Spacetime submanifolds of Euclidean conformal gauge theory

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1 Summary

We have a coordinate solution for the connection flat biconformal spaces. However, these coordinates are unrelated to two natural structures on the space: the symplectic form and the induced Killing metric. Combining these structures when the original space is Euclidean leads to the identification of natural canonically conjugate, Lorentzian submanifolds.

The original frame field, \((e^a, f_a)\) canonical, but not orthogonal, and are not adapted to the Lorentzian submanifolds. Here we find a frame field \((\xi^a, \psi_a)\) for the space satisfying the following conditions:

1. The forms \((\xi^a, \psi_a)\) are canonically conjugate.
2. \((\xi^a, \psi_a)\) are orthogonal.
3. Each set of basis forms, \(\{\xi^a\}\) and \(\{\psi_a\}\) is involute, and each set spans a metric submanifold.

Once we have such a basis, we find adapted coordinates on each of the two metric submanifolds. Holding either set of adapted coordinates constant then projects to the conjugate manifold.

We find that a frame field will be orthonormal and canonical if it is related to the original basis by

\[
\begin{align*}
\chi^a &= e^a + \frac{1}{2} h^{ab} f_b \\
\psi_a &= h_{ab} e^b - \frac{1}{2} f_b
\end{align*}
\]

where \(h_{ab}\) is symmetric and \(h^{ab}\) is its inverse. This result is unique up to transformations that do not mix \(\chi^a\) and \(\psi_a\) with each other.

It is possible, but lengthy, to show that these basis forms are in involution if and only if the induced metric \(h_{ab}\) is given by \(h_{ab} = \lambda (y_a y_b - \frac{1}{2} \eta_{ab} y^2)\). Then the basis takes the form

\[
\begin{align*}
\chi^a &= \left(1 - \frac{1}{2\lambda}\right) d\xi^a + \frac{1}{2} h^{ac} dy_c \\
\psi_a &= \left(1 + \frac{1}{2\lambda}\right) h_{ab} d\psi^b - \frac{1}{2} dy_b
\end{align*}
\]

in the coordinates of the original solution, and the involutions are given by

\[
\begin{align*}
d\chi^a &= 0 \\
d\psi_a &= -(2\lambda + 1) (y_b \delta^c_a + y_a \delta^c_b - \eta_{ab} y^2) \psi_c d\psi^b
\end{align*}
\]

In the case where \(\lambda = \frac{1}{2}\) the connection is

\[
\begin{align*}
\omega^a_b &= -\frac{1}{u^2} \Delta^a_{cd} u_c (du^d + dv^d) \\
\xi^a &= -du^a \\
\psi_a &= h_{ab} d\psi^b \\
\omega &= \frac{u_a}{u^2} (du^d + dv^d)
\end{align*}
\]
where the spacetime metric is

\[ h_{ab} = \frac{1}{(u^2)^2} \left( 2u_au_b - u^2 \eta_{ab} \right) \]

## 2 Minkowski coordinates for flat biconformal space

One solution for flat biconformal space is given by

\[ \omega^a_b = 2\Delta^{ac}_{db} \mathbf{e}^d \]
\[ \mathbf{e}^a = \mathbf{d}x^a \]
\[ \mathbf{f}_a = \mathbf{d}y_a - \left( y_ay_b - \frac{1}{2} \eta_{ab} y^2 \right) \mathbf{e}^b \]
\[ \omega = -y_\alpha \mathbf{d}x^\alpha \]

We would like to use this to find an orthogonal canonical basis for the metric submanifolds.

