta_{db}\left(\eta^{ce}W_{ce}\right)\end{aligned}$$

so that

$$0=R_{bd}+\left(n-2\right)W_{bd}+\eta_{db}\left(\eta^{ce}W_{ce}\right)$$

Take one further contraction, and solve for the trace of W_{ab},

$$\begin{aligned} 0 &= R + (n-2)\,\eta^{ab} W_{ab} + n\eta^{ce} W_{ce} \\ \eta^{ab} W_{ab} &= -\frac{1}{2(n-1)} R \end{aligned}$$

substituting, we have

$$0 = R_{ab} + (n-2)\,W_{ab} - \frac{1}{2(n-1)} R\eta_{ab}$$

Solving for W_{ab},

$$\begin{aligned} W_{ab} &= -\frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R\eta_{ab}\right) \\ &= \mathscr{R}_{ab} \end{aligned}$$

so that the special conformal gauge field is given by the Eisenhart tensor of the Weyl connection. This solves completely for the special conformal field, W_{ab}, in terms of the spin connection and its derivatives. We may therefore remove the special conformal gauge field from the rest of the problem.

4.5 The remaining field equations

Summarizing the results of the preceeding section, we have the remaining field equations, where the Weyl 1-form and auxiliary field satisfy

$$\begin{aligned} \mathbf{d}\omega &= 0 \\ \omega_a &= \mathscr{R}_{ab}\omega^b = \mathscr{R}_{ab} \\ \mathscr{R}_{ab} &= -\frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R\eta_{ab}\right) \\ \mathbf{R}^a_b &= \mathbf{d}\omega^a_b - \omega^c_b \omega^a_c \end{aligned}$$

We also have the structure equations, which may now be simplified by eliminating the special conformal gauge field.

Starting with the dilatation equation, we have

$$\begin{aligned} \Omega &= \mathbf{d}\omega - \omega^a \omega_a \\ &= \mathbf{d}^2\varphi - \omega^a \omega^b \mathscr{R}_{ab} \\ &= 0 \end{aligned}$$

since the Eisenhart tensor is symmetric. This shows that, regardless of the value of the constants α and β in the original action, the dilatation vanishes.

The structure equation for the special conformal transformations simply shows that the "new" curvature, Ω_a, is determined from the Riemann curvature as the covariant exterior derivative of the Eisenhart tensor,

$$\begin{aligned} \Omega_a &= \mathbf{d}\omega_a - \omega^c_a \omega_c - \omega_a \omega \\ &= \mathbf{D}\mathscr{R}_a \end{aligned}$$

Finally, the Lorentz equation simplifies,

$$\begin{aligned} \Omega^a_b &= \mathbf{d}\omega^a_b - \omega^c_b \omega^a_c - 2\Delta^{ad}_{cb}\omega_d \omega^c \\ &= \mathbf{R}^a_b - 2\Delta^{ad}_{cb}\mathscr{R}_d \omega^c \\ &= \mathbf{C}^a_b + 2\Delta^{ad}_{cb}\left(\mathscr{R}_d + \frac{1}{2}\Omega_{de}\omega^e\right)\omega^c - 2\Delta^{ad}_{cb}\mathscr{R}_d \omega^c \\ &= \mathbf{C}^a_b + \frac{1}{2} 2\Delta^{ad}_{cb}\Omega_{de}\omega^e \omega^c \\ &= \mathbf{C}^a_b \end{aligned}$$

Collecting these results we have the following:

Connection The connection is that of a trivial Weyl geometry

$$\begin{aligned} \mathbf{d}\omega^a &= \omega^b \omega_b^a + \omega \omega^a \\ \mathbf{d}\omega &= 0 \end{aligned}$$

Field equations The field equations, including results from the algebraic equations, are

$$\begin{aligned} C^a{}_{bcd;a} &= 0 \\ 2\mathscr{R}^{ab} C_{acbd} &= \Theta_{cd} \end{aligned}$$

where C^a_{bcd} is the Weyl curvature tensor of $\mathbf{R}^a_b = \mathbf{d}\omega^a_b - \omega^c_b \omega^a_c$.

Bianchi identities We have the Bianchi identities:

$$\begin{aligned} C^a_{[bcd]} &= 0 \\ 0 &= C^a_{bde;f} + C^a_{bef;d} + C^a_{bef;d} \\ &\quad + 2\Delta^{ac}_{eb}\left(\mathscr{R}_{cd;f} - \mathscr{R}_{cf;d}\right) + 2\Delta^{ac}_{fb}\left(\mathscr{R}_{ce;d} - \mathscr{R}_{cd;e}\right) + 2\Delta^{ac}_{db}\left(\mathscr{R}_{cf;e} - \mathscr{R}_{ce;f}\right) \\ \mathbf{DD}\mathscr{R}_a &= 0 \end{aligned}$$

with the consequent trace relations,

$$C^a_{bde;a} = (n-3)\left(\mathscr{R}_{bd;e} - \mathscr{R}_{be;d}\right)$$

4.5.1 A little digression

It is possible that the third Bianchi identity is related to the remaining algebraic field equation. This third relation, which follows from the special conformal transformations, expands to give

$$\begin{aligned} 0 &= \mathbf{DD}\mathscr{R}_a \\ &= \mathbf{D}\left(\mathbf{d}\mathscr{R}_a - \omega^b_a \mathscr{R}_b - \mathscr{R}_a \omega\right) \\ &= \mathbf{d}\left(\mathbf{d}\mathscr{R}_a - \omega^b_a \mathscr{R}_b - \mathscr{R}_a \omega\right) - \omega^c_a\left(\mathbf{d}\mathscr{R}_c - \omega^b_c \mathscr{R}_b - \mathscr{R}_c \omega\right) \\ &\quad + \left(\mathbf{d}\mathscr{R}_a - \omega^b_a \mathscr{R}_b - \mathscr{R}_a \omega\right)\omega \\ &= -\mathbf{d}\omega^b_a \mathscr{R}_b + \omega^b_a \mathbf{d}\mathscr{R}_b - \mathbf{d}\mathscr{R}_a \omega - \mathscr{R}_a \mathbf{d}\omega - \omega^c_a \mathbf{d}\mathscr{R}_c + \omega^c_a \omega^b_c \mathscr{R}_b + \omega^c_a \mathscr{R}_c \omega \\ &\quad + \mathbf{d}\mathscr{R}_a \omega - \omega^b_a \mathscr{R}_b \omega - \mathscr{R}_a \omega\omega \\ &= -\left(\mathbf{d}\omega^b_a - \omega^c_a \omega^b_c\right)\mathscr{R}_b \\ &= -\mathbf{R}^b_a \mathscr{R}_b \end{aligned}$$

Expanding the curvature in terms of the Weyl curvature and the Eisenhart tensor, this becomes,

$$\begin{aligned} 0 &= \mathbf{R}^b_a \mathscr{R}_b \\ &= \mathbf{C}^b_a \mathscr{R}_b + 2\Delta^{bc}_{da} \mathscr{R}_c \omega^d \mathscr{R}_b \\ &= \mathbf{C}^b_a \mathscr{R}_b + \mathscr{R}_a \omega^d \mathscr{R}_d - \eta_{da}\eta^{bc} \mathscr{R}_c \omega^d \mathscr{R}_b \\ &= \mathbf{C}^b_a \mathscr{R}_b \end{aligned}$$

In components

$$0 = C_{abcd}\mathscr{R}^a_e + C_{abde}\mathscr{R}^a_c + C_{abec}\mathscr{R}^a_d$$

The traces vanish identically, so this identity is distinct from the condition required by the remaining algebraic field equation:

$$\mathscr{R}^{ac} C_{abcd} = \frac{1}{2} T_{bd}$$

4.5.2 Combine the Lorentz field equation and the Bianchi identity

The structure equations are those of a trivial Weyl geometry, while the curvature field equation

$$C^{a}{}_{bcd;a} = (n-3)\left(\mathscr{R}_{bc;d} - \mathscr{R}_{bd;c}\right) = 0$$

provides the integrability condition for the Ricci tensor to be built purely from a conformal factor. Therefore, solutions to these equations include the conformal equivalence class of Ricci flat spacetimes. However, we have two constraints on the allowed curvatures:

$$\begin{aligned} \mathbf{C}_{a}^{b}\mathscr{R}_{b} &= 0 \\ \mathscr{R}^{ac}C_{abcd} &= \frac{1}{2}T_{bd} \end{aligned}$$

While they are built purely from the conformal factor and its derivatives, the Ricci tensor and the Eisenhart tensor are, in general, nonvanishing. Therefore, these remaining constraints are nontrivial: the class of spacetimes satisfying the restrictions is a probably a proper subset of the class of Ricci flat spacetimes.

