1 Potentials for spheres

1.1 Point charge outside a grounded conducting sphere

We place image charges inside the grounded ($\Phi = 0$) sphere. By symmetry these must be cylindrically symmetric. But we guess that we can do it with a single charge, which then must lie along the vector pointing to the outside charge q. Let the coordinates be as in Jackson. Then the potential from q and its image q' is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|x\mathbf{n} - y\mathbf{n}'|} + \frac{q'}{|x\mathbf{n} - y'\mathbf{n}'|}\right)$$

On the spherical boundary, where $x = a$, we must have zero potential for all $\mathbf{n}$,

$$\begin{aligned} 0 &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|a\mathbf{n} - y\mathbf{n}'|} + \frac{q'}{|a\mathbf{n} - y'\mathbf{n}'|}\right) \\ 0 &= \frac{q}{\sqrt{a^2 + y^2 - 2ay\mathbf{n}\cdot\mathbf{n}'}} + \frac{q'}{\sqrt{a^2 + y'^2 - 2ay'\mathbf{n}\cdot\mathbf{n}'}} \end{aligned}$$

Here it is clear that q and q' must have opposite signs. Now, squaring and cross-multiplying,

$$\begin{aligned} q^2\left(a^2 + y'^2 - 2ay'\mathbf{n}\cdot\mathbf{n}'\right) &= q'^2\left(a^2 + y^2 - 2ay\mathbf{n}\cdot\mathbf{n}'\right) \\ q^2\left(a^2 + y'^2\right) - 2q^2ay'\mathbf{n}\cdot\mathbf{n}' &= q'^2\left(a^2 + y^2\right) - 2q'^2ay\mathbf{n}\cdot\mathbf{n}' \end{aligned}$$

For this to hold for all $\mathbf{n}$, the $\mathbf{n}\cdot\mathbf{n}'$ terms must match as well, so we must satisfy two relations,

$$\begin{aligned} 2q^2ay'\mathbf{n}\cdot\mathbf{n}' &= 2q'^2ay\mathbf{n}\cdot\mathbf{n}' \\ q^2\left(a^2 + y'^2\right) &= q'^2\left(a^2 + y^2\right) \end{aligned}$$

From the first, we see that

$$q'^2 = q^2\frac{y'}{y}$$

Substitititing into the second equation gives

$$\begin{aligned} q^2\left(a^2 + y'^2\right) &= q^2\frac{y'}{y}\left(a^2 + y^2\right) \\ a^2 + y'^2 &= y'\left(\frac{a^2 + y^2}{y}\right) \\ y'^2 - y'\left(\frac{a^2 + y^2}{y}\right) + a^2 &= 0 \end{aligned}$$

and solving the quadratic we find

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{a^2 + y^2}{y} \pm \sqrt{\left(\frac{a^2 + y^2}{y}\right)^2 - 4a^2}\right) \\ &= \frac{1}{2y}\left(a^2 + y^2 \pm \sqrt{\left(a^2 + y^2\right)^2 - 4a^2y^2}\right) \\ &= \frac{1}{2y}\left(a^2 + y^2 \pm \sqrt{\left(a^2 - y^2\right)^2}\right) \\ &= \frac{1}{2y}\left(a^2 + y^2 \pm \left(a^2 - y^2\right)\right) \\ &= \begin{cases} \frac{a^2}{y} \\ y \end{cases} \end{aligned}$$

The second solution is not allowed because this would place q' outside of the sphere. Therefore, we have

$$\begin{aligned} y' &= \frac{a^2}{y} \\ q' &= -q\frac{a}{y} \end{aligned}$$

The potential is therefore,

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|x\mathbf{n} - y\mathbf{n}'|} - \frac{a}{y\left|x\mathbf{n} - \frac{a^2}{y}\mathbf{n}'\right|}\right)$$

We easily check that when $\mathbf{x} = a\mathbf{n}$,

$$\begin{aligned} \Phi(a\mathbf{n}) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|a\mathbf{n} - y\mathbf{n}'|} - \frac{a}{y\left|a\mathbf{n} - \frac{a^2}{y}\mathbf{n}'\right|}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{a^2 + y^2 - 2ay\mathbf{n}\cdot\mathbf{n}'}} - \frac{a}{y\sqrt{a^2 + \frac{a^4}{y^2} - \frac{2a^3}{y}\mathbf{n}\cdot\mathbf{n}'}}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{a^2 + y^2 - 2ay\mathbf{n}\cdot\mathbf{n}'}} - \frac{1}{\sqrt{y^2 + a^2 - 2ay\mathbf{n}\cdot\mathbf{n}'}}\right) = 0 \end{aligned}$$