Finally, setting \( y_a = \frac{\beta z_a}{z^2} \) and defining new coordinates

\[ u^a = \left( 1 - \frac{1}{2\lambda} \right) x^a - \frac{1}{\beta \lambda} z^a \]
\[ v^a = \left( 1 + \frac{1}{2\lambda} \right) x^a + \frac{1}{\beta \lambda} z^a \]

the solution for the new basis takes the simple form

\[ \xi^a = -\mathbf{d}u^a \]
\[ \psi_a = h_{ab} \mathbf{d}v^b \]

where, in terms of \( z_a \),

\[ h_{ab} = \frac{\beta^2 \lambda}{(z^2)^2} \left( z_az_b - \frac{1}{2} \right) \eta_{ab} \]

The symplectic form is \( \xi^a \psi_a \). In terms of the \((u^a, v^b)\) coordinates, the symplectic form is

\[ \xi^a \psi_a = -\mathbf{d}u^a h_{ab} \mathbf{d}v^b \]
\[ \neq \mathbf{d}v^a \mathbf{d}u_a \]

so although the coordinate differentials appear canonical, they are only canonical with respect to the submanifold metric, \( h_{ab} \).

## 3 Orthogonal canonical bases

Let

\[ \chi^a = \mathbf{e}^a + \alpha h^{ab} \mathbf{f}_b \]
\[ \psi_a = \beta h_{ab} \mathbf{e}^b - \gamma \mathbf{f}_b \]

Then the inner product gives

\[ \langle \chi^a, \chi^b \rangle = \langle \mathbf{e}^a + \alpha h^{ac} \mathbf{e}_c + \alpha h^{bd} \mathbf{f}_d \rangle \]
\[ = \alpha h^{ab} + \alpha h^{ba} \]
\[\langle \chi^a, \psi_b \rangle = \left\langle e^a + \alpha h^{ac} f_c, \beta h_{bd} e^d - \gamma f_b \right\rangle \]
\[= \alpha \beta h^{ad} h_{bd} - \gamma \delta^a_b \]
\[\langle \psi_a, \psi_b \rangle = \left\langle \beta h_{ac} e^c - \gamma f_a, \beta h_{bd} e^d - \gamma f_b \right\rangle \]
\[= -\gamma \beta h_{ab} - \gamma \beta h_{ba} \]

so orthogonality requires
\[0 = \alpha \beta h^{ad} h_{bd} - \gamma \delta^a_b \]
\[h^{ad} h_{bd} = \frac{\gamma}{\alpha \beta} \delta^a_b \]

Without loss of generality, we normalize \( h_{ab} \) by choosing \( \gamma = \alpha \beta \), so that \( h_{ab} h^{cb} = \delta^c_a \).

Canonical conjugacy requires
\[e^a f_a = \chi^a \psi_a \]
\[= (e^a + \alpha h^{ac} f_c) \left( \beta h_{ad} e^d - \alpha \beta f_a \right) \]
\[= e^a \beta h_{ad} e^d + \alpha \beta h^{ac} h_{ad} e^d - \alpha \beta e^a f_a - \alpha^2 \beta h^{ac} f_a \]

The \( e^a e^d \) and \( f_c f_a \) terms must vanish, so symmetry of \( h_{ab} \) is required,
\[h_{ab} = h_{ba} \]

Then
\[e^a f_a = \chi^a \psi_a \]
\[= \alpha \beta e^a f_a + \alpha \beta h^{ac} h_{ad} e^d f_c \]
\[= 2 \alpha \beta e^a f_a \]

so we require
\[2 \alpha \beta = 2 \gamma = 1 \]

These specifications result in
\[\chi^a = e^a + \frac{1}{\alpha} h^{ab} f_b \]
\[\psi_a = \frac{1}{2 \alpha} h_{ab} e^b - \frac{1}{2} f_b \]

and
\[\langle \chi^a, \chi^b \rangle = 2 \alpha h^{ab} \]
\[\langle \chi^a, \psi_b \rangle = 0 \]
\[\langle \psi_a, \psi_b \rangle = -\frac{1}{2 \alpha} h_{ab} \]

Since \( 2 \alpha h^{ab} \) is inverse to \( \frac{1}{2 \alpha} h_{ab} \), we may as well absorb the remaining constant into the definition of \( h_{ab} \).

\[\langle \chi^a, \chi^b \rangle = h^{ab} \]
\[\langle \chi^a, \psi_b \rangle = 0 \]
\[\langle \psi_a, \psi_b \rangle = -h_{ab} \]

