Waveguides and resonant cavities

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Essentially, a waveguide is a conducting tube of uniform cross-section and a cavity is a waveguide with end caps. The dimensions of the guide or cavity are chosen to transmit, hold or amplify particular forms of electromagnetic wave.

We will consider the case of a hollow tube extended in the z direction, with arbitrary but constant cross-sectional shape in the xy-plane. We consider possible wave solutions matching these boundary conditions.

1 Wave equations

As we have shown, the Maxwell equations,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0\\
\boldsymbol{\nabla}\cdot\mathbf{D} &= 0\\
\boldsymbol{\nabla}\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t} &= 0
\end{aligned}$$

give rise to wave equations,

$$\begin{aligned}
\Box\mathbf{E} &= 0\\
\Box\mathbf{B} &= 0
\end{aligned}$$

We assume that the electric and magnetic fields have spatial and time dependence of the form

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}(x,y)\,e^{\pm ikz-i\omega t}\\
\mathbf{B} &= \mathbf{B}(x,y)\,e^{\pm ikz-i\omega t}
\end{aligned}$$

for waves traveling in the $\pm z$ direction. Unlike our plane wave solutions, however, the fields must satisfy boundary conditions in the x and y directions, along the sides of the waveguide.

Separating the del operator into longitudinal (z) and transverse parts,

$$\boldsymbol{\nabla}=\boldsymbol{\nabla}_t+\hat{\mathbf{k}}\frac{\partial}{\partial z},$$

the d'Alembertian becomes

$$\begin{aligned}
\Box &= -\mu\epsilon\frac{\partial^2}{\partial t^2}+\nabla^2\\
&= -\mu\epsilon\frac{\partial^2}{\partial t^2}+\nabla_t^2+\frac{\partial^2}{\partial z^2}
\end{aligned}$$

Substituting the assumed time and z-dependence into the wave equations gives

$$\begin{aligned}
0 &= \left(\nabla_t^2-\mu\epsilon\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{E}\\
&= \left(\nabla_t^2-\mu\epsilon\,(-i\omega)^2+(ik)^2\right)\mathbf{E}
\end{aligned}$$

and, with the corresponding result for $\mathbf{B}$,

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\mathbf{E} \\ 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\mathbf{B} \end{aligned}$$

2 General solution: Separating transverse and longitudinal components

We can simplify the problem by separating the transverse, $\mathbf{E}_t$, and longitudinal, $\hat{\mathbf{k}}E_z$, parts, then treating E_z and B_z as sources for the transverse parts. We can use the z-component of wave equation together with the boundary conditions to solve for the "sources", E_z, B_z. Defining

$$\begin{aligned} \boldsymbol{\nabla} &= \boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z} \\ \mathbf{E} &= \mathbf{E}_t + \hat{\mathbf{k}}E_z \\ \mathbf{E}_t &= \left(\hat{\mathbf{k}} \times \mathbf{E}\right) \times \hat{\mathbf{k}} \end{aligned}$$

and similarly for $\mathbf{B}$, the source-free Maxwell equations become

$$\begin{aligned} \boldsymbol{\nabla}_t \cdot \mathbf{B}_t \pm ikB_z &= 0 \\ \boldsymbol{\nabla}_t \cdot \mathbf{E}_t \pm ikE_z &= 0 \\ \left(\boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) - i\omega\left(\mathbf{B}_t + \hat{\mathbf{k}}B_z\right) &= 0 \\ \left(\boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{B}_t + \hat{\mathbf{k}}B_z\right) + i\epsilon\mu\omega\left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) &= 0 \end{aligned}$$

The first two equations have the form we are after,

$$\begin{aligned} \boldsymbol{\nabla}_t \cdot \mathbf{B}_t &= \mp ikB_z \\ \boldsymbol{\nabla}_t \cdot \mathbf{E}_t &= \mp ikE_z \end{aligned}$$

and we turn out attention to the curl equations.