1.2 Charge density on the surface of the sphere

Consider properties of this solution. We know that

$$\frac{\partial\Phi}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

so we can compute the surface charge density on the sphere,

$$\begin{aligned} \sigma &= -\varepsilon_0\frac{\partial\Phi}{\partial n} \\ &= -\varepsilon_0 \left.\frac{\partial\Phi}{\partial x}\right|_{x=a} \\ &= -\varepsilon_0\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left.\left(\frac{1}{\sqrt{x^2 + y^2 - 2xy\mathbf{n}\cdot\mathbf{n}'}} - \frac{a}{y\sqrt{x^2 - \frac{2a^2x}{y}\mathbf{n}\cdot\mathbf{n}' + \frac{a^4}{y^2}}}\right)\right|_{x=a} \\ &= -\frac{q}{4\pi}\left(-\frac{1}{2}\right)\left.\left(\frac{2x - 2y\mathbf{n}\cdot\mathbf{n}'}{(x^2 + y^2 - 2xy\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{a\left(2x - \frac{2a^2}{y}\mathbf{n}\cdot\mathbf{n}'\right)}{y\left(x^2 - \frac{2a^2x}{y}\mathbf{n}\cdot\mathbf{n}' + \frac{a^4}{y^2}\right)^{3/2}}\right)\right|_{x=a} \\ &= \frac{q}{4\pi}\left(\frac{a - y\mathbf{n}\cdot\mathbf{n}'}{(a^2 + y^2 - 2ay\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{a^2 - \frac{a^3}{y}\mathbf{n}\cdot\mathbf{n}'}{y\left(a^2 - \frac{2a^3}{y}\mathbf{n}\cdot\mathbf{n}' + \frac{a^4}{y^2}\right)^{3/2}}\right) \\ &= \frac{q}{4\pi}\left(\frac{a - y\mathbf{n}\cdot\mathbf{n}'}{(a^2 + y^2 - 2ay\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{a - \frac{a}{y}\mathbf{n}\cdot\mathbf{n}'}{(y^2 - 2ay\mathbf{n}\cdot\mathbf{n}' + a^2)^{3/2}}\right) \end{aligned}$$

Try again

$$
\begin{aligned}
\sigma &= -\varepsilon_0 \left.\frac{\partial \Phi}{\partial x}\right|_{x=a} = -\varepsilon_0 \frac{\partial}{\partial x}\frac{1}{4\pi\varepsilon_0}\left.\left(\frac{q}{|x\mathbf{n}-y\mathbf{n}'|}+\frac{q'}{|x\mathbf{n}-y'\mathbf{n}'|}\right)\right|_{x=a} \\
&= -\frac{q}{4\pi}\frac{\partial}{\partial x}\left.\left(\frac{1}{\sqrt{x^2+y^2-2xy\mathbf{n}\cdot\mathbf{n}'}}-\frac{a}{y\sqrt{x^2+\frac{a^4}{y^2}-\frac{2a^2x}{y}\mathbf{n}\cdot\mathbf{n}'}}\right)\right|_{x=a} \\
&= \frac{1}{2}\frac{q}{4\pi}\left.\frac{2x-2y\mathbf{n}\cdot\mathbf{n}'}{(x^2+y^2-2xy\mathbf{n}\cdot\mathbf{n}')^{3/2}}-\frac{a\left(2x-\frac{2a^2}{y}\mathbf{n}\cdot\mathbf{n}'\right)}{y\left(x^2+\frac{a^4}{y^2}-\frac{2a^2x}{y}\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}}\right|_{x=a} \\
&= \frac{q}{4\pi}\left(\frac{a-y\mathbf{n}\cdot\mathbf{n}'}{(a^2+y^2-2ay\mathbf{n}\cdot\mathbf{n}')^{3/2}}-\frac{ay^3\left(a-\frac{a^2}{y}\mathbf{n}\cdot\mathbf{n}'\right)}{ya^3\left(y^2+a^2-2ay\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}}\right) \\
&= \frac{q}{4\pi}\left(\frac{a-y\mathbf{n}\cdot\mathbf{n}'}{(a^2+y^2-2ay\mathbf{n}\cdot\mathbf{n}')^{3/2}}-\frac{\frac{ay^2}{a^2}-y\mathbf{n}\cdot\mathbf{n}'}{(y^2+a^2-2ay\mathbf{n}\cdot\mathbf{n}')^{3/2}}\right) \\
&= \frac{aq}{4\pi}\left(1-\frac{y^2}{a^2}\right)\left(a^2+y^2-2ay\mathbf{n}\cdot\mathbf{n}'\right)^{-3/2}
\end{aligned}
$$