The relative sign between the two submanifolds is essential to maintain the zero signature of the biconformal metric.
The final form of any canonical orthogonal basis is thus
\[\chi^a = e^a + \frac{1}{2} h^{ab} f_b \]
\[\psi_a = h_{ab} e^b - \frac{1}{2} f_b \]
4 Writing the flat solution in an orthogonal canonical basis

Now, substituting the solution, we have

\[ \chi^a = e^a + \frac{1}{2} h^{ab} f_b \]

\[ \chi^a = dx^a + \frac{1}{2} h^{ab} (dy_b - (y_b y_c - \frac{1}{2} \eta_{bc} y^2) dx^c) \]

\[ \chi^a = \left( 6^a - \frac{1}{2} h^{ab} \left( y_b y_c - \frac{1}{2} \eta_{bc} y^2 \right) \right) dx^c + \frac{1}{2} h^{ab} dy_b \]

\[ \psi_a = h_{ab} dx^b = \left( \eta_{ab} y - \frac{1}{2} \frac{\xi}{\lambda} \left( y^2 - \frac{1}{2} \lambda \eta_{ab} y^2 \right) \right) dx^b \]

\[ \psi_a = \left( h_{ab} + \frac{1}{2} \left( y_a y_b - \frac{1}{2} \eta_{ab} y^2 \right) \right) dx^b - \frac{1}{2} dy_b \]

This is orthogonal and canonical for any choice of \( h_{ab} \). Detailed analysis [Time] shows that the only involute choice is

\[ h_{ab} = \lambda \left( y_a y_b - \frac{1}{2} \eta_{ab} y^2 \right) \]

in which case \( h^{ab} = \frac{4}{\lambda (y^2)} \left( y^a y^b - \frac{1}{2} \eta^{ab} y^2 \right) \), and the basis becomes

\[ \chi^a = \left( 1 - \frac{1}{2} \lambda \right) dx^a + \frac{1}{2} h^{ab} dy_b \]

\[ \psi_a = \left( 1 + \frac{1}{2} \lambda \right) h_{ab} dx^b - \frac{1}{2} dy_b \]

This is not quite symmetric, except in the limit \( \lambda \to \infty \).

5 Involution of the submanifold basis

Consider the exterior derivatives of the new basis forms:

\[ d\chi^a = \frac{1}{2} dh^{ab} dy_b \]

\[ d\chi^a = \frac{1}{2} d \left( \frac{4}{\lambda (y^2)} \left( y^a y^b - \frac{1}{2} \eta^{ab} y^2 \right) \right) dy_b \]

\[ d\chi^a = \frac{1}{2} \left( -\frac{16 \eta^{ab} dy_c}{\lambda (y^2)^2} \left( y^a y^b - \frac{1}{2} \eta^{ab} y^2 \right) + \frac{4}{\lambda (y^2)^2} \left( dy^a y^b + y^a dy^b - \eta^{ab} y^c dy_c \right) \right) dy_b \]

\[ d\chi^a = \frac{2}{\lambda (y^2)^2} \left( y^a dy_c + dy^a y^b dy_b \right) \]

\[ d\chi^a = \frac{2}{\lambda (y^2)^2} \left( \eta^{ab} y^a dy_b - y^a dy_b y^b \right) \]

\[ d\chi^a = \frac{2}{\lambda (y^2)^2} \left( \eta^{ab} y^a dy_b - y^a dy_b y^b \right) \]

so that \( \chi^a \) is closed, hence exact. For \( \psi_a \) we find

\[ d\psi_a = \left( 1 + \frac{1}{2} \lambda \right) dh_{ab} dx^b \]
Collecting terms,

\[
\left( \lambda + \frac{1}{2} \right) d \left( y_a y_b - \frac{1}{2} y^2 \eta_{ab} \right) d x^b
\]

\[
= \left( \lambda + \frac{1}{2} \right) \left( d y_a y_b + y_a \eta_{ac} y^c x^c - y_c \eta_{ab} \right) d x^b
\]