4.6 Structures and new features

There are no new features in the auxiliary conformal gauging. The Weyl vector is pure gauge, and the special conformal gauge field, ω_a, has been eliminated – hence the name *auxiliary*. The Killing metric of the conformal group, restricted to the base manifold, vanishes entirely, and we must introduce the original metric of the representation space, η_{ab}, by hand.

Numerous other treatments of this gauging, with a variety of choices for the action, are found in the literature. A review of these is given in the accompanying papers (Wheeler; Wehner and Wheeler).

5 Biconformal gauging

We now consider same condition in the biconformal gauging.

$$\begin{aligned} \mathbf{d}\omega_{b}^{a} &= \omega_{b}^{c}\omega_{c}^{a} + 2\Delta_{cb}^{ad}\omega_{d}\omega^{c} + \Omega_{b}^{a} \\ \mathbf{d}\omega^{a} &= \omega^{b}\omega_{b}^{a} + \omega\omega^{a} + \Omega^{a} \\ \mathbf{d}\omega_{a} &= \omega_{a}^{b}\omega_{b} + \omega_{a}\omega + \Omega_{a} \\ \mathbf{d}\omega &= \omega^{a}\omega_{a} + \Omega \end{aligned}$$

We take the quotient $\left\{M_{b}^{a}, P_{a}, K^{a}, D\right\} / \left\{M_{b}^{a}, D\right\}$, resulting in a $2n$-dimensional manifold. We expect n of these dimensions to correspond to spacetime, with the remaining n to remaining to be interpreted. The horizontal directions are spanned by the gauge fields of translations, ω^a, and special conformal transformations, ω_a, (often, in this context, called co-translations). Horizontal curvatures are now expanded in all $2n$ gauge forms, for example,

$$\Omega_{b}^{a} = \frac{1}{2}\Omega_{bcd}^{a}\omega^{c}\omega^{d} + \Omega_{bd}^{ac}\omega_{c}\omega^{d} + \frac{1}{2}\Omega_{b}^{acd}\omega_{c}\omega_{d}$$

Each of the connection 1-forms is a function of $2n$ coordinates, $\left(x^{\alpha}, y_{\beta}\right)$, and is a linear combination of $\left(\mathbf{d}x^{\alpha}, \mathbf{d}y_{\beta}\right)$,

$$\omega_{b}^{a} = \omega_{b\alpha}^{a}\left(x^{\beta}, y_{\sigma}\right)\mathbf{d}x^{\alpha} + \omega_{b}^{a\alpha}\left(x^{\beta}, y_{\sigma}\right)\mathbf{d}y_{\alpha}$$

5.1 New structures

In sharp constrast to the previous gauge theories, the biconformal gauging leads to new structures:

1. Uniform linear action in any dimension

2. Symplectic structure

3. Natural metric
4. SL(2,R) and complex structure

These structures, which follow entirely from the properties of the conformal group, lead to further results. Most importantly, the presence of a symplectic form tells us how to interpret the extra dimensions. Unlike higher dimensional gravity theories which require compactification or other dimensional reduction to make physical comparisons, the biconformal gauging gives a type of phase space. Physical questions may be addressed directly in this phase space, or on any suitable configuration subspace, without compactification. New, geometric insights into both Hamiltonian mechanics and quantum mechanics are revealed when they are formulated on these biconformal spaces.

Two further results which we explore in more detail in the next sub-Section are:

1. The linear action leads to general relativity on an n-dimensional submanifold.
2. By combining the metric and symplectic structures, we can derive the existence of time in initially Euclidean models.

5.1.1 Uniform linear action

We find the most general action linear in the biconformal curvatures.

The gauge transformations of the connection forms are simpler in the biconformal gauging because we no longer have either translational or special conformal transformations. Therefore, $\Lambda^a = \Lambda_a = 0$, and the connection transforms as

$$\begin{aligned} \tilde{\omega}^a_b &= \Lambda^a_c \omega^c_d \bar{\Lambda}^d_b - \mathbf{d}\Lambda^a_c \bar{\Lambda}^c_b \\ \tilde{\omega}^a &= \Lambda^a_c \omega^c \bar{\Lambda} \\ \tilde{\omega}_a &= \Lambda \omega_d \bar{\Lambda}^d_a \\ \tilde{\omega} &= \omega - \mathbf{d}\Lambda \bar{\Lambda} \end{aligned}$$

Notice that the basis forms, (ω^a, ω_b) transform as tensors under the remaining Lorentz and dilatational symmetries.

Each of the curvatures now transforms as a separate tensor:

$$\begin{aligned} \tilde{\Omega}^a_b &= \Lambda^a_c \Omega^c_d \bar{\Lambda}^d_b \\ \tilde{\Omega}^a &= \Lambda^a_c \Omega^c \bar{\Lambda} \\ \tilde{\Omega}_a &= \Lambda \Omega_d \bar{\Lambda}^d_a \\ \tilde{\Omega} &= \Omega \end{aligned}$$

It is therefore consistent to set the torsion to zero,

$$\Omega^a = 0 \tag{5}$$

This is consistent with our usual expectation for spacetime, and also guarantees the existence of a momentum submanifold of biconformal space.

Second, we assume the minimum condition consistent with the existence of a spacetime submanifold. This minimum condition will be discussed below. We show that under these conditions the low-energy field equations reduce to the conformal Einstein equation. The proof is lengthy and will only be summarized here. Details will be found in another set of notes, [**?**].

As with the Poincaré and Weyl gauge theories, and unlike the auxiliary conformal gauge theory, it is possible to use an action linear in the biconformal curvatures. The most general curvature-linear action in n-dimensions is

$$S = \int \varepsilon_{ac\cdots d}{}^{be\cdots f} \left(\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma \omega^a \omega_b \right) \omega^c \cdots \omega^d \omega_e \cdots \omega_f$$

There can be no linear torsion or co-torsion (Ω_a) terms.

Exercise: Show that the resulting field equations, with vanishing torsion, are:

$$
\begin{aligned}
0 &= \beta\left(\Omega_{db}^{\ \ d}-\Omega_{bd}^{\ \ d}\right)\\
0 &= \Omega_{a}^{\ \ ba}\\
0 &= 2\alpha\Delta_{eb}^{cf}\delta_{dg}^{ab}\Omega_{ac}^{\ \ d}\\
0 &= \alpha\Omega_{bac}^{a}+\beta\Omega_{bc}\\
0 &= 2\left(\alpha\Omega_{cd}^{ec}+\beta\Omega_{d}^{e}\right)\delta_{eb}^{ad}+\Lambda_{b}^{a}\\
0 &= \alpha\Omega_{a}^{bac}+\beta\Omega^{bc}\\
0 &= 2\left(\alpha\Omega_{dc}^{ce}+\beta\Omega_{d}^{e}\right)\delta_{eb}^{ad}+\Lambda_{b}^{a}
\end{aligned}
$$

Answer: See Notes: Solution to the biconformal field equations

5.1.2 Symplectic structure

The structure equations of the conformal group show that almost all biconformal spaces have a symplectic form. The dilatation equation is

$$\mathbf{d}\omega=\omega^{a}\omega_{a}+\Omega$$

In all known solutions to the field equations, the dilatation takes the form

$$\Omega=-\kappa\omega^{a}\omega_{a}$$

so that the structure equation becomes

$$\mathbf{d}\omega=(1-\kappa)\,\omega^{a}\omega_{a}$$

The 2-form $\omega^a\omega_a$ is necessarily non-degenerate, and the equality shows that it is also exact. Therefore, $\omega^a\omega_a$ is a closed, non-degenerate 2-form. This is the requirement for it to be symplectic.

