Jackson, Chapter 5 notes

Magnetic field of a current loop using elliptic integrals

The vector potential is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

For a current loop with total current I, the current density is

$$\begin{aligned}\mathbf{J}(\mathbf{x}') &= \lambda\delta(r'-a)\,\delta(\cos\theta')\,\hat{\boldsymbol{\varphi}} \\ &= J_\varphi\hat{\boldsymbol{\varphi}} \\ &= J_\varphi\left(\hat{\mathbf{j}}\cos\varphi - \hat{\mathbf{i}}\sin\varphi\right)\end{aligned}$$

and we fix λ by integrating over the x^+z plane to get the total current, I. With $\hat{\mathbf{n}} = \hat{\mathbf{j}}$ the unit normal to the xz plane,

$$\begin{aligned}I &= \int \mathbf{n}\cdot\mathbf{J}(\mathbf{x}')\,d^2x' \\ &= \int_0^\infty rdr\int_0^\pi d\theta J_y(\mathbf{x}')\,d^2x' \\ &= \lambda\int_0^\infty rdr\int_0^\pi d\theta\delta(r'-a)\,\delta(\cos\theta')\,d^2x' \\ &= \lambda a\end{aligned}$$

so that

$$\mathbf{J}(\mathbf{x}') = \frac{I}{a}\delta(r'-a)\,\delta(\cos\theta')\,\hat{\boldsymbol{\varphi}}$$

Notice that Jackson inserts a factor of $\sin\theta'$ here, but this makes no difference since whenever $\delta(\cos\theta')$is nonzero, $\sin\theta' = 1$.

Now compute the vector potential.

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{\mu_0 I}{4\pi a}\int\frac{\delta(r'-a)\,\delta(\cos\theta')}{|\mathbf{x}-\mathbf{x}'|}r'^2\sin\theta' dr'd\varphi'd\theta'\hat{\boldsymbol{\varphi}}' \\ &= \frac{\mu_0 Ia}{4\pi}\int\frac{\hat{\mathbf{j}}\cos\varphi' - \hat{\mathbf{i}}\sin\varphi'}{|\mathbf{x}-\mathbf{x}'|}d\varphi' \\ &= A_\varphi\hat{\boldsymbol{\varphi}}\end{aligned}$$

Since the problem is azimuthally symmetric, there can be no φ-dependence in the potential. We may therefore choose any convenient angle φ for the *observation* point, $\mathbf{x}$. Choosing $\varphi = 0$, we compute $\mathbf{A}(r,\theta,0)$. The denominator is given by

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{r^2+a^2-2ar\mathbf{n}\cdot\mathbf{n}'}$$

where

$$\begin{aligned}\mathbf{n} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\ &= \frac{1}{r}\left(r\sin\theta\cos\varphi\hat{\mathbf{i}} + r\sin\theta\sin\varphi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}}\right) \\ &= \sin\theta\cos\varphi\hat{\mathbf{i}} + \sin\theta\sin\varphi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\ \mathbf{n}' &= \sin\theta'\cos\varphi'\hat{\mathbf{i}} + \sin\theta'\sin\varphi'\hat{\mathbf{j}} + \cos\theta'\hat{\mathbf{k}}\end{aligned}$$

Taking the dot product,

$$\begin{aligned}\mathbf{n}\cdot\mathbf{n}' &= \left(\sin\theta\cos\varphi\hat{\mathbf{i}}+\sin\theta\sin\varphi\hat{\mathbf{j}}+\cos\theta\hat{\mathbf{k}}\right)\cdot\left(\sin\theta'\cos\varphi'\hat{\mathbf{i}}+\sin\theta'\sin\varphi'\hat{\mathbf{j}}+\cos\theta'\hat{\mathbf{k}}\right)\\ &= \sin\theta\cos\varphi\sin\theta'\cos\varphi'+\sin\theta\sin\varphi\sin\theta'\sin\varphi'+\cos\theta\cos\theta'\\ &= \sin\theta\sin\theta'\cos\left(\varphi-\varphi'\right)+\cos\theta\cos\theta'\end{aligned}$$

Therefore,

$$\left|\mathbf{x}-\mathbf{x}'\right|=\sqrt{r^2+a^2-2ar\left(\sin\theta\sin\theta'\cos\left(\varphi-\varphi'\right)+\cos\theta\cos\theta'\right)}$$

and since we are evaluating at $\varphi=0$, and the current loop lies at $\theta'=\frac{\pi}{2}$, this reduces to

$$\left|\mathbf{x}-\mathbf{x}'\right|=\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}$$

We therefore have the potential in the form

$$\mathbf{A}\left(r,\theta,0\right) = \frac{\mu_0 Ia}{4\pi}\int\frac{\hat{\mathbf{j}}\cos\varphi'-\hat{\mathbf{i}}\sin\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}d\varphi'$$

Next, notice that

$$\frac{\cos\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}$$

is even in φ' while

$$\frac{\sin\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}$$

is odd. Since we are integrating over the full range of φ', the integral over the $\sin\varphi'$ term will vanish, leaving us with

$$\mathbf{A}\left(r,\theta,0\right) = \frac{\mu_0 Ia}{4\pi}\hat{\mathbf{j}}\int_0^{2\pi}\frac{\cos\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}d\varphi'$$

We now consider this integral.

