Electromagnetic energy and momentum

Conservation of energy: the Poynting vector

In previous chapters of Jackson we have seen that the energy density of the electric (eq. 4.89 in Jackson) and magnetic (eq. 5.148) fields may be written as

$$\begin{aligned} u_E &= \frac{1}{2}\mathbf{E}\cdot\mathbf{D} \\ u_M &= \frac{1}{2}\mathbf{B}\cdot\mathbf{H} \end{aligned}$$

respectively. We assume that the total electromagnetic energy density is the sum of these,

$$u = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right)$$

We will assume that any medium is linear, so that $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$.

First, compute the work done by the electromagnetic fields on a system of particles. For a single particle, we have the Lorentz force law,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

and this provides the entire basis for extending mechanical energy and momentum to field energy and field momentum. The rate at which energy is delivered to a particle is $\mathbf{F}\cdot\mathbf{v} = q\left(\mathbf{v}\cdot\mathbf{E} + \mathbf{v}\cdot(\mathbf{v}\times\mathbf{B})\right) = q\mathbf{v}\cdot\mathbf{E}$. Now generalize to a continuous system of particles. The total rate of doing work for many particles is

$$\begin{aligned} \frac{dW}{dt} &= \sum \mathbf{F}_i\cdot\mathbf{v}_i \\ &= \sum q_i\mathbf{v}_i\cdot\mathbf{E}\left(x_i\right) \end{aligned}$$

In the continuum limit the sum becomes an integral while $q\mathbf{v}\to\rho\mathbf{v}\to\mathbf{J}$,

$$\frac{dW}{dt} = \int_V \mathbf{J}\cdot\mathbf{E}\, d^3x$$

Now we use Maxwell's equations to express this in terms of the fields.

$$\int_V \mathbf{J}\cdot\mathbf{E}\, d^3x = \int_V \left(\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{E}\, d^3x$$

We need to rearrange the curl term,

$$\begin{aligned} \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H}) &= \sum_{j,k,l} E_j\varepsilon_{jkl}\frac{\partial}{\partial x_k}H_l \\ &= \sum_{j,k,l}\varepsilon_{jkl}E_j\frac{\partial}{\partial x_k}H_l \\ &= \sum_{j,k,l}\varepsilon_{jkl}\left[\frac{\partial}{\partial x_k}\left(E_jH_l\right) - H_l\frac{\partial}{\partial x_k}E_j\right] \\ &= \sum_{j,k,l}\varepsilon_{jkl}\frac{\partial}{\partial x_k}\left(E_jH_l\right) - \sum_{j,k,l}\varepsilon_{jkl}H_l\frac{\partial}{\partial x_k}E_j \\ &= \frac{\partial}{\partial x_k}\sum_{j,k,l}\varepsilon_{jkl}\left(E_jH_l\right) + \sum_{j,k,l}\varepsilon_{lkj}H_l\frac{\partial}{\partial x_k}E_j \\ &= -\frac{\partial}{\partial x_k}\sum_{j,k,l}\varepsilon_{kjl}\left(E_jH_l\right) + \sum_{j,k,l}\varepsilon_{lkj}H_l\frac{\partial}{\partial x_k}E_j \\ &= -\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{H}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) \end{aligned}$$

Now replace the curl of the electric field, $\boldsymbol{\nabla}\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}$ and substitute back into the energy equation,

$$\begin{aligned}\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x &= \int_V\left(\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)d^3x\\ &= -\int_V\left(\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)d^3x\end{aligned}$$

Using linearity of the fields, the last two terms we write as the rate of change of the energy density,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{\mu}\mathbf{B}\cdot\mathbf{B}\right)\\ &= \frac{1}{2}\left(2\epsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{2}{\mu}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)\\ &= \mathbf{E}\cdot\frac{\partial(\epsilon\mathbf{E})}{\partial t}+\left(\frac{1}{\mu}\mathbf{B}\right)\cdot\frac{\partial\mathbf{B}}{\partial t}\\ &= \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x &= \int_V\left(\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)d^3x\\ &= -\int_V\left(\frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})\right)d^3x\\ 0 &= \int_V\left(\mathbf{J}\cdot\mathbf{E}+\frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})\right)d^3x\end{aligned}$$

Since the volume is arbitrary, the integrand must vanish at each point,

$$\frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})=-\mathbf{J}\cdot\mathbf{E}$$

The $\mathbf{J}\cdot\mathbf{E}$ term is the energy given up by the fields to do work on the particles. If there is no work done on any particles, then this vanishes and we have the continuity equation,

$$\frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{S}=0$$

where the electromagnetic current is given by the Poynting vector,

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}$$

which is interpreted to be the energy flowing across a unit area per second. This is clearer in the integral form if we use the divergence theorem on the Poynting term,

$$\begin{aligned}\frac{dW_{mech}}{dt} &= \int_V \mathbf{J}\cdot\mathbf{E}\,d^3x\\ &= -\int_V\frac{\partial u}{\partial t}\,d^3x-\int_V\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})\;d^3x\\ &= -\frac{d}{dt}\int_V u\,d^3x-\int_V\boldsymbol{\nabla}\cdot\mathbf{S}\,d^3x\\ &= -\frac{dW_{EM}}{dt}-\oint_S\mathbf{n}\cdot\mathbf{S}\,d^2x\end{aligned}$$