The presence of a symplectic form tells us that the $2n$-dimensional base manifold should be understood as a phase space (in classical or quantum mechanical applications) or a co-tangent bundle when we build a gravity theory. This turns out to work extremely well. For classical mechanics or quantum mechanics, we take the action functional to be proportional to the integral of the Weyl vector,

$$S=\int\omega$$

Then the dilatational gauge freedom provides the usual ability to change the Lagrangian by a total derivative. With the Newtonian assumption of universal time, ω becomes

$$\omega=H\left(x^{i},p_{j},t\right)dt-p_{i}dx^{i}$$

and we immediately get Hamilton's equations as the extrema. Along these extrema, no physical size change is ever observable.

5.1.3 Natural metric

As we have seen, the Killing metric of the conformal group is

$$K_{\Sigma\Lambda}=\left(\begin{array}{cccc}\frac{1}{2}\Delta_{db}^{ac} & 0 & 0 & 0\\ 0 & 0 & \delta_{b}^{a} & 0\\ 0 & \delta_{a}^{b} & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$

When $K_{\Sigma\Lambda}$ is restricted to the base manifold it now remains non-degenerate:

$$K_{\Sigma\Lambda}=\left(\begin{array}{cc}0 & \delta_{b}^{a}\\ \delta_{a}^{b} & 0\end{array}\right)$$

We therefore have the inner products

$$
\begin{aligned}
\left\langle \omega^a, \omega^b \right\rangle &= 0 \\
\langle \omega^a, \omega_b \rangle &= \delta^a_b \\
\langle \omega_a, \omega_b \rangle &= 0
\end{aligned}
$$

This is in sharp contrast to all of the other gauge theories of general relativity that we have examined. In the other gauge theories, the induced Killing metric vanishes entirely. It is only by keeping both translation sectors of the conformal group (i.e., translations of the origin and translations of the point at infinity) on the base manifold that the Killing metric is nontrivial.

5.1.4 SL(2,R) and complex structure

If we introduce of two copies of the metric, η_{ab}, of the original n-dimensional space as an additional structure on the base manifold, we get a conformal equivalence class of matrices,

$$
G_{AB} = \begin{pmatrix} e^{2\varphi}\eta_{ab} & \\ & e^{-2\varphi}\eta^{ab} \end{pmatrix}
$$

since the original metric depends on the conformal factor. If we combine this, and its inverse, with the induced Killing metric and the symplectic form,

$$
\begin{aligned}
K_{AB} &= \begin{pmatrix} & \delta^a_b \\ \delta^a_b & \end{pmatrix} \\
\Omega_{AB} &= \begin{pmatrix} & -\delta^a_b \\ \delta^a_b & \end{pmatrix}
\end{aligned}
$$

we readily build the mixed tensors

$$
\begin{aligned}
K^A{}_B &= G^{AC}K_{CB}\begin{pmatrix} & \delta^a_b \\ \delta^a_b & \end{pmatrix} \\
&= \begin{pmatrix} & e^{-2\varphi}\eta^{ab} \\ e^{2\varphi}\eta_{ab} & \end{pmatrix} \\
\Omega^A{}_B &= G^{AC}K_{CB} \\
&= \begin{pmatrix} & -e^{-2\varphi}\eta^{ab} \\ e^{2\varphi}\eta_{ab} & \end{pmatrix}
\end{aligned}
$$

These commute to give

$$
Q = [K, \Omega] = 2\begin{pmatrix} \delta^a_b & \\ & -\delta^b_a \end{pmatrix}
$$

and we easily see that K, Ω, Q generate $SL(2,R)$. Finally, notice that

$$
\Omega^A{}_C \Omega^C{}_B = -\delta^A_B
$$

This gives biconformal spaces complex structure. The complex structure should allow us to establish an isomorphism between the biconformal space and twistor space. However, without the conformal gauging, we do not readily see the symplectic and metric structures.

It is well known that twistor space admits both Lorentzian and Euclidean submanifolds. However, using the symplectic form and Killing metric to form orthogonal configuration and momentum submanifolds, the relationship between this different signature submanifolds becomes unique.

5.2 Summary of biconformal solution

Solving the field equations and structure equations is lengthy (see Notes). After certain gauge and coordinate choices, the connection takes the final form

$$\begin{aligned} \omega_b^a &= \alpha_b^a - 2\Delta_{db}^{ac} W_c \mathbf{e}^d & (6)\\ \omega^a &= \mathbf{e}^a(x) & (7)\\ \omega_a &= a(\mathbf{f}_a + \mathbf{b}_a) & (8)\\ \omega &= W_c \mathbf{e}^c = -y_\beta \mathbf{d}x^\beta & (9) \end{aligned}$$

where α_b^a is the usual Lorentzian spin connection compatible with the solder form, $\mathbf{e}^a$,

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \alpha_b^a$$

and

$$\begin{aligned} a &= (1-\kappa)^{-1} & (10)\\ \kappa &= -\frac{1}{(n-1)}\left(1 + \frac{\gamma n^2}{(\alpha(n-1)-\beta)}\right) & (11)\\ \mathbf{f}_a &= e_a{}^\beta(x)\ \mathbf{d}y_\beta & (12)\\ \mathbf{b}_a &= \mathscr{R}_a - e_a{}^\nu y_\mu \Gamma_\nu^\mu - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{e}^c & (13) \end{aligned}$$

It is important to notice that all dependence on y_α is explicit: the only undetermined field is the solder form, $\mathbf{e}^a(x)$, and this depends only on x^α. The full $2n$-dimensional geometry is determined by the solution on an n-dimensional subspace.

Notice that the strange looking $\Gamma_\nu^\mu = \Gamma_{\nu\alpha}^\mu \mathbf{d}x^\alpha$ term in $\mathbf{b}_a$ may be written α-covariantly as

$$\begin{aligned} D_\nu y_\mu &= \frac{\partial}{\partial x^\nu}\ y_\mu - y_\alpha \Gamma_{\mu\nu}^\alpha = -y_\alpha \Gamma_{\mu\nu}^\alpha\\ \mathbf{D}y_\mu &= \mathbf{d}_x\ y_\mu - y_\alpha \Gamma_\mu^\alpha = -y_\alpha \Gamma_\mu^\alpha \end{aligned}$$

with the understanding that the μ index of $\mathbf{D}y_\mu$ labels n different functions, but does not transform as a vector. There is a sense in which y_μ *does* transform as a vector? The submanifolds spanned by y_μ are flat Riemannian geometries. If we assume the corresponding manifolds are R^n, then the coordinate y_α doubles as a vector. Then this term is fully covariant.

We also have

$$\mathscr{R}_{ab} \equiv -\frac{1}{(n-2)}\left(R_{ab} - \frac{1}{2(n-1)} R\eta_{ab}\right) \qquad (14)$$

The curvatures are then

$$\begin{aligned} \Omega_b^a &= \frac{1}{2}\Omega_{bcd}^a \mathbf{e}^c \mathbf{e}^d - 2\kappa \Delta_{db}^{ac} \omega_c \mathbf{e}^d & (15)\\ &= \mathbf{R}_b^a - 2\Delta_{db}^{ac} \mathscr{R}_c \mathbf{e}^d - 2a\kappa \Delta_{db}^{ac} \mathbf{f}_c \mathbf{e}^d & (16)\\ &= \mathbf{C}_b^a - 2a\kappa \Delta_{db}^{ac} \mathbf{f}_c \mathbf{e}^d & (17)\\ \Omega^a &= 0 & (18)\\ \Omega_a &= \mathbf{D}_{(x,\alpha)} \mathscr{R}_a + W_b \mathbf{C}_a^b & (19)\\ &= \mathbf{d}\mathscr{R}_a - \alpha_a^b \mathscr{R}_b + W_b \mathbf{C}_a^b & (20)\\ \Omega &= -\kappa \mathbf{e}^a \omega_a = -a\kappa \mathbf{e}^a \mathbf{f}_a = -a\kappa \mathbf{d}x^\beta \mathbf{d}y_\beta & (21) \end{aligned}$$

Finally, the structure equations take the form

$$\begin{aligned} \mathbf{d}\omega_b^a &= \omega_b^c \omega_c^a + 2\Delta_{cb}^{ad} \omega_d \mathbf{e}^c + \mathbf{C}_b^a & (22)\\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \omega_b^a + \omega \mathbf{e}^a & (23)\\ \mathbf{d}\omega_a &= \omega_a^b \omega_b + \omega_a \omega + \mathbf{D}_{(x,\alpha)} \mathscr{R}_a + W_b \mathbf{C}_a^b & (24)\\ \mathbf{d}\omega &= \mathbf{e}^a \omega_a & (25) \end{aligned}$$

where

$$\mathbf{C}^a_b = \mathbf{R}^a_b - 2\Delta^{ac}_{db}\mathscr{R}_c\mathbf{e}^d$$

is the Weyl curvature.