Now substitute

\[d y_a = (2 \lambda + 1) \left( y_a y_b - \frac{1}{2} \eta_{ab} y^2 \right) d x^b - 2 \psi_a\]

to find

\[
d \psi_a = \left( \lambda + \frac{1}{2} \right) \left( d y_a y_b + y_a \eta_{ac} y^c x^c - 2 \psi_a \right) y_b d x^b
\]

\[
+ \left( \lambda + \frac{1}{2} \right) y_a \left( 2 \lambda + 1 \right) \left( y_b y_c - \frac{1}{2} \eta_{bc} y^2 \right) d x^c - 2 \psi_a \right) d x^b
\]

\[
- \left( \lambda + \frac{1}{2} \right) y^c \left( 2 \lambda + 1 \right) \left( y_c y_d - \frac{1}{2} \eta_{cd} y^2 \right) d x^d - 2 \psi_c \right) \eta_{ab} d x^b
\]

\[
= \left( \lambda + \frac{1}{2} \right) \left( \frac{1}{2} (2 \lambda + 1) \eta_{ac} y_c y_b d x^c + 2 \psi_a y_b d x^b \right)
\]

\[
+ \left( \lambda + \frac{1}{2} \right) \left( \frac{1}{2} (2 \lambda + 1) \eta_{bc} y^2 x^c d x^b - 2 \psi_a \psi_b d x^b \right)
\]

\[
- \left( \lambda + \frac{1}{2} \right) \left( \frac{1}{2} (2 \lambda + 1) y^2 y_d \eta_{ab} d x^d - 2 \psi_a \psi_b \eta_{ab} d x^b \right)
\]

Collecting terms,

\[
d \psi_a = \frac{1}{4} (2 \lambda + 1)^2 y^2 \left( \eta_{ac} y_b + \eta_{bc} y_a + \eta_{ab} y_c \right) d x^b d x^c
\]

\[
- \left( 2 \lambda + 1 \right) \left( y_b \delta_a^c + y_a \delta_b^c - \eta_{ab} \right) \psi_x d x^b
\]

\[
= \left( 2 \lambda + 1 \right) \left( y_b \delta_a^c + y_a \delta_b^c - \eta_{ab} \right) \psi_x d x^b
\]

and we see that \( \psi_a \) is also involute.

### 6 Coordinates constant on the submanifolds

We would like to find independent coordinates on each submanifold, which, when held constant, project to the conjugate submanifold.

Let \( y_a = \frac{\beta z_a}{z^2} \). Then

\[
d y_a = \frac{\beta d z_a}{z^2} - \frac{2 \beta z_a z_b d z^b}{(z^2)^2}
\]

\[
= \frac{2 \beta}{(z^2)^2} \left( \frac{1}{2} z^2 \eta_{ab} - z_a z_b \right) d z^b
\]

\[
= - \frac{2}{\beta} \left( y_a y_b - \frac{1}{2} y^2 \eta_{ab} \right) d z^b
\]

\[
= - \frac{2}{\beta \lambda} h_{ab} d z^b
\]