Let

$$I \equiv \int_0^{2\pi}\frac{\cos\varphi' d\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}$$

and make the change of variable $\xi=\pi-\varphi'$. Then

$$\begin{aligned}\cos\varphi' &= \cos\left(\pi-\xi\right)\\ &= -\cos\xi\end{aligned}$$

and I becomes

$$\begin{aligned}I &= \int_\pi^{-\pi}\frac{\left(-\cos\xi\right)\left(-d\xi\right)}{\sqrt{r^2+a^2-2ar\sin\theta\left(-\cos\xi\right)}}\\ &= -\int_{-\pi}^{\pi}\frac{\cos\xi d\xi}{\sqrt{r^2+a^2+2ar\sin\theta\cos\xi}}\end{aligned}$$

We wish to cast this in terms of the complete elliptic integrals, defined by

$$\begin{aligned}K\left(k\right) &= \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\\ E\left(k\right) &= \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta\end{aligned}$$

To do this, we need to replace the $\cos\xi$ with something in terms of $\sin^2\alpha$. We can accomplish this with the half-angle formula,

$$\begin{aligned}\cos\xi &= \cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2} \\ &= 1 - 2\sin^2\frac{\xi}{2}\end{aligned}$$

Substituting,

$$\begin{aligned}I &= -\int_{-\pi}^{\pi} \frac{\left(1-2\sin^2\frac{\xi}{2}\right)d\xi}{\sqrt{r^2+a^2+2ar\sin\theta\left(1-2\sin^2\frac{\xi}{2}\right)}} \\ &= -\int_{-\pi}^{\pi} \frac{\left(1-2\sin^2\frac{\xi}{2}\right)d\xi}{\sqrt{r^2+a^2+2ar\sin\theta-4ar\sin\theta\sin^2\frac{\xi}{2}}} \\ &= -\frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}\int_{-\pi}^{\pi} \frac{\left(1-2\sin^2\frac{\xi}{2}\right)d\xi}{\sqrt{1-\left(\frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}\right)\sin^2\frac{\xi}{2}}}\end{aligned}$$

This gets the denominator in the right form, and we may define

$$k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}$$

so that

$$\begin{aligned}I &= -\int_{-\pi}^{\pi} \frac{\left(1-2\sin^2\frac{\xi}{2}\right)d\xi}{\sqrt{r^2+a^2+2ar\sin\theta\left(1-2\sin^2\frac{\xi}{2}\right)}} \\ &= -\int_{-\pi}^{\pi} \frac{\left(1-2\sin^2\frac{\xi}{2}\right)d\xi}{\sqrt{r^2+a^2+2ar\sin\theta-4ar\sin\theta\sin^2\frac{\xi}{2}}} \\ &= -\frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}\int_{-\pi}^{\pi} \frac{\left(1-2\sin^2\frac{\xi}{2}\right)d\xi}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}}\end{aligned}$$

Now we need to deal with the $\sin^2\frac{\xi}{2}$ term in the numerator. Notice that we may write

$$\begin{aligned}\frac{1}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}} - \sqrt{1-k^2\sin^2\frac{\xi}{2}} &= \frac{1}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}} - \frac{1-k^2\sin^2\frac{\xi}{2}}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}} \\ &= \frac{k^2\sin^2\frac{\xi}{2}}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}}\end{aligned}$$

so we may rewrite I as

$$I = -\frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}\int_{-\pi}^{\pi}\left[\frac{1}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}} - \frac{2\sin^2\frac{\xi}{2}}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}}\right]d\xi$$

$$= -\frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}\int_{-\pi}^{\pi}\left[\frac{1}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}}-\frac{2}{k^2}\left(\frac{1}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}}-\sqrt{1-k^2\sin^2\frac{\xi}{2}}\right)\right]d\xi$$

$$= -\frac{1}{k^2\sqrt{r^2+a^2+2ar\sin\theta}}\int_{-\pi}^{\pi}\left[\frac{k^2-2}{\sqrt{1-k^2\sin^2\frac{\xi}{2}}}+2\sqrt{1-k^2\sin^2\frac{\xi}{2}}\right]d\xi$$

Finally, let $\zeta = \frac{\xi}{2}$,

$$I = -\frac{2}{k^2\sqrt{r^2+a^2+2ar\sin\theta}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{k^2-2}{\sqrt{1-k^2\sin^2\zeta}}+2\sqrt{1-k^2\sin^2\zeta}\right]d\zeta$$

$$= -\frac{4}{k^2\sqrt{r^2+a^2+2ar\sin\theta}}\int_{0}^{\frac{\pi}{2}}\left[\frac{k^2-2}{\sqrt{1-k^2\sin^2\zeta}}+2\sqrt{1-k^2\sin^2\zeta}\right]d\zeta$$

$$= -\frac{4}{k^2\sqrt{r^2+a^2+2ar\sin\theta}}\left[\left(k^2-2\right)K(k)+2E(k)\right]$$

and substitute into the potential,

$$\mathbf{A}(r,\theta,0) = \frac{\mu_0 I a}{4\pi}\hat{\mathbf{j}}\int_0^{2\pi}\frac{\cos\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}d\varphi'$$

$$= -\frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}\frac{\mu_0 I a}{\pi k^2}\left[\left(k^2-2\right)K(k)+2E(k)\right]\hat{\mathbf{j}}$$

This agrees with eq.5.37 in Jackson.