The derivation of this solution and its interpretation as General Relativity may be found in the accompanying papers.

5.3 General relativity on biconformal spaces

With the solution of the torsion-free field equations, we have reduced the structure equations to:

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + 2\Delta^{ad}_{cb}\omega_d\mathbf{e}^c + \mathbf{C}^a_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega^a_b + \omega\mathbf{e}^a \\
\mathbf{d}\omega_a &= \omega^b_a\omega_b + \omega_a\omega + \mathbf{D}_{(x,\alpha)}\mathscr{R}_a + W_b\mathbf{C}^b_a \\
\mathbf{d}\omega &= \mathbf{e}^a\omega_a
\end{aligned}$$

In order to find this solution, we make use of the involution of the solder form evident in the second equation. Using this to set the solder form to zero, and using the form of the soluiton, we have a submanifold described by the simple equation

$$\mathbf{d}\omega_a = 0$$

This equation was used to introduce the y_α coordinates.

A second, complementary submanifold may be found by requiring involution of ω_a. If this occurs then we can consistently set

$$\omega_a = 0$$

which reduces the structure equations of the submanifold to

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b\omega^a_c + \mathbf{C}^a_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega^a_b + \omega\mathbf{e}^a \\
0 &= \mathbf{D}_{(x,\alpha)}\mathscr{R}_a + W_b\mathbf{C}^b_a \\
\mathbf{d}\omega &= 0
\end{aligned}$$

The final two equations are the familiar conditions for the metric to be conformal to a Ricci flat metric. The first, second and final equations also imply this condition, since the first requires the curvature of the spin connection to give the Weyl curvature instead of the Riemann curvature. This happens only if the Ricci tensor is purely conformal.

The induced metric on these submanifolds vanishes, since the Killing metric gives

$$\begin{aligned}
\left\langle \omega^a, \omega^b \right\rangle &= 0 \\
\left\langle \omega^a, \omega_b \right\rangle &= \delta^a_b \\
\left\langle \omega_a, \omega_b \right\rangle &= 0
\end{aligned}$$

This suggests that we look for metric submanifolds.

5.4 The existence of time

We have proved that the symplectic form, the metric, and the demand for orthogonal momentum and configuration submanifolds, imply the existence of time in 4-dimensions. The result also holds for n odd. When n is even and greater than 4, Euclidean models still require the existence of time in the gauge theory.

If we turn the result around a little, we find that the conformal gauge theory of a Euclidean space of any dimension always leads to an orthogonal pair of momentum/configuration submanifolds with induced Lorentz metric.

Starting from the solution above, the result is not difficult to show. The direct proof, found in http://arxiv.org/pdf/0811.0112v1, is more difficult.

5.4.1 Canonical, orthogonal submanifolds

The restriction of the conformal Killing metric to the base manifold of the biconformal gauging is non-degenerate, but anti-diagonal. We seek a new set of canonically conjugate basis forms such that there is an unambiguous induced metric on the configuration space. The only way to guarantee that this induced metric is independent of momentum is to require the configuration and momentum submanifolds to be orthogonal to one another. Since the Killing metric is non-degenerate, this implies that the momentum submanifold will also have an induced metric.

It is straightforward to show that any canonically conjugate, orthogonal pair of submanifolds may be spanned by forms (χ^a, ψ_b) related to the original basis by

$$\begin{aligned}
\chi^a &= \omega^b + \frac{1}{2} h^{bc} \omega_c \\
\eta_a &= \frac{1}{2} \omega_a - h_{ab} \omega^b \\
\omega^a &= \frac{1}{2} \left(\chi^a - h^{ab} \eta_b \right) \\
\omega_a &= \eta_a + h_{ab} \chi^b
\end{aligned}$$

where h_{ab} is symmetric and nondegenerate and h^{ab} is its inverse. The inner products of the new basis forms are then orthogonal

$$\begin{aligned}
\left\langle \chi^a, \chi^b \right\rangle &= h^{ab} \\
\langle \chi^a, \psi_b \rangle &= 0 \\
\langle \psi_a, \psi_b \rangle &= -h_{ab}
\end{aligned}$$

with $\pm h_{ab}$ the new metric. They are also canonical since

$$\mathbf{d}\omega = \chi^a \psi_a$$

5.4.2 New coordinates

We can learn something about the induced metric by choosing corresponding coordinates. From the solution for the basis

$$\begin{aligned}
\omega^a &= \mathbf{e}^a(x) \\
\omega_a &= a\left(\mathbf{f}_a + \mathbf{b}_a\right)
\end{aligned}$$

and the Killing metric, we find the line element

$$\begin{aligned}
ds^2 &= 2\delta^b_a \omega^a \omega_b \\
&= 2a\left(e^a_\mu dx^\mu\right)\left(e^\nu_a dy_\nu + b_{a\nu} dx^\nu\right) \\
&= 2a\left(dx^\mu dy_\mu + b_{\mu\nu} dx^\mu dx^\nu\right)
\end{aligned}$$

Coordinates on the orthogonal submanifolds will make the metric block diagonal (The submanifold condition depends on the existence of a pair of involutions, which do turn out to exist and can be derived directly. This calculation is very lengthy, and will not be presented here).

Define a coordinate z_μ such that

$$dz_\mu = dy_\mu + b_{\mu\nu} dx^\nu$$

Then in terms of z_μ, the line element is

$$ds^2 = 2a dx^\mu dz_\mu$$

Clearly, x^μ and z_μ are null directions. Setting

$$\begin{aligned}
dr_\mu &= h_{\mu\nu} dx^\nu + \frac{1}{2} dz_\mu \\
ds_\mu &= h_{\mu\nu} dx^\nu - \frac{1}{2} dz_\mu
\end{aligned}$$

we have

$$\begin{aligned}
h^{\mu\nu}dr_\mu dr_\nu &= h^{\mu\nu}\left(h_{\mu\alpha}dx^\alpha + \frac{1}{2}dz_\mu\right)\left(h_{\nu\beta}dx^\beta + \frac{1}{2}dz_\nu\right) \\
&= h_{\nu\beta}dx^\nu dx^\beta + dx^\nu dz_\nu + \frac{1}{4}h^{\mu\nu}dz_\mu dz_\nu \\
h^{\mu\nu}ds_\mu ds_\nu &= h^{\mu\nu}\left(h_{\mu\alpha}dx^\alpha - \frac{1}{2}dz_\mu\right)\left(h_{\nu\beta}dx^\beta - \frac{1}{2}dz_\nu\right) \\
&= h_{\nu\beta}dx^\nu dx^\beta - dx^\mu dz_\mu + \frac{1}{4}h^{\mu\nu}dz_\mu dz_\nu
\end{aligned}$$

and therefore,

$$\begin{aligned}
ds^2 &= 2a dx^\mu dz_\mu \\
&= a\left(h^{\mu\nu}dr_\mu dr_\nu - h^{\mu\nu}ds_\mu ds_\nu\right)
\end{aligned}$$

Therefore, r_α and s_α make the metric block diagonal,

$$g_{MN} = \frac{1}{a}\begin{pmatrix} h_{\mu\nu} & \\ & -h_{\mu\nu} \end{pmatrix}$$

Any coordinate transformation of the form $\left(\tilde{r}_\alpha(r), \tilde{s}_\beta(s)\right)$ preserves this block diagonal form.