Therefore, the charge density on the sphere is

$$
\begin{aligned}
\sigma &= \frac{aq}{4\pi}\left(1-\frac{y^2}{a^2}\right)\left(a^2+y^2-2ay\mathbf{n}\cdot\mathbf{n}'\right)^{-3/2} \\
\sigma &= -\frac{aq}{4\pi}\frac{y^2}{a^2}\left(1-\frac{a^2}{y^2}\right)\frac{1}{y^3}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\mathbf{n}\cdot\mathbf{n}'\right)^{-3/2} \\
\sigma &= -\frac{q}{4\pi ay}\left(1-\frac{a^2}{y^2}\right)\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\mathbf{n}\cdot\mathbf{n}'\right)^{-3/2}
\end{aligned}
$$

To find the total charge induced on the sphere, we integrate this over the whole sphere. If we write $\mathbf{n}\cdot\mathbf{n}' = \cos\gamma$ there is no reason not to choose the direction of $\mathbf{n}'$ to be along the z-axis so that $\gamma = \theta$. Then the integral becomes

$$
\begin{aligned}
Q &= \int_0^{2\pi} d\varphi \int_0^{\pi} \sigma a^2 \sin\theta d\theta \\
&= -\frac{qa^2}{4\pi ay}\left(1-\frac{a^2}{y^2}\right)\int_0^{2\pi} d\varphi \int_0^{\pi}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\mathbf{n}\cdot\mathbf{n}'\right)^{-3/2}\sin\theta d\theta \\
&= -\frac{qa}{2y}\left(1-\frac{a^2}{y^2}\right)\int_{-1}^{1}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\xi\right)^{-3/2} d\xi
\end{aligned}
$$

where we have set $\xi = \cos\theta$. Then we need

$$\begin{aligned}
\int_{-1}^{1}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\xi\right)^{-3/2}d\xi &= \frac{y}{a}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\xi\right)^{-1/2}\Bigg|_{-1}^{1} \\
&= \frac{y}{a}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\right)^{-1/2}-\frac{y}{a}\left(1+\frac{a^2}{y^2}+\frac{2a}{y}\right)^{-1/2} \\
&= \frac{y}{a}\left(\frac{1}{1-\frac{a}{y}}-\frac{1}{1+\frac{a}{y}}\right) \\
&= \frac{2}{1-\frac{a^2}{y^2}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q &= -\frac{qa}{2y}\left(1-\frac{a^2}{y^2}\right)\frac{2}{1-\frac{a^2}{y^2}} \\
&= -\frac{qa}{y} \\
&= q'
\end{aligned}$$

This result may be seen immediately using Gauss's law.

1.3 Conducting sphere in a uniform electric field

We can solve the problem of a conducting sphere in a uniform electric field by writing the uniform field as a limit of a pair of opposite charges. If we have charges $-Q, Q$ located at $\mathbf{k}d$ and $-\mathbf{k}d$, respectively then their potential is given by

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\mathbf{x}+d\mathbf{k}|}-\frac{Q}{|\mathbf{x}-d\mathbf{k}|}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{x^2+d^2+2dz}}-\frac{Q}{\sqrt{x^2+d^2-2dz}}\right)
\end{aligned}$$

Consider the limit of this potential as both $d\to\infty$ and $Q\to\infty$, with $\lambda \equiv \frac{Q}{d^2}$ held constant. Then the limiting potential is