We have

$$\frac{dW_{mech}}{dt}+\frac{dW_{EM}}{dt} = -\oint_S \mathbf{n}\cdot\mathbf{S}\,d^2x$$

showing that the time rate of change of the total mechanical and electromagnetic energy in a region, V, is minus the rate at which energy flows out over the bounding surface, S. This is the reason that $\mathbf{S}$ has the interpretation as an energy flux.

Conservation of momentum: the Maxwell stress tensor

Conservation of momentum has led us to energy flux. A similar consideration of conservation of momentum leads us to momentum fluxes and stresses. Return to the Lorentz force law for a continuous system of particles,

$$\begin{aligned}\mathbf{F}_{mech} &= \frac{d\mathbf{P}_{mech}}{dt}\\ &= \int_V (\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})\,d^3x\end{aligned}$$

where we have taken $q\to\rho d^3x$ and $q\mathbf{v}\to\mathbf{J}d^3x$. Now replace the current density as before, but, because Jackson leaves the general case as an exercise, we work in vacuum,

$$\begin{aligned}\frac{d\mathbf{P}_{mech}}{dt} &= \int_V \left(\epsilon_0\mathbf{E}\,(\boldsymbol{\nabla}\cdot\mathbf{E})+\left(\epsilon_0 c^2\boldsymbol{\nabla}\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}\right)d^3x\\ &= \epsilon_0\int_V \left(\mathbf{E}\,(\boldsymbol{\nabla}\cdot\mathbf{E})-\mathbf{B}\times\left(c^2\boldsymbol{\nabla}\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial t}\right)\right)d^3x\\ &= \epsilon_0\int_V \left(\mathbf{E}\,(\boldsymbol{\nabla}\cdot\mathbf{E})-c^2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}\times\frac{\partial\mathbf{E}}{\partial t}\right)d^3x\end{aligned}$$

We can write the last term in terms of the Poynting vector,

$$\begin{aligned}\mathbf{B}\times\frac{\partial\mathbf{E}}{\partial t} &= \frac{\partial}{\partial t}(\mathbf{B}\times\mathbf{E})-\frac{\partial\mathbf{B}}{\partial t}\times\mathbf{E}\\ &= -\mu_0\frac{\partial\mathbf{S}}{\partial t}+(\boldsymbol{\nabla}\times\mathbf{E})\times\mathbf{E}\end{aligned}$$

and this makes the remaining expression almost symmetrical in the electric and magnetic fields,

$$\begin{aligned}\frac{d\mathbf{P}_{mech}}{dt} &= \epsilon_0\int_V \left(\mathbf{E}\,(\boldsymbol{\nabla}\cdot\mathbf{E})-c^2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})-\mu_0\frac{\partial\mathbf{S}}{\partial t}+(\boldsymbol{\nabla}\times\mathbf{E})\times\mathbf{E}\right)d^3x\\ \frac{d\mathbf{P}_{mech}}{dt}+\frac{d}{dt}\epsilon_0\mu_0\int_V \mathbf{S}\,d^3x &= \epsilon_0\int_V \big(\mathbf{E}\,(\boldsymbol{\nabla}\cdot\mathbf{E})-c^2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})-\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})\big)\,d^3x\end{aligned}$$

The right side of this equation would be symmetric in $\mathbf{E}$ and $c\mathbf{B}$ if we had an additional term of the form $c^2\mathbf{B}\,(\boldsymbol{\nabla}\cdot\mathbf{B})$. But this term is zero anyway because the divergence of the magnetic field vanishes, so we can add it in without changing anything,

$$\frac{d\mathbf{P}_{mech}}{dt}=-\int_V\frac{\partial}{\partial t}\left(\frac{1}{c^2}\mathbf{S}\right)d^3x+\epsilon_0\int_V\big(\mathbf{E}\,(\boldsymbol{\nabla}\cdot\mathbf{E})+c^2\mathbf{B}\,(\boldsymbol{\nabla}\cdot\mathbf{B})-c^2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})-\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})\big)\,d^3x$$

We would like to make the right side of this relationship look like the continuity equation again, but this time the density is a vector quantity, the momentum density,

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S}$$

For each component of momentum we want a corresponding divergence. This will make the right side take the form

$$\frac{\partial g_i}{\partial t} + \sum_j \frac{\partial}{\partial x_j} T_{ij}$$

so that, in the absence of mechanical work, the momentum density is conserved.