5.4.3 Integrability of the new coordinates

There are two integrability relations we must address in order for the r_α and s_α coordinates to exist. First, we require integrability of the coordinate transformation,

$$\mathbf{d}z_\mu = \mathbf{d}y_\mu + b_{\mu\nu}\mathbf{d}x^\nu$$

or equivalently, involution of

$$\begin{aligned}
e_a^{\ \mu}\mathbf{d}z_\mu &= e_a^{\ \mu}\mathbf{d}y_\mu + e_a^{\ \mu}b_{\mu\nu}\mathbf{d}x^\nu \\
&= \mathbf{f}_a + \mathbf{b}_a \\
&= \frac{1}{a}\boldsymbol{\omega}_a
\end{aligned}$$

The special conformal structure equation,

$$\mathbf{d}\omega_a = \omega_a^b\omega_b + \omega_a\omega + \mathbf{D}_{(x,\alpha)}\mathscr{R}_a + W_b\mathbf{C}_a^b$$

together with the form of the solution, show that ω_a is involute if and only if

$$\left(\mathbf{D}_{(x,\alpha)}\mathscr{R}_a + W_b\mathbf{C}_a^b\right)\Big|_{\omega_a=0} = 0$$

If this condition holds, then the structure equations reduce to

$$\begin{aligned}
\mathbf{d}\omega_b^a &= \omega_b^c\omega_c^a + \mathbf{C}_b^a \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega_b^a + \omega\mathbf{e}^a \\
\left(\mathbf{D}_{(x,\alpha)}\mathscr{R}_a + W_b\mathbf{C}_a^b\right)\Big|_{\omega_a=0} &= 0 \\
\mathbf{d}\omega &= 0
\end{aligned}$$

This describes a trivial Weyl geometry with Ricci tensor of purely conformal type, so in order to define the configuration space the metric must be conformal to a solution of the Einstein equation.

Now consider the second integrability condition. We need both

$$\begin{aligned}
\mathbf{d}r_\mu &= h_{\mu\nu}\mathbf{d}x^\nu + \frac{1}{2}\mathbf{d}z_\mu \\
&= h_{\mu\nu}\mathbf{d}x^\mu + \frac{1}{2}\left(\mathbf{d}y_\mu + b_{\mu\alpha}\mathbf{d}x^\alpha\right) \\
&= h_{\mu\nu}\mathbf{d}x^\mu + \frac{1}{2}\mathbf{d}y_\mu + \frac{1}{2}b_{\mu\alpha}\mathbf{d}x^\alpha \\
\mathbf{d}s_\mu &= h_{\mu\nu}\mathbf{d}x^\nu - \frac{1}{2}\mathbf{d}y_\mu - \frac{1}{2}b_{\mu\alpha}\mathbf{d}x^\alpha
\end{aligned}$$

integrable. We can achieve this algebraically by setting

$$h_{\mu\nu} = \sigma b_{\mu\nu}$$

with σ constant, since this reduces the differentials to

$$\begin{aligned}
\mathbf{d}r_\mu &= \frac{1}{2}\mathbf{d}y_\mu + \left(\sigma + \frac{1}{2}\right) b_{\mu\alpha}\mathbf{d}x^\alpha \\
\mathbf{d}s_\mu &= -\frac{1}{2}\mathbf{d}y_\mu + \left(\sigma - \frac{1}{2}\right) b_{\mu\alpha}\mathbf{d}x^\alpha
\end{aligned}$$

and we have already seen that $b_{\mu\alpha}\mathbf{d}x^\alpha$ is closed.

5.4.4 The signature of the induced metric

Now consider the form of $\mathbf{b}_a$:

$$\mathbf{b}_a = \mathscr{R}_a - e_a{}^{\nu} y_\mu \Gamma^\mu_\nu - y_a y_c \mathbf{e}^c + \frac{1}{2}\eta_{ac}\left(\eta^{gh} y_g y_h\right)\mathbf{e}^c$$

It is sufficient to consider the flat case. Then we may set the curvature and connection to zero, leaving

$$\begin{aligned}
h_{\mu\nu} &= \sigma b_{\mu\nu} \\
&= \frac{\sigma}{2}\left(-2y_\mu y_\nu + \eta_{\mu\nu}\left(\eta^{\rho\sigma} y_\rho y_\sigma\right)\right)
\end{aligned}$$

Set $\sigma = 2$.

We have shown (see http://arxiv.org/pdf/0811.0112v1) that:

1. The metric always takes this form.
2. The signature of the induced metric is consistent across the entire phase space only if the original metric, $\eta_{\mu\nu}$, is Euclidean or of signature zero.

Consider the Euclidean case. We begin with a Euclidean metric, $\eta_{\mu\nu} = diag\,(1,1,\ldots,1)$, and construct its conformal group. We then gauge that group biconformally. The resulting symplectic space has canonical, orthogonal submanifolds with induced metric

$$h_{\mu\nu} = 2y_\mu y_\nu - \eta_{\mu\nu}\left(\eta^{\rho\sigma} y_\rho y_\sigma\right)$$

Consider the copy of the configuration space at y_α. Perform a rotation so that

$$y_\alpha = \rho\,(1,0,\ldots,0)$$

Then the metric is

$$\begin{aligned}
h_{\mu\nu} &= -2y_\mu y_\nu + \eta_{\mu\nu}\left(\eta^{\rho\sigma} y_\rho y_\sigma\right) \\
&= -2\rho^2 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \rho^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}
\end{aligned}$$

$$
= \quad \rho^2 \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}
$$

which is Lorentzian. This means that the conformal gauge theory of Euclidean spaces is described by a Lorentzian phase space. It is possible to map back and forth between the original Euclidean space and the Lorentzian configuration space. Any physical theory that we can construct in the Lorentzian space has a corresponding, equivalent formulation in Euclidean space. In this sense, we may view the existence of time as a consequence of the conformal gauge theory, rather than an intrinsic property of the original space.

It is probably most correct in these models to regard the $2n$-dimensional biconformal space as "the world". This is consistent with both classical and quantum mechanics, which take their most elegant forms in phase space. Indeed, phase space is necessary for the description of quantum mechanics - without both position and momentum, we have no uncertainty principle.

A Variation of the modified linear action

Exercise: Find the variational field equations from the action

$$
S = \int \kappa^2 \left(\Omega^{ab} + \Lambda \mathbf{e}^a \mathbf{e}^b \right) \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} + \kappa^2 \beta \Omega^* \Omega + \mathbf{D}\kappa^* \mathbf{D}\kappa + m^2 \kappa^* \kappa
$$

Answer: The special conformal gauge field is varied above.

Varying the spin connection, leads to:

$$
\begin{aligned}
\delta S &= \int \kappa^{\frac{n-2}{2}} \mathbf{D}\delta\omega^{ab} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
&= \int \mathbf{D}\left(\kappa^{\frac{n-2}{2}} \delta\omega^{ab} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \right) + \delta\omega^{ab} \mathbf{D}\kappa^{\frac{n-2}{2}} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
&\quad + (n-2) \int \delta\omega^{ab} \kappa^{\frac{n-2}{2}} \mathbf{D}\mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
&= \int \mathbf{d}\left(\kappa^{\frac{n-2}{2}} \delta\omega^{ab} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \right) + \delta\omega^{ab} \mathbf{D}\kappa^{\frac{n-2}{2}} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
&\quad + (n-2) \int \delta\omega^{ab} \kappa^{\frac{n-2}{2}} \mathbf{D}\mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
&= \int \delta\omega^{ab} \mathbf{D}\kappa^{\frac{n-2}{2}} \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} + (n-2) \int \delta\omega^{ab} \kappa^{\frac{n-2}{2}} \mathbf{D}\mathbf{e}^c \mathbf{e}^e \cdots \mathbf{e}^d e_{abce\cdots d}
\end{aligned}
$$

Since the torsion, $\Omega^a = \mathbf{D}\mathbf{e}^a$ must vanish in order for the action to be invariant, we drop the second term, resulting in the field equation

$$
\begin{aligned}
0 &= \frac{n-2}{2} \kappa^{\frac{n-4}{2}} \mathbf{D}\kappa \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
0 &= \mathbf{D}\kappa \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d}
\end{aligned}
$$

Wedging with $\mathbf{e}^f$ and taking the dual,

$$
\begin{aligned}
0 &= D_g \kappa \mathbf{e}^f \mathbf{e}^g \mathbf{e}^c \cdots \mathbf{e}^d e_{abc\cdots d} \\
0 &= D_g \kappa e^{fgc\cdots d} e_{abc\cdots d} \\
&= -(n-2)! D_g \kappa \left(\delta^f_a \delta^g_b - \delta^f_b \delta^g_a \right) \\
&= -(n-2)! \left(\delta^f_a D_b \kappa - \delta^f_b D_a \kappa \right)
\end{aligned}
$$

The trace on fa then gives

$$
0 = -(n-2)(n-1) D_b \kappa
$$

and κ is constant.