Then, in terms of \( z_a \),

\[
\chi^a = \left( 1 - \frac{1}{2 \lambda} \right) d x^a + \frac{1}{2} h^{ab} d y_b
\]
\[
(1 - \frac{1}{2\lambda}) dx^a + \frac{1}{2} h_{ab} \left( -\frac{2}{\beta \lambda} h_{bc} \, dz^c \right) = (1 - \frac{1}{2\lambda}) dx^a - \frac{1}{\beta \lambda} dz^a
= \mathbf{d} \left( (1 - \frac{1}{2\lambda}) x^a - \frac{1}{\beta \lambda} z^a \right)
\]
which is indeed exact, and
\[
\psi_a = \left( 1 + \frac{1}{2\lambda} \right) h_{ab} dx^b - \frac{1}{2} dy_b
= \left( \lambda + \frac{1}{2} \right) \left( y_{ab} \frac{1}{2} \eta_{ab} \right)^2 \, dx^b - \frac{1}{2} (\frac{2\beta}{(z^2)^2}) \left( \frac{1}{2} \sqrt{\frac{\beta}{\lambda}} \delta^b_a - z_{a \beta} \right) \, dz^b
= \left( \lambda + \frac{1}{2} \right) \frac{\beta^2}{(z^2)^2} \left( z_{a \beta} \frac{1}{2} \eta_{ab} \right)^2 \, dx^b + \frac{\beta}{(z^2)^2} \left( z_{a \beta} \frac{1}{2} \eta_{ab} \right) \, dz^b
= \frac{\beta}{(z^2)^2} \left( z_{a \beta} \frac{1}{2} \eta_{ab} \right) \left( \beta \left( \lambda + \frac{1}{2} \right) \, dx^b + dz^b \right)
\]
which is involute. At this point it is natural to define coordinates,
\[
u^a = \left( 1 - \frac{1}{2\lambda} \right) x^a - \frac{1}{\beta \lambda} z^a
\]
\[
u^a = \left( 1 + \frac{1}{2\lambda} \right) x^a + \frac{1}{\beta \lambda} z^a
\]
so that
\[
\chi^a = \mathbf{d}\nu^a
\]
\[
\psi_a = \beta^2 \lambda \left( \frac{1}{2} \eta_{ab} \right) \, dz^b
\]
which are independent as long as we do not have
\[
(1 - \frac{1}{2\lambda}) x^a - \frac{1}{\beta \lambda} z^a = \alpha \left( 1 + \frac{1}{2\lambda} \right) x^a + \frac{1}{\beta \lambda} z^a
\]
which requires both
\[
(1 - \frac{1}{2\lambda}) x^a = \alpha \left( 1 + \frac{1}{2\lambda} \right) x^a
\]
\[-\frac{1}{\beta \lambda} z^a = \frac{\alpha}{\beta \lambda} z^a
\]
and therefore
\[
\alpha = -1 = \frac{1 - \frac{1}{2\lambda}}{1 + \frac{1}{2\lambda}}
\]
\[-\frac{1}{2\lambda} = 1 - \frac{1}{2\lambda}
\[-1 = 1
\]
which can never happen.

Now, recall that
\[-2\beta \left( z_a z_b - \frac{1}{2} \lambda \right) = -2 \frac{\beta \lambda}{(z^2)^2} h_{ab} \]
\[\beta^2 \lambda \left( z_a z_b - \frac{1}{2} \lambda \right) = h_{ab} \]
and we have
\[\chi^a = du^a \quad \psi_a = h_{ab} dv^b \]

Written in terms of \(u^a, v^b\), we can solve for \(z^a\),
\[\left( 1 + \frac{1}{2\lambda} \right) u^a = \left( 1 + \frac{1}{2\lambda} \right) \left( 1 - \frac{1}{2\lambda} \right) x^a - \frac{1}{\beta \lambda} \left( 1 + \frac{1}{2\lambda} \right) z^a \]
\[\left( 1 - \frac{1}{2\lambda} \right) v^a = \left( 1 - \frac{1}{2\lambda} \right) \left( 1 + \frac{1}{2\lambda} \right) x^a + \frac{1}{\beta \lambda} \left( 1 - \frac{1}{2\lambda} \right) z^a \]

Subtracting,
\[\left( 1 - \frac{1}{2\lambda} \right) v^a - \left( 1 + \frac{1}{2\lambda} \right) u^a = \frac{1}{\beta \lambda} \left( 1 + \frac{1}{2\lambda} \right) z^a + \frac{1}{\beta \lambda} \left( 1 - \frac{1}{2\lambda} \right) z^a \]
\[\frac{\beta}{4} (2\lambda - 1) v^a - \frac{\beta}{4} (2\lambda + 1) u^a = z^a \]

so the metric \(h_{ab}\) takes the form
\[h_{ab} = \frac{\beta^2 \lambda}{(z^2)^2} \left( z_a z_b - \frac{1}{2} \lambda \right) = \frac{\beta^2 \lambda}{(z^2)^2} \left( z_a z_b - \frac{1}{2} \lambda \right) \eta_{ab} \]

Now, check the orthonormality and the canonical conjugacy. With
\[\xi^a = -du^a \quad \psi_a = h_{ab} dv^b \]
we know the symplectic form is
\[\xi^a \psi_a = -du^a h_{ab} dv^b \neq dv^a du_a \]

which only appears canonical if we use \(h_{ab}\) to lower the index on \(d u^a\). But in the broader picture, we should use \(\eta_{ab}\) to lower the index, as expected. So \( -x^a \) and \( y_a \) are canonically conjugate coordinates, while \( du^a \) and \( h_{ab} dv^b \) are canonically conjugate.