We recognize the divergence term with a simple rearrangement. Consider the electric field parts alone, since the magnetic parts will rearrange in the same way. We have

$$\begin{aligned}
\left[\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{E}\right) - \mathbf{E}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right)\right]_i &= E_i \sum_j \frac{\partial}{\partial x_j} E_j - \sum_{jk} \varepsilon_{ijk} E_j \left(\boldsymbol{\nabla}\times\mathbf{E}\right)_k \\
&= E_i \sum_j \frac{\partial}{\partial x_j} E_j - \sum_{jklm} \varepsilon_{ijk} E_j \varepsilon_{klm} \frac{\partial}{\partial x_l} E_m \\
&= E_i \sum_j \frac{\partial}{\partial x_j} E_j - \sum_{jlm} \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) E_j \frac{\partial}{\partial x_l} E_m \\
&= E_i \sum_j \frac{\partial}{\partial x_j} E_j - \sum_j \left(E_j \frac{\partial}{\partial x_i} E_j - E_j \frac{\partial}{\partial x_j} E_i \right) \\
&= E_i \sum_j \frac{\partial}{\partial x_j} E_j + \sum_j E_j \frac{\partial}{\partial x_j} E_i - \sum_j E_j \frac{\partial}{\partial x_i} E_j \\
&= \sum_j \frac{\partial}{\partial x_j} \left(E_i E_j\right) - \frac{1}{2}\frac{\partial}{\partial x_i} \sum_j E_j E_j \\
&= \sum_j \frac{\partial}{\partial x_j} \left(E_i E_j\right) - \frac{1}{2} \sum_j \frac{\partial}{\partial x_j} \left(\delta_{ij}\mathbf{E}^2\right) \\
&= \sum_j \frac{\partial}{\partial x_j} \left(E_i E_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2 \right)
\end{aligned}$$

The same calculation holds for the magnetic field so the full integrand becomes

$$\begin{aligned}
\varepsilon_0 \left(\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{E}\right) + c^2\mathbf{B}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right) - c^2\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{B}\right) - \mathbf{E}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right)\right)_i &= \sum_j \frac{\partial}{\partial x_j} \left[\epsilon_0 \left(E_i E_j + c^2 B_i B_j - \frac{1}{2}\delta_{ij}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)\right)\right] \\
&= \sum_j \frac{\partial}{\partial x_j} T_{ij}
\end{aligned}$$

where we have defined the Maxwell stress tensor:

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j + c^2 B_i B_j - \frac{1}{2}\delta_{ij}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)\right)$$

The final conservation law follows by using the moving the time derivative to the left and using the divergence theorem on the right. Setting the time derivative term to the time rate of change of total field momentum,

$$\int \frac{\partial}{\partial t}\left(\frac{1}{c^2}\mathbf{S}\right) d^3x = \int_V \frac{\partial}{\partial t}\mathbf{g}\, d^3x$$

$$
\begin{aligned}
&= \frac{d}{dt}\int_V \mathbf{g}\, d^3x \\
&= \frac{d\mathbf{P}_{EM}}{dt}
\end{aligned}
$$

and applying the divergence theorem to each component of T_{ij},

$$
\int_V \sum_j \frac{\partial}{\partial x_j} T_{ij}\, d^3x = \oint_S \sum_j n_j T_{ij}\, d^2x
$$

we have

$$
\frac{d\left(\mathbf{P}_{mech}\right)_i}{dt} + \frac{d\left(\mathbf{P}_{EM}\right)_i}{dt} = \epsilon_0 \oint_S \sum_j n_j T_{ij}\, d^2x
$$

For each component of momentum, the right side gives the momentum flowing across the bounding surface. Notice that the right side of the conservation law has a positive sign, even though n_j is the outward normal. This is because the way we have defined T_{ij} gives the negative of the momentum flux.

To check this sign, consider a plane electromagnetic wave with electric field in the x direction, magnetic field in the y direction, and propagating in the positive z direction. Then T_{ij} has components

$$
\begin{aligned}
T_{ij} &\equiv \epsilon_0 \left(E_i E_j + c^2 B_i B_j - \frac{1}{2}\delta_{ij}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)\right) \\
&= \epsilon_0 \begin{pmatrix} \frac{1}{2}E^2 & & \\ & \frac{1}{2}c^2B^2 & \\ & & -\frac{1}{2}\left(E^2 + c^2B^2\right) \end{pmatrix}
\end{aligned}
$$

Contracting with the unit vector $\mathbf{n} = \mathbf{k}$ in the direction of propagation gives

$$
\sum_j T_{ij} n_j = T_{i3} = -\frac{1}{2}\delta_{i3}\left(E^2 + c^2B^2\right)
$$

which has sign opposite from the expected increase in energy.

The energy and momentum fluxes, $\mathbf{S}$ and T_{ij}, completely characterize the electromagnetic contribution to energy and momentum. As such, they provide the electromagnetic source for the Einstein equation. For example, solving the combined Maxwell-Einstein equations gives the electromagnetic and gravitational field in the neighborhood of a neutron star or black hole, where there exist powerful magnetic fluxes produced by the rotating star.