Notice that the integrability condition has changed slightly. We now have

$$\begin{aligned} 0 &= \mathbf{d}^2\kappa \\ &= \mathbf{d}\omega\kappa - \omega\mathbf{d}\kappa \\ &= (\omega^a\omega_a + \Omega)\,\kappa - \omega\omega\kappa \\ &= (\omega^a\omega_a + \Omega)\,\kappa \end{aligned}$$

so the dilatation must be

$$\begin{aligned} \Omega &= -\omega^a\omega_a \\ \frac{1}{2}\Omega_{ab}\mathbf{e}^a\mathbf{e}^b &= \frac{1}{2}(\omega_{ba} - \omega_{ab})\,\mathbf{e}^a\mathbf{e}^b \\ \Omega_{ab} &= \omega_{ba} - \omega_{ab} \end{aligned}$$

Solder form variation With the action given by

$$\begin{aligned} S &= \int \kappa^2\left(\Omega^{ab}\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} + \Lambda\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d}\right) \\ &\quad + \int \kappa^2\beta\Omega^*\Omega - \frac{1}{n!}\left(g^{ab}D_a\kappa D_b\kappa + m^2\kappa^2\right)e_{c\cdots d}\mathbf{e}^c\cdots\mathbf{e}^d \end{aligned}$$

it is easiest to vary some terms differently than others. for the first two terms:

$$\begin{aligned} \delta S_1 &= \delta\int\kappa^2\left(\Omega^{ab} + \Lambda\mathbf{e}^a\mathbf{e}^b\right)\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} \\ &= \int\kappa^2\left(\delta\Omega^{ab}\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} + (n-2)\,\Omega^{ab}\delta\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} + n\Lambda\delta\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d}\right) \\ &= \int\kappa^2\left(-2\eta^{bh}\Delta^{ae}_{fh}\omega_e\delta\mathbf{e}^f\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} + (n-2)\,\Omega^{ab}\delta\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} + n\Lambda\delta\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d}\right) \\ &= \int\delta\mathbf{e}^f\kappa^2\left(2\eta^{bh}\Delta^{ae}_{fh}\omega_e\mathbf{e}^c\cdots\mathbf{e}^d e_{abc\cdots d} + (n-2)\,\Omega^{ab}\mathbf{e}^c\cdots\mathbf{e}^d e_{abfc\cdots d} + n\Lambda\mathbf{e}^b\mathbf{e}^c\cdots\mathbf{e}^d e_{fbc\cdots d}\right) \end{aligned}$$

For the remaining terms, we have

$$\begin{aligned} \Omega_{\alpha\beta} &= \partial_\alpha\omega_\beta - \partial_\beta\omega_\alpha - e_\alpha{}^b\omega_{b\beta} + e_\beta{}^b\omega_{b\alpha} \\ \delta\Omega_{\alpha\beta} &= -\delta e_\alpha{}^b\omega_{b\beta} + \delta e_\beta{}^b\omega_{b\alpha} \end{aligned}$$

$$S_2 = \int\kappa^2\beta\Omega_{\alpha\beta}\Omega_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^\alpha\mathbf{d}x^\beta\mathbf{d}x^\lambda\cdots\mathbf{d}x^\tau - \frac{1}{n!}\left(g^{\alpha\beta}D_\alpha\kappa D_\beta\kappa + m^2\kappa^2\right)\sqrt{-g}\varepsilon_{\lambda\cdots\tau}\mathbf{d}x^\mu\cdots\mathbf{d}x^\nu$$

where $g_{\alpha\beta} = \eta_{ab}e_\alpha{}^a e_\beta{}^b$, and therefore

$$\begin{aligned} \delta g^{\zeta\chi} &= -g^{\zeta\mu}g^{\chi\nu}\delta g_{\mu\nu} \\ &= -g^{\zeta\mu}g^{\chi\nu}\left(\eta_{ab}\delta e_\mu{}^a e_\nu{}^b + \eta_{ab}e_\mu{}^a\delta e_\nu{}^b\right) \\ &= -\left(\eta_{ab}g^{\zeta\xi}g^{\chi\phi}e_\phi{}^b + \eta_{ab}g^{\zeta\theta}g^{\chi\xi}e_\theta{}^a\right)\delta e_\xi{}^a \end{aligned}$$

Varying,

$$\begin{aligned} \delta S_2 &= \int 2\kappa^2\beta\Omega_{\alpha\beta}\delta\Omega_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^\alpha\mathbf{d}x^\beta\mathbf{d}x^\lambda\cdots\mathbf{d}x^\tau \\ &\quad + \int 2\kappa^2\beta\Omega_{\alpha\beta}\Omega_{\mu\nu}\delta g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^\alpha\mathbf{d}x^\beta\mathbf{d}x^\lambda\cdots\mathbf{d}x^\tau + \kappa^2\beta\Omega_{\alpha\beta}\Omega^{\rho\sigma}\delta\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^\alpha\mathbf{d}x^\beta\mathbf{d}x^\lambda\cdots\mathbf{d}x^\tau \\ &\quad + \frac{1}{n!}\int\delta e_\xi{}^a\left(\eta_{ab}g^{\alpha\xi}g^{\beta\phi}e_\phi{}^b + \eta_{ab}g^{\alpha\theta}g^{\beta\xi}e_\theta{}^a\right)D_\alpha\kappa D_\beta\kappa\sqrt{-g}\varepsilon_{\lambda\cdots\tau}\mathbf{d}x^\mu\cdots\mathbf{d}x^\nu \\ &\quad - \frac{1}{2}\frac{1}{n!}\int\left(g^{\alpha\beta}D_\alpha\kappa D_\beta\kappa + m^2\kappa^2\right)g_{\zeta\chi}\left(\eta_{ab}g^{\zeta\xi}g^{\chi\phi}e_\phi{}^b + \eta_{ab}g^{\zeta\theta}g^{\chi\xi}e_\theta{}^a\right)\delta e_\xi{}^a\sqrt{-g}\varepsilon_{\lambda\cdots\tau}\mathbf{d}x^\mu\cdots\mathbf{d}x^\nu \end{aligned}$$