The curl terms expand as

$$\begin{aligned} \left(\boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) &= \boldsymbol{\nabla}_t \times \mathbf{E}_t + \boldsymbol{\nabla}_t \times \hat{\mathbf{k}}E_z + \hat{\mathbf{k}} \times \frac{\partial \mathbf{E}_t}{\partial z} + \hat{\mathbf{k}} \times \hat{\mathbf{k}}\frac{\partial E_z}{\partial z} \\ &= \boldsymbol{\nabla}_t \times \mathbf{E}_t + \hat{\mathbf{i}} \times \hat{\mathbf{k}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{j}} \times \hat{\mathbf{k}}\frac{\partial E_z}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \mathbf{E}_t}{\partial z} \\ &= \boldsymbol{\nabla}_t \times \mathbf{E}_t - \hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \mathbf{E}_t}{\partial z} \end{aligned}$$

and similarly for the magnetic terms. Thus

$$\begin{aligned} \boldsymbol{\nabla}_t \times \mathbf{E}_t - \hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{E}_t - i\omega\left(\mathbf{B}_t + \hat{\mathbf{k}}B_z\right) &= 0 \\ \boldsymbol{\nabla}_t \times \mathbf{B}_t - \hat{\mathbf{j}}\frac{\partial B_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial B_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{B}_t + i\varepsilon\mu\omega\left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) &= 0 \end{aligned}$$

Noting that $\boldsymbol{\nabla}_t \times \mathbf{E}_t$ lies in the z-direction, we separate the transverse and longitudinal parts of each equation,

$$\begin{aligned}
-\hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{E}_t - i\omega\mathbf{B}_t &= 0 \\
\boldsymbol{\nabla}_t \times \mathbf{E}_t - i\omega B_z\hat{\mathbf{k}} &= 0 \\
-\hat{\mathbf{j}}\frac{\partial B_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial B_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{B}_t + i\epsilon\mu\omega\mathbf{E}_t &= 0 \\
\boldsymbol{\nabla}_t \times \mathbf{B}_t + i\epsilon\mu\omega E_z\hat{\mathbf{k}} &= 0
\end{aligned}$$

We can still simplify the first and third. Since they are transverse, we lose no information by taking the cross product with $\hat{\mathbf{k}}$. For the first this becomes

$$\begin{aligned}
0 &= \hat{\mathbf{k}} \times \left(-\hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{E}_t - i\omega\mathbf{B}_t\right) \\
&= \hat{\mathbf{i}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{j}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times ik\mathbf{E}_t\right) - i\omega\hat{\mathbf{k}} \times \mathbf{B}_t \\
&= \boldsymbol{\nabla}_t E_z \pm (-ik\mathbf{E}_t) - i\omega\hat{\mathbf{k}} \times \mathbf{B}_t \\
0 &= -\boldsymbol{\nabla}_t E_z \pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \mathbf{B}_t
\end{aligned}$$

and similarly for the third.

3 General solution: Separating $\mathbf{E}_t$ and $\mathbf{B}_t$

In the resulting pair of equations,

$$\begin{aligned}
\pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \mathbf{B}_t &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{B}_t - i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t &= \boldsymbol{\nabla}_t B_z
\end{aligned}$$

the transverse $\mathbf{E}_t$ and $\mathbf{B}_t$ are still coupled. To separate them, solve the second for $\mathbf{B}_t$,

$$\mathbf{B}_t = \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t\right)$$

and substitute into the first,

$$\begin{aligned}
\pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \mathbf{B}_t &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \left(\pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t\right)\right) &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{E}_t \pm \frac{\omega}{k}\hat{\mathbf{k}} \times \left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t\right) &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{E}_t \pm i\epsilon\mu\frac{\omega^2}{k}\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{E}_t\right) &= \boldsymbol{\nabla}_t E_z \mp \frac{\omega}{k}\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \\
\pm\left(ik - i\epsilon\mu\frac{\omega^2}{k}\right)\mathbf{E}_t &= \boldsymbol{\nabla}_t E_z \mp \frac{\omega}{k}\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \\
\pm\frac{i}{k}\left(k^2 - \epsilon\mu\omega^2\right)\mathbf{E}_t &= \boldsymbol{\nabla}_t E_z \mp \frac{\omega}{k}\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z
\end{aligned}$$

and therefore,

$$\mathbf{E}_t = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right)$$

Substituting this back into the expression for $\mathbf{B}_t$,

$$\begin{aligned}
\mathbf{B}_t &= \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}}\times\mathbf{E}_t\right) \\
&= \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z - \frac{\epsilon\mu\omega}{\epsilon\mu\omega^2-k^2}\hat{\mathbf{k}}\times\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\right) \\
&= \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z - \frac{\epsilon\mu\omega}{\epsilon\mu\omega^2-k^2}\left(\pm k\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z + \omega\boldsymbol{\nabla}_t B_z\right)\right) \\
&= \frac{1}{ik}\left(-\frac{\epsilon\mu\omega}{\epsilon\mu\omega^2-k^2}k\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \mp \frac{\epsilon\mu\omega}{\epsilon\mu\omega^2-k^2}\omega\boldsymbol{\nabla}_t B_z \pm \boldsymbol{\nabla}_t B_z\right) \\
&= \frac{i}{k}\left(\frac{\epsilon\mu\omega}{\epsilon\mu\omega^2-k^2}k\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \pm \left(\frac{\epsilon\mu\omega}{\epsilon\mu\omega^2-k^2}\omega - 1\right)\boldsymbol{\nabla}_t B_z\right) \\
&= \frac{i}{\epsilon\mu\omega^2-k^2}\left(\epsilon\mu\omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \pm k\boldsymbol{\nabla}_t B_z\right)
\end{aligned}$$