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\lim_{d,Q}\left(\frac{Q}{\sqrt{x^2+d^2+2dz}}-\frac{Q}{\sqrt{x^2+d^2-2dz}}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\lim_{d,Q}\left(\frac{Q}{d\sqrt{1+\frac{x^2}{d^2}+\frac{2z}{d}}}-\frac{Q}{d\sqrt{1+\frac{x^2}{d^2}-\frac{2z}{d}}}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\lim_{d,Q}\left(\frac{Q}{d}\left(1-\frac{x^2}{2d^2}-\frac{z}{d}+\ldots\right)-\frac{Q}{d}\left(1-\frac{x^2}{2d^2}+\frac{z}{d}+\ldots\right)\right) \\
&= \frac{1}{4\pi\varepsilon_0}\lim_{d,Q}\frac{Q}{d}\left(-\frac{2z}{d}-\ldots\right) \\
&= -\frac{\lambda z}{2\pi\varepsilon_0}
\end{aligned}$$

The resulting electric field is

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi = \frac{\lambda}{2\pi\varepsilon_0}\mathbf{k}$$

so we choose $\lambda = 2\pi\varepsilon_0 E_0$ to get the uniform field $\mathbf{E} = E_0\mathbf{k}$.

Now consider what happens when we insert the conducting sphere. We need to maintain the potential constant on the spherical surface. Since we are interested in the external field, we may include two image charges in the interior of the sphere, with the positions and magnitudes of the charges given by the results of the previous subsections. Therefore, we place a charge $Q' = -\frac{Qa}{d}$ at $-\mathbf{k}\frac{a^2}{d}$ and a second image charge $Q' = \frac{Qa}{d}$ at $\mathbf{k}\frac{a^2}{d}$. Now, for each value of d and Q, the sphere remains at constant potential.

The potential due to all four charges is now

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\mathbf{x}+d\mathbf{k}|} - \frac{aQ}{d\left|\mathbf{x}+\frac{a^2}{d}\mathbf{k}\right|} + \frac{aQ}{d\left|\mathbf{x}-\frac{a^2}{d}\mathbf{k}\right|} - \frac{Q}{|\mathbf{x}-d\mathbf{k}|}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{x^2+d^2+2dz}} - \frac{aQ}{d\sqrt{x^2+\frac{a^4}{d^2}+\frac{2a^2}{d}z}} + \frac{aQ}{d\sqrt{x^2+\frac{a^4}{d^2}-\frac{2a^2}{d}z}} - \frac{Q}{\sqrt{x^2+d^2-2dz}}\right)
\end{aligned}$$

and we take the same limit, with $\lambda = \frac{Q}{d^2} = 2\pi\varepsilon_0 E_0$. The first and fourth terms give the uniform field as before, so

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\lim_{d,Q}\left(\frac{Q}{|\mathbf{x}+d\mathbf{k}|} - \frac{aQ}{d\left|\mathbf{x}+\frac{a^2}{d}\mathbf{k}\right|} + \frac{aQ}{d\left|\mathbf{x}-\frac{a^2}{d}\mathbf{k}\right|} - \frac{Q}{|\mathbf{x}-d\mathbf{k}|}\right) \\
&= -E_0 z + \frac{1}{4\pi\varepsilon_0}\lim_{d,Q}\left(-\frac{aQ}{d\left|\mathbf{x}+\frac{a^2}{d}\mathbf{k}\right|} + \frac{aQ}{d\left|\mathbf{x}-\frac{a^2}{d}\mathbf{k}\right|}\right) \\
&= -E_0 z + \frac{aQ}{4\pi\varepsilon_0 d^2}\lim_{d,Q}\left(-\frac{d}{r\sqrt{1+\frac{a^4}{d^2r^2}+\frac{2a^2}{dr^2}z}} + \frac{d}{r\sqrt{1+\frac{a^4}{d^2r^2}-\frac{2a^2}{dr^2}z}}\right) \\
&= -E_0 z + \frac{1}{2}aE_0\lim_{d,Q}\left(-\frac{d}{r}\left(1-\frac{a^4}{2d^2r^2}-\frac{a^2}{dr^2}z\right) + \frac{d}{r}\left(1-\frac{a^4}{2d^2r^2}+\frac{a^2}{dr^2}z\right)\right) \\
&= -E_0 z + \frac{1}{2}aE_0\lim_{d,Q}\frac{d}{r}\left(\frac{a^2}{dr^2}z+\frac{a^2}{dr^2}z\right) = -E_0 z + \frac{1}{2}aE_0\frac{2a^2}{r^3}z \\
&= -\left(1-\frac{a^3}{r^3}\right)E_0 r\cos\theta
\end{aligned}$$

This has the correct limit, $E_0 z$, as $r \to \infty$.