Substituting the variations,

$$\begin{aligned}
\delta S_2 &= \int 2\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{\alpha\beta}\left(-\delta^{\xi}_{\mu}\omega_{a\nu}+\delta^{\xi}_{\nu}\omega_{a\mu}\right)g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\lambda}\cdots\mathbf{d}x^{\tau}\\
&-\int 2\delta e_{\xi}{}^{a}\kappa^2\beta\left(\eta_{ab}\Omega_{\alpha\beta}\Omega^{\xi\sigma}g^{\rho\phi}e_{\phi}{}^{b}+\eta_{ab}\Omega_{\alpha\beta}\Omega^{\theta\sigma}g^{\rho\xi}e_{\theta}{}^{b}\right)\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\lambda}\cdots\mathbf{d}x^{\tau}\\
&+\frac{1}{2}\int\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{\alpha\beta}\Omega^{\rho\sigma}\left(\eta_{ab}g^{\xi\phi}e_{\phi}{}^{b}+\eta_{ab}g^{\theta\xi}e_{\theta}{}^{b}\right)\sqrt{-g}\varepsilon_{\rho\sigma\lambda\cdots\tau}\mathbf{d}x^{\alpha}\mathbf{d}x^{\beta}\mathbf{d}x^{\lambda}\cdots\mathbf{d}x^{\tau}\\
&+\frac{1}{n!}\int\delta e_{\xi}{}^{a}\left(\eta_{ab}g^{\alpha\xi}g^{\beta\phi}e_{\phi}{}^{b}+\eta_{ab}g^{\alpha\theta}g^{\beta\xi}e_{\theta}{}^{a}\right)D_{\alpha}\kappa D_{\beta}\kappa\sqrt{-g}\varepsilon_{\lambda\cdots\tau}\mathbf{d}x^{\mu}\cdots\mathbf{d}x^{\nu}\\
&-\frac{1}{2}\frac{1}{n!}\int\left(g^{\alpha\beta}D_{\alpha}\kappa D_{\beta}\kappa+m^2\kappa^2\right)g_{\zeta\chi}\left(\eta_{ab}g^{\zeta\xi}g^{\chi\phi}e_{\phi}{}^{b}+\eta_{ab}g^{\zeta\theta}g^{\chi\xi}e_{\theta}{}^{a}\right)\delta e_{\xi}{}^{a}\sqrt{-g}\varepsilon_{\lambda\cdots\tau}\mathbf{d}x^{\mu}\cdots\mathbf{d}x^{\nu}
\end{aligned}$$

Convert to orthonormal,

$$\begin{aligned}
\delta S_2 &= \int 4\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{cd}\delta^{\xi}_{i}\omega_{ag}\eta^{gh}\eta^{ij}e_{hje\cdots f}\mathbf{e}^{c}\mathbf{e}^{d}\mathbf{e}^{e}\cdots\mathbf{e}^{f}\\
&-\int 2\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{cd}\Omega^{bi}\left(\delta^{g}_{a}\delta^{\xi}_{b}+\eta_{ab}\eta^{g\xi}\right)e_{gie\cdots f}\mathbf{e}^{c}\mathbf{e}^{d}\mathbf{e}^{e}\cdots\mathbf{e}^{f}\\
&+\int\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{cd}\Omega^{gh}\delta^{\xi}_{a}e_{ghe\cdots f}\mathbf{e}^{c}\mathbf{e}^{d}\mathbf{e}^{e}\cdots\mathbf{e}^{f}\\
&+\frac{1}{n!}\int\delta e_{\xi}{}^{a}2\eta^{c\xi}D_{c}\kappa D_{a}\kappa e_{e\cdots f}\mathbf{e}^{e}\cdots\mathbf{e}^{f}\\
&-\frac{1}{2}\frac{1}{n!}\int\left(\eta^{cd}D_{c}\kappa D_{d}\kappa+m^2\kappa^2\right)2\delta^{\xi}_{a}\delta e_{\xi}{}^{a}e_{e\cdots f}\mathbf{e}^{e}\cdots\mathbf{e}^{f}
\end{aligned}$$

Now define $\Phi=\frac{1}{n!}e_{a\cdots b}\mathbf{e}^{a}\cdots\mathbf{e}^{b}$ so that $\mathbf{e}^{a}\cdots\mathbf{e}^{b}=e^{a\cdots b}\Phi$, and recombine with the first term

$$\begin{aligned}
0 &= \int e^{\xi}_{k}\delta e_{\xi}{}^{a}\kappa^2\mathbf{e}^{k}\left(2\eta^{bh}\Delta^{fe}_{ah}\omega_{e}\mathbf{e}^{c}\cdots\mathbf{e}^{d}e_{fbc\cdots d}+(n-2)\,\Omega^{fb}\mathbf{e}^{c}\cdots\mathbf{e}^{d}e_{abfc\cdots d}+n\Lambda\mathbf{e}^{b}\mathbf{e}^{c}\cdots\mathbf{e}^{d}e_{abc\cdots d}\right)\\
&+\int 4\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{cd}e^{\xi}_{i}\omega_{ag}\eta^{gh}\eta^{ij}e_{hje\cdots f}e^{cde\cdots f}\Phi\\
&-\int 2\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{cd}\Omega^{bi}\left(\delta^{g}_{a}e^{\xi}_{b}+\eta_{ab}\eta^{gk}e^{\xi}_{k}\right)e_{gie\cdots f}e^{cde\cdots f}\Phi\\
&+\int\delta e_{\xi}{}^{a}\kappa^2\beta\Omega_{cd}\Omega^{gh}e^{\xi}_{a}e_{ghe\cdots f}e^{cde\cdots f}\Phi\\
&-\int\delta e_{\xi}{}^{a}2\eta^{cd}e^{\xi}_{d}D_{c}\kappa D_{a}\kappa\Phi\\
&+\frac{1}{2}\int\left(\eta^{cd}D_{c}\kappa D_{d}\kappa+m^2\kappa^2\right)2e^{\xi}_{a}\delta e_{\xi}{}^{a}\Phi
\end{aligned}$$

and the field equation is

$$\begin{aligned}
0 &= \kappa^2\left(2\eta^{bh}\Delta^{fe}_{ah}\omega_{eg}e^{kgc\cdots d}e_{fbc\cdots d}+(n-2)\,\Omega^{fb}{}_{gh}e^{kghc\cdots d}e_{abfc\cdots d}+n\Lambda e^{kbc\cdots d}e_{abc\cdots d}\right)\\
&+4\kappa^2\beta\Omega_{cd}\omega_{ag}\eta^{gh}\eta^{kj}e_{hje\cdots f}e^{cde\cdots f}\\
&-2\kappa^2\beta\Omega_{cd}\Omega^{bi}\left(\delta^{g}_{a}\delta^{k}_{b}+\eta_{ab}\eta^{gk}\right)e_{gie\cdots f}e^{cde\cdots f}+\kappa^2\beta\Omega_{cd}\Omega^{gh}\delta^{k}_{a}e_{ghe\cdots f}e^{cde\cdots f}\\
&-2\eta^{ck}D_{c}\kappa D_{a}\kappa+\delta^{k}_{a}\left(\eta^{cd}D_{c}\kappa D_{d}\kappa+m^2\kappa^2\right)
\end{aligned}$$

Simplify the Levi-Civita pairs,

$$\begin{aligned}
0 &= \kappa^2\left(\left(-(n-2)!2\Delta^{fe}_{ah}\omega_{eg}\left(\delta^{k}_{f}\eta^{gh}-\eta^{kh}\delta^{g}_{f}\right)\right)-n(n-1)!\delta^{k}_{a}\Lambda\right)\\
&-\kappa^2 2\,(n-2)!\Omega^{fb}{}_{gh}\left(\delta^{k}_{a}\delta^{g}_{b}\delta^{h}_{f}+\delta^{k}_{b}\delta^{g}_{f}\delta^{h}_{a}+\delta^{k}_{f}\delta^{g}_{a}\delta^{h}_{b}\right)\\
&+4\kappa^2\beta\Omega_{cd}\omega_{ag}\eta^{gh}\eta^{kj}\left(-(n-2)!\left(\delta^{c}_{h}\delta^{d}_{j}-\delta^{c}_{j}\delta^{d}_{h}\right)\right)
\end{aligned}$$

$$-2\kappa^2\beta\Omega_{cd}\Omega^{bi}\left(\delta_a^g\delta_b^k+\eta_{ab}\eta^{gk}\right)\left(-(n-2)!\left(\delta_g^c\delta_i^d-\delta_i^c\delta_g^d\right)\right)+\kappa^2\beta\Omega_{cd}\Omega^{gh}\delta_a^k\left(-(n-2)!\left(\delta_g^c\delta_h^d-\delta_h^c\delta_g^d\right)\right)$$
$$-2\eta^{ck}D_c\kappa D_a\kappa+\delta_a^k\left(\eta^{cd}D_c\kappa D_d\kappa+m^2\kappa^2\right)$$

so that

$$\begin{aligned}
0 &= \kappa^2\left(2(n-2)!\left(\omega^k{}_a-\delta_a^k\eta^{ge}\omega_{eg}\right)-n(n-1)!\delta_a^k\Lambda\right)\\
&-\kappa^2 2(n-2)!\left(-\delta_a^k\Omega^{fb}{}_{fb}+2\Omega^{fk}{}_{fa}\right)\\
&-8\kappa^2\beta(n-2)!\Omega^{gk}\omega_{ag}\\
&+2(n-2)!\kappa^2\beta\left(\Omega_{ad}\Omega^{kd}-\Omega_{ca}\Omega^{kc}+\Omega_{cd}\Omega^{bd}\eta_{ab}\eta^{ck}-\Omega_{cd}\Omega^{bc}\eta_{ab}\eta^{dk}\right)+2(n-2)!\kappa^2\beta\delta_a^k\Omega_{cd}\Omega^{cd}\\
&-2\eta^{ck}D_c\kappa D_a\kappa+\delta_a^k\left(\eta^{cd}D_c\kappa D_d\kappa+m^2\kappa^2\right)
\end{aligned}$$