and we have solved for the transverse fields in terms of the longitudinal ones:

$$\begin{aligned}
\mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2-k^2}\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right) \\
\mathbf{B}_t &= \frac{i}{\epsilon\mu\omega^2-k^2}\left(\epsilon\mu\omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \pm k\boldsymbol{\nabla}_t B_z\right)
\end{aligned}$$

To have a complete solution, we must check the remaining Maxwell equations,

$$\begin{aligned}
\boldsymbol{\nabla}_t\cdot\mathbf{B}_t &= \mp ikB_z \\
\boldsymbol{\nabla}_t\cdot\mathbf{E}_t &= \mp ikE_z \\
\boldsymbol{\nabla}_t\times\mathbf{E}_t &= i\omega B_z\hat{\mathbf{k}} \\
\boldsymbol{\nabla}_t\times\mathbf{B}_t &= -i\epsilon\mu\omega E_z\hat{\mathbf{k}}
\end{aligned}$$

The divergence equations become

$$\boldsymbol{\nabla}_t\cdot\mathbf{E}_t = \frac{i}{\epsilon\mu\omega^2-k^2}\left(\pm k\boldsymbol{\nabla}_t\cdot\boldsymbol{\nabla}_t E_z - \omega\boldsymbol{\nabla}_t\cdot\left(\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\right)$$

Using the reduced form of the wave equation,

$$0 = \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\mathbf{E}$$

together with

$$\begin{aligned}
\boldsymbol{\nabla}_t\cdot\left(\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right) &= \sum_{i,k}\nabla_i^t\varepsilon_{i3k}\nabla^t B_z \\
&= -\left(\nabla_1^t\nabla_2^t B_z - \nabla_2^t\nabla_1^t B_z\right) \\
&= 0
\end{aligned}$$

the right side simplifies,

$$\begin{aligned}
\boldsymbol{\nabla}_t\cdot\mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2-k^2}\left(\pm k\left(k^2-\mu\epsilon\omega^2\right)E_z\right) \\
&= \mp ikE_z
\end{aligned}$$

as required.

For the curl,

$$\begin{aligned}
\boldsymbol{\nabla}_t \times \mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2} \boldsymbol{\nabla}_t \times \left(\pm k \boldsymbol{\nabla}_t E_z - \omega \hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \right) \\
&= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2} \boldsymbol{\nabla}_t \times \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \right)
\end{aligned}$$

Sorting out the double curl,

$$\begin{aligned}
\left[\boldsymbol{\nabla}_t \times \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \right) \right]_i &= \sum_{j,k} \varepsilon_{ijk} \nabla_j^t \left[\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \right]_k \\
&= \sum_{j,k,m} \varepsilon_{ijk} \nabla_j^t \left(\varepsilon_{k3m} \nabla_m^t B_z \right) \\
&= \sum_{j,k,m} \varepsilon_{ijk} \varepsilon_{3mk} \nabla_j^t \nabla_m^t B_z \\
&= \sum_{j,m} \left(\delta_{i3} \delta_{jm} - \delta_{im} \delta_{j3} \right) \nabla_j^t \nabla_m^t B_z \\
&= \left(\delta_{i3} \sum_j \nabla_j^t \nabla_j^t B_z - \nabla_3^t \nabla_i^t B_z \right) \\
\boldsymbol{\nabla}_t \times \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \right) &= \hat{\mathbf{k}} \boldsymbol{\nabla}_t^2 B_z
\end{aligned}$$

The second term vanishes because $\boldsymbol{\nabla}_t$ has no z component. Therefore, using the wave equation again,

$$\begin{aligned}
\boldsymbol{\nabla}_t \times \mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2} \hat{\mathbf{k}} \nabla_t^2 B_z \\
&= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2} \hat{\mathbf{k}} \left(k^2 - \mu\epsilon\omega^2 \right) B_z \\
&= i\omega B_z \hat{\mathbf{k}}
\end{aligned}$$

The corresponding divergence and curl of $\mathbf{B}_t$ are left as exercises for the reader.