## 7 The connection in involute coordinates

The final coordinate transformation is
\[u^a = \left( 1 - \frac{1}{2\lambda} \right) x^a - \frac{1}{\beta \lambda} z^a \]
\[v^a = \left( 1 + \frac{1}{2\lambda} \right) x^a + \frac{1}{\beta \lambda} z^a \]
with inverse

\[
x^a = \frac{1}{2} (u^a + v^a)
\]
\[
y_a = \frac{\beta z_a}{z^2}
\]
\[
= \frac{4 ((2 \lambda - 1) v^a - (2 \lambda + 1) u^a)}{(2 \lambda - 1)^2 v^a v_a - 2 (4 \lambda^2 - 1) v^a u_a + (2 \lambda + 1)^2 u^a u_a}
\]
\[
z^a = \frac{\beta}{4} ((2 \lambda - 1) v^a - (2 \lambda + 1) u^a)
\]
\[
z^2 = \frac{\beta^2}{4} \left((2 \lambda - 1)^2 v^a v_a - 2 (4 \lambda^2 - 1) v^a u_a + (2 \lambda + 1)^2 u^a u_a\right)
\]

These take a simpler form if we set \( \lambda = \frac{1}{2} \). Then

\[
u^a = \frac{1}{\beta \lambda} x^a
\]
\[
v^a = 2 x^a + \frac{2}{\beta} z^a
\]
\[
= 2 x^a - 2 \lambda u^a
\]

This means that the metric depends on \( u^a \) only, so that the momentum space is truly the cotangent space. The inverse is

\[
x^a = \frac{1}{2} (u^a + v^a)
\]
\[
y_a = \frac{\beta z_a}{z^2}
\]
\[
= -2 u^a
\]
\[
z^a = -\frac{\beta}{2} u^a
\]
\[
z^2 = \frac{\beta^2}{4} u^2
\]

The connection,

\[
\omega^a_b = 2 \Delta_{ac}^b \psi^c
\]
\[
e^a = dx^a
\]
\[
f_a = dy_a - \left(y_a y_b - \frac{1}{2} \eta_{ab}\right) e^b
\]
\[
\omega = -y_a dx^a
\]

takes the final form

\[
\omega^a_b = \frac{8 \lambda}{2 \lambda + 1} \frac{1}{(2 \lambda - 1)^2 v^a v_a - 2 (4 \lambda^2 - 1) v^a u_a + (2 \lambda + 1)^2 u^a u_a} \Delta_{abc}^b \left((2 \lambda - 1) v_c - (2 \lambda + 1) u_c\right) (du^d + dv^d)
\]
\[
\xi^a = -du^a
\]
\[
\psi_a = h_{ab} dv^b
\]
\[
= -\frac{2 (2 \lambda - 1) v_a - (2 \lambda + 1) u_a}{(2 \lambda - 1)^2 v^a v_a - 2 (4 \lambda^2 - 1) v^a u_a + (2 \lambda + 1)^2 u^a u_a} (du^d + dv^d)
\]

where

\[
h_{ab} = \frac{\beta^2 \lambda}{(z^2)^2} \left(z_a z_b - \frac{1}{2} z^2 \eta_{ab}\right)
\]

\]

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In the case where $\lambda = \frac{1}{2}$ the connection is therefore,

\begin{align*}
\omega^e_b &= -\frac{1}{u^2} \Delta_{ab}^{ac} u_c \left( du^d + dv^d \right) \\
\xi^a &= -du^a \\
\psi_a &= h_{ab} dv^b \\
\omega &= \frac{u_a}{u^2} \left( du^a + dv^a \right)
\end{align*}

where the spacetime metric is

\[
h_{ab} = \frac{1}{(u^2)^2} \left( 2u_a u_b - u^2 \eta_{ab} \right)
\]

References