Lower k to b, and define:

$$\begin{aligned}
\Theta_{ab} &= \Omega_{ad}\Omega_b{}^d-\frac{1}{4}\eta_{ab}\Omega_{cd}\Omega^{cd}\\
T_{ab} &= D_a\kappa D_b\kappa-\frac{1}{2}\eta_{ab}\left(\eta^{cd}D_c\kappa D_d\kappa+m^2\kappa^2\right)
\end{aligned}$$

Then

$$\omega_{ba}-\eta_{ab}\eta^{cd}\omega_{cd} = 2\left(\Omega^c{}_{bca}-\frac{1}{2}\eta_{ab}\Omega^{cd}{}_{cd}\right)+\frac{1}{2}n(n-1)\eta_{ab}\Lambda+4\beta\omega_{ac}\Omega^c{}_b-4\beta\Theta_{ab}+\frac{1}{\kappa^2}T_{ab}$$

Take the trace with η^{ab},

$$-(n-1)\eta^{ab}\omega_{ab} = -(n-2)\Omega^{cd}{}_{cd}+\frac{1}{2}n^2(n-1)\Lambda-4\beta\omega_{ac}\Omega^{ac}-4\beta\eta^{ab}\Theta_{ab}+\frac{1}{\kappa^2}\eta^{ab}T_{ab}$$

so that

$$\begin{aligned}
\omega_{ba} &= 2\left(\Omega^c{}_{bca}-\frac{1}{2}\eta_{ab}\Omega^{cd}{}_{cd}\right)+\frac{1}{2}n(n-1)\eta_{ab}\Lambda+4\beta\omega_{ac}\Omega^c{}_b-4\beta\Theta_{ab}+\frac{1}{\kappa^2}T_{ab}\\
&\quad+\frac{1}{n-1}\eta_{ab}\left((n-2)\Omega^{cd}{}_{cd}-\frac{1}{2}n^2(n-1)\Lambda+4\beta\omega_{ac}\Omega^{ac}+4\beta\eta^{ab}\Theta_{ab}-\frac{1}{\kappa^2}\eta^{ab}T_{ab}\right)
\end{aligned}$$

Notice that ω_{ab} still occurs on the right. Find the antisymmetric part using $\omega_{ba}-\omega_{ab}=\Omega_{ab}$:

$$\begin{aligned}
\omega_{ba}-\omega_{ab} &= 2\left(\Omega^c{}_{bca}-\Omega^c{}_{acb}\right)+4\beta\omega_{ac}\Omega^c{}_b-4\beta\omega_{bc}\Omega^c{}_a\\
\Omega_{ab} &= 2\left(\Omega^c{}_{bca}-\Omega^c{}_{acb}\right)+4\beta\omega_{ac}\Omega^c{}_b-4\beta\omega_{bc}\Omega^c{}_a
\end{aligned}$$

Using

$$\Omega^c_{bcd}-\Omega^c_{dcb}=-(n-2)\Omega_{bd}$$

this becomes

$$\Omega_{ab} = -\frac{4\beta}{2n-5}\left(\omega_{ac}\Omega^c{}_b-\omega_{bc}\Omega^c{}_a\right)$$

The antisymmetric part of ω_{ab} drops out of the left side, so we have only the symmetric part. Write this as an operator acting on the dilatation and try to invert the operator:

$$0 = \left(\delta_a^c\delta_b^d+\frac{4\beta}{2n-5}\left(\omega_a{}^c\delta_b^d-\delta_a^d\omega_b{}^c\right)\right)\Omega_{cd}$$

where $\omega_a{}^c$ is built from the symmetric part of the gauge field only.

Is this invertible? Suppose $\frac{4\beta}{2n-5} = -\frac{1}{2}$ and $\omega_a{}^c = \delta_a^c$. Then

$$\begin{aligned}
\delta_a^c\delta_b^d + \frac{4\beta}{2n-5}\left(\omega_a{}^c\delta_b^d - \delta_a^d\omega_b{}^c\right) &= \delta_a^c\delta_b^d - \frac{1}{2}\left(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c\right) \\
&= \frac{1}{2}\left(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c\right)
\end{aligned}$$

and this will annihilate the antisymmetric dilatation. Therefore, it is not always invertible. On the other hand, this represents a set of measure zero from among all solutions, so we won't be surprised if we must have $\Omega_{ab} = 0$.

Weyl vector variation Finally, we vary the Weyl vector. We have

$$\begin{aligned}
\delta S &= \int 2\kappa^2\beta\delta^*\Omega + 2\delta\mathbf{D}\kappa^*\mathbf{D}\kappa \\
&= \int 2\kappa^2\beta\mathbf{d}\delta\omega^*\Omega + 2\left(-\left(\frac{n-2}{2}\right)\delta\omega\kappa\right)^*\mathbf{D}\kappa \\
&= \int \delta\omega\left(2\kappa^2\beta\mathbf{d}^*\Omega - (n-2)\kappa^*\mathbf{D}\kappa\right)
\end{aligned}$$

Therefore, wedging with $\mathbf{e}^h$ and taking an additional dual of the field equation,

$$\begin{aligned}
0 &= 2\kappa^2\beta\mathbf{d}^*\Omega - (n-2)\kappa^*\mathbf{D}\kappa \\
0 &= \frac{2\kappa\beta}{(n-2)!}\Omega_{ab;g}\eta^{ac}\eta^{bd}e_{cde\cdots f}e^{hge\cdots f} - \frac{n-2}{(n-1)!}D_a\kappa\eta^{ab}e_{be\cdots f}e^{he\cdots f} \\
&= 2\kappa\beta\Omega_{ab;g}\eta^{ac}\eta^{bd}\left(\delta_c^h\delta_d^g - \delta_c^g\delta_d^h\right) + (n-2)D_a\kappa\eta^{ah} \\
&= -4\kappa\beta\left(\Omega^c{}_{b;c}\eta^{bh}\right) + (n-2)D_a\kappa\eta^{ah}
\end{aligned}$$

and therefore,

$$4\kappa\beta\left(\Omega^c{}_{a;c}\right) = (n-2)D_a\kappa$$

Collecting the remaining equations:

$$\begin{aligned}
\omega_{ba} &= 2\left(\Omega^c{}_{bca} - \frac{1}{2}\eta_{ab}\Omega^{cd}{}_{cd}\right) + \frac{1}{2}n(n-1)\eta_{ab}\Lambda + 4\beta\omega_{ac}\Omega^c{}_b - 4\beta\Theta_{ab} + \frac{1}{\kappa^2}T_{ab} \\
&\quad + \frac{1}{n-1}\eta_{ab}\left((n-2)\Omega^{cd}{}_{cd} - \frac{1}{2}n^2(n-1)\Lambda + 4\beta\omega_{ac}\Omega^{ac} + 4\beta\eta^{ab}\Theta_{ab} - \frac{1}{\kappa^2}\eta^{ab}T_{ab}\right) \\
D_a\kappa &= 0 \\
4\kappa\beta\left(\Omega^c{}_{a;c}\right) &= (n-2)D_a\kappa
\end{aligned}$$

Of course, the remaining equation is inconsistent. These three do not present a sensible set of equations either, since no constraint is placed on the Lorentz curvature.