4 Characteristics of solutions

Our general method of solution is now to solve

$$\begin{aligned}
0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2 \right) E_z \\
0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2 \right) B_z
\end{aligned}$$

with the appropriate boundary conditions, then use these solutions to solve

$$\begin{aligned}
\mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2} \left(\pm k \boldsymbol{\nabla}_t E_z - \omega \hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z \right) \\
\mathbf{B}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2} \left(\epsilon\mu\omega \hat{\mathbf{k}} \times \boldsymbol{\nabla}_t E_z \pm k \boldsymbol{\nabla}_t B_z \right)
\end{aligned}$$

for the transverse parts. We consider three special cases, depending on one or both of E_z and B_z vanishing:

1. If both E_z and B_z vanish, then both the electric and magnetic fields are purely transverse. These solutions are called TEM waves (Transverse Electric and Magnetic).

2. If E_z vanishes, then the electric field is purely transverse. These solutions are called TE waves (Transverse Electric).

3. If B_z vanishes, then the magnetic field is purely transverse. These solutions are called TM waves (Transverse Magnetic).

Generic waves are a combination of all three. This method sketched above works for TE and TM waves (see below), but in the case where $E_z = B_z = 0$ we have $\epsilon\mu\omega^2 - k^2 = 0$ and this solution fails. We treat this special TEM case first, then use the technique to discusss TE and TM waves.

4.1 Transverse electromagnetic waves: TEM

Transverse electromagnetic waves in a waveguide are those with no z-component to the fields,

$$\begin{aligned} E_z &= 0 \\ B_z &= 0 \end{aligned}$$

In this case, our original equations have zero source,

$$\begin{aligned} \boldsymbol{\nabla}_t \cdot \mathbf{B}_{TEM} &= 0 \\ \boldsymbol{\nabla}_t \cdot \mathbf{E}_{TEM} &= 0 \\ \boldsymbol{\nabla}_t \times \mathbf{E}_{TEM} &= 0 \\ \boldsymbol{\nabla}_t \times \mathbf{B}_{TEM} &= 0 \end{aligned}$$

so that the transverse electric and magnetic fields satisfy the Laplace equation of electrostatics in 2-dimensions. This means that our general solution above cannot be used because $\mu\epsilon\omega^2 - k^2 = 0$. Instead, the relevant solution to the 2-dimensional Laplace equation is determined purely by its boundary conditions. Since a closed conductor allows no field inside, TEM waves cannot exist inside a completely enclosed, perfectly conducting cavity.

We also have

$$\begin{aligned} \pm k\mathbf{E}_{TEM} + \omega\hat{\mathbf{k}} \times \mathbf{B}_{TEM} &= 0 \\ \pm k\mathbf{B}_{TEM} - \epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_{TEM} &= 0 \end{aligned}$$

Combining these last two equations,

$$\begin{aligned} k^2\mathbf{E}_{TEM} + \epsilon\mu\omega^2\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{E}_{TEM}\right) &= 0 \\ \left(k^2 - \epsilon\mu\omega^2\right)\mathbf{E}_{TEM} &= 0 \end{aligned}$$

we see that we must have

$$k = k_0 = \sqrt{\epsilon\mu}\omega$$

and the magnetic field satisfies

$$\mathbf{B}_{TEM} = \pm\sqrt{\epsilon\mu}\hat{\mathbf{k}} \times \mathbf{E}_{TEM}$$

TEM waves are the dominant mode in a coaxial cable: inner and outer cylindrical conductors held at opposite potential lead to a radial electric field, while opposite currents on the conductors lead to an azimuthal magnetic field. These are transverse to the direction along the cable, so waves propagate along the cable between the conductors.

4.2 Transverse electric, TE, modes and transverse magnetic, TM, modes

4.2.1 Boundary conditions

Modes driven by nonzero E_z and/or B_z fall into two categories. To see why, consider a perfectly conducting boundary.

For a perfectly conducting waveguide, we find the boundary conditions using the assumption that free charges move instantly to produce whatever surface charge density, Σ, and surface current density, $\mathbf{K}$, are required to make the electric and magnetic fields vanish inside the conductor. The full boundary conditions are therefore

$$\begin{aligned} \mathbf{n}\cdot\mathbf{D} &= \Sigma \\ \mathbf{n}\times\mathbf{H} &= \mathbf{K} \\ \mathbf{n}\times\mathbf{E} &= 0 \\ \mathbf{n}\cdot\mathbf{B} &= 0 \end{aligned}$$

There can therefore be no tangential component of the electric field at the surface, and no normal component of the magnetic field.