2 Green function for a sphere

The general electrostatic problem for fields exterior to a sphere may be solved using the potential above and the Green function approach. Recall that the Green function for Dirichlet boundary conditions is required to satisfy

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

and

$$G(\mathbf{x}, \mathbf{x}')\big|_S = 0$$

on the boundary S. But the potential for a single charge outside a grounded conducting sphere,

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|x\mathbf{n}-y\mathbf{n}'|} - \frac{a}{y\left|x\mathbf{n}-\frac{a^2}{y}\mathbf{n}'\right|}\right)$$

satisfies

$$-\nabla^2\Phi(\mathbf{x}) = \frac{q}{\varepsilon_0}\left(\delta^3(x\mathbf{n}-y\mathbf{n}') - \frac{a}{y}\delta^3\left(x\mathbf{n}-\frac{a^2}{y}\mathbf{n}'\right)\right)$$

which reduces outside the sphere to

$$-\nabla^2\Phi(\mathbf{x}) = \frac{q}{\varepsilon_0}\delta^3(x\mathbf{n}-y\mathbf{n}')$$

since the second delta function is only nonzero inside the sphere.

This potential also satisfies the necessary boundary condition, so we have $G(\mathbf{x},\mathbf{x}') = \frac{4\pi\varepsilon_0}{q}\Phi$,

$$\begin{aligned}
G(\mathbf{x},\mathbf{x}') &= \frac{1}{|x\mathbf{n}-x'\mathbf{n}'|} - \frac{a}{x'\left|x\mathbf{n}-\frac{a^2}{x'}\mathbf{n}'\right|} \\
&= \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \\
&= \frac{1}{\sqrt{x^2+x'^2-2xx'\mathbf{n}\cdot\mathbf{n}'}} - \frac{a}{x'\sqrt{x^2+\frac{a^4}{x'^2}-\frac{2a^2x}{x'}\mathbf{n}\cdot\mathbf{n}'}}
\end{aligned}$$

with

$$\begin{aligned}
\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{x'=a} &= -\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial x'}\right|_{x'=a} \\
&= -\frac{\partial}{\partial x'}\left(\frac{1}{\sqrt{x^2+x'^2-2xx'\mathbf{n}\cdot\mathbf{n}'}} - \frac{a}{\sqrt{x^2x'^2+a^4-2a^2xx'\mathbf{n}\cdot\mathbf{n}'}}\right)\Bigg|_{x'=a} \\
&= -\left(-\frac{1}{2}\right)\left(\frac{2x'-2x\mathbf{n}\cdot\mathbf{n}'}{(x^2+x'^2-2xx'\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{a\left(2x^2x'-2a^2x\mathbf{n}\cdot\mathbf{n}'\right)}{(x^2x'^2+a^4-2a^2xx'\mathbf{n}\cdot\mathbf{n}')^{3/2}}\right)\Bigg|_{x'=a} \\
&= \frac{a-x\mathbf{n}\cdot\mathbf{n}'}{(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{a\left(ax^2-a^2x\mathbf{n}\cdot\mathbf{n}'\right)}{(a^2x^2+a^4-2a^3x\mathbf{n}\cdot\mathbf{n}')^{3/2}} \\
&= \frac{a-x\mathbf{n}\cdot\mathbf{n}'}{(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{a^2\left(x^2-ax\mathbf{n}\cdot\mathbf{n}'\right)}{a^3(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}} \\
&= \frac{1}{(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}}\left(a-x\mathbf{n}\cdot\mathbf{n}'-\frac{1}{a}x^2+x\mathbf{n}\cdot\mathbf{n}'\right) \\
&= \frac{a}{(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}}\left(1-\frac{x^2}{a^2}\right) \\
&= -\frac{x^2-a^2}{a(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}}
\end{aligned}$$

The solution to the general Dirichlet problem, in which we are given a charge density $\rho(\mathbf{x}')$ in the region outside a sphere of radius a, with the potential $\Phi(a\mathbf{n}')$ specified on the surface of the sphere. The the potential everywhere outside the sphere, V, is given by

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(a\mathbf{n}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}d^2x' \\
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x;^2}\mathbf{x}'\right|}\right)d^3x' + \frac{1}{4\pi}\oint_S \Phi(a\mathbf{n}')\frac{x^2-a^2}{a(x^2+a^2-2ax\mathbf{n}\cdot\mathbf{n}')^{3/2}}d^2x'
\end{aligned}$$