For the longitudinal electric field, the boundary condition is

$$\left(\mathbf{n}\times\hat{\mathbf{k}}\right)E_z = 0$$

so that $E_z = 0$ at the boundary. For the boundary condition on B_z we start with our separation of the Maxwell equations into longitudinal and transverse components, where we found

$$ik\mathbf{B}_t - i\epsilon\mu\omega\hat{\mathbf{k}}\times\mathbf{E}_t = \boldsymbol{\nabla}_t B_z$$

Consider the normal component of this equation,

$$\begin{aligned} ik\mathbf{n}\cdot\mathbf{B}_t - i\epsilon\mu\omega\mathbf{n}\cdot\left(\hat{\mathbf{k}}\times\mathbf{E}_t\right) &= \mathbf{n}\cdot\boldsymbol{\nabla}_t B_z \\ ik\left(\mathbf{n}\cdot\mathbf{B}_t\right) - i\epsilon\mu\omega\hat{\mathbf{k}}\cdot\left(\mathbf{E}_t\times\mathbf{n}\right) &= \frac{\partial B_z}{\partial n} \end{aligned}$$

The left side of this equation vanishes by the boundary conditions, so we must have

$$\frac{\partial B_z}{\partial n} = 0$$

at the surface as well.

Therefore, we seek solutions to the 2-dimensional wave equations

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)E_z \\ 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)B_z \end{aligned}$$

with boundary conditions

$$\begin{aligned} E_z\big|_S &= 0 \\ \left.\frac{\partial B_z}{\partial n}\right|_S &= 0 \end{aligned}$$

Since each of these components also satisfies the original wave equation,

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)E_z \\ 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)B_z \end{aligned}$$

we have well-defined eigenvalue problems for E_z and B_z. Since the transverse direction is a bounded region, we expect a discrete set of eigenvalues. Because the boundary conditions are different but the equations the same, uniqueness guarantees that the spectrum of allowed values will be different for E_z and B_z. This means that at a given resonant frequency, in general only one or the other source field will be excited. This divides the solutions into two types,

1. TE waves: The electric field is transverse, i.e., $E_z = 0$ everywhere while B_z satisfies the boundary condition $\left.\frac{\partial B_z}{\partial n}\right|_S = 0$.
2. TM waves: The magnetic field is transverse so that $B_z = 0$ everywhere while E_z satisfies the boundary condition $\left.E_z\right|_S = 0$.

A general solution for the field in a waveguide or cavity is a superposition of TE, TM and TEM waves.

4.2.2 The transverse fields

First, we simplify our solutions for transverse fields in the TE and TM cases.

For TM waves, we set $B_z = 0$. Then

$$\begin{aligned}
\mathbf{E}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\boldsymbol{\nabla}_t E_z \\
\mathbf{B}_t &= \frac{i\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t E_z\right)
\end{aligned}$$

Taking the curl of the transverse electric field,

$$\begin{aligned}
\hat{\mathbf{k}} \times \mathbf{E}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t E_z \\
&= \pm\frac{ik}{i\epsilon\mu\omega}\mathbf{B}_t \\
&= \pm\frac{k}{\epsilon\omega}\mathbf{H}_t
\end{aligned}$$

so that

$$\mathbf{H}_t = \pm\frac{\epsilon\omega}{k}\hat{\mathbf{k}} \times \mathbf{E}_t$$

For TE waves, $E_z = 0$ we get a similar result,

$$\begin{aligned}
\mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\
\mathbf{B}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\left(\boldsymbol{\nabla}_t B_z\right)
\end{aligned}$$

so substituting the second equation into the first,

$$\begin{aligned}
\mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\
&= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \frac{\epsilon\mu\omega^2 - k^2}{\pm ik}\mathbf{B}_t\right) \\
&= \frac{-i\omega}{\pm ik}\hat{\mathbf{k}} \times \mathbf{B}_t
\end{aligned}$$

so taking the cross product with $\hat{\mathbf{k}}$,

$$\begin{aligned}
\hat{\mathbf{k}} \times \mathbf{E}_t &= \mp\frac{\omega}{k}\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{B}_t\right) \\
&= \pm\frac{\omega}{k}\mathbf{B}_t
\end{aligned}$$

so that

$$\mathbf{H}_t = \pm\frac{k}{\mu\omega}\hat{\mathbf{k}} \times \mathbf{E}_t$$

Both of these relationships, for TM and TE waves, have the form

$$\mathbf{H}_t = \pm\frac{1}{Z}\hat{\mathbf{k}} \times \mathbf{E}_t$$

where, using $k_0 = \sqrt{\varepsilon\mu}\omega$,

$$Z = \begin{cases} \frac{k}{\epsilon\omega} = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}} & TM\ modes \\ \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}} & TE\ modes \end{cases}$$

The quantity Z is called the wave impedence.

This gives solutions of the form:

$$\begin{aligned} \mathbf{E}_t &= \pm\frac{ik}{\epsilon\mu\omega^2 - k^2}\boldsymbol{\nabla}_t E_z \\ \mathbf{H}_t &= \pm\frac{1}{Z}\hat{\mathbf{k}} \times \mathbf{E}_t \end{aligned}$$

with $Z = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}}$ for TM and

$$\begin{aligned} \mathbf{E}_t &= -\frac{i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\ \mathbf{H}_t &= \pm\frac{1}{Z}\hat{\mathbf{k}} \times \mathbf{E}_t \end{aligned}$$

with $Z = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}}$ for TE.

4.2.3 The eigenvalue problem

We want to find solutions for TE and TM modes. These will differ from the TEM mode due to the presence of either nonzero E_z or nonzero B_z, which provide the source for the transverse fields.

Denote either of these source fields by ψ,

$$\psi = \begin{cases} E_z & TM\ modes \\ B_z & TE\ modes \end{cases}$$

and define

$$\gamma^2 \equiv \mu\epsilon\omega^2 - k^2$$

Then the reduced wave equation for ψ becomes an eigenvalue problem,

$$\nabla_t^2\psi = -\gamma^2\psi$$

with boundary conditons

$$\begin{aligned} \psi|_S &= 0 \quad TM \\ \left.\frac{\partial\psi}{\partial n}\right|_S &= 0 \quad TE \end{aligned}$$

Since these boundary conditions are periodic, the wave equation will have a discrete spectrum of allowed values for the constant, $\gamma = \gamma_\lambda$. These correspond to discrete values of the wavelength, or equivalently, wave number k, given by

$$k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$$

The Laplacian will have wave modes (rather than exponential modes) only for $\gamma^2 > 0$. This leads to a cutoff frequency, ω_λ,

$$\begin{aligned}\mu\epsilon\omega^2 &> \gamma_\lambda^2 \\ \omega \geq \omega_\lambda &\equiv \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}}\end{aligned}$$

Expressing k_λ in terms of frequency, we have

$$\begin{aligned}k_\lambda &= \sqrt{\mu\epsilon\omega^2 - \gamma_\lambda^2} \\ &= \sqrt{\mu\epsilon}\sqrt{\omega^2 - \frac{\gamma_\lambda^2}{\mu\varepsilon}} \\ &= \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\lambda^2}\end{aligned}$$

so the wave vector becomes imaginary, leading to attenuation, for frequencies lower than ω_λ.

Now consider a fixed frequency, as we allow ω_λ to range over the possible eigenvalues. There is some maximum allowed cutoff frequency, corresponding to some minimum value of k_λ and therefore some maximum wavelength that can propagate at that frequency. As a result, only a finite number of possible wavelengths can propagate. It is possible to choose the dimensions of the waveguide so that at the desired frequency (or frequency range), only a single wavelength can propagate.

Once we have solved for ψ, we can find the transverse fields from the expressions above.

4.3 Example: TE modes in a rectangular waveguide

Suppose we have TE modes in a rectangular waveguide. Let the cross-section of the guide run from $x = 0$ to $x = a$, and from $y = 0$ to $y = b$. Then, with $\gamma = H_z$,we solve

$$\begin{aligned}0 &= \left(\nabla_t^2 + \gamma^2\right)\psi \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)\psi\end{aligned}$$

with boundary conditon

$$\left.\frac{\partial\psi}{\partial n}\right|_S = 0 \quad TE$$

The boundary condition in each direction may be satisfied at the origin by a cosine, so we have

$$\psi = H_0 \cos\alpha x \cos\beta y$$

and fitting the boundary conditions at $x = a$ and at $y = b$ requires $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$. Therefore, the eigenfunctions are

$$\psi_{mn} = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

with eigenvalues

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The cutoff frequency follows from

$$\begin{aligned}\omega_{mn} &= \frac{\gamma_{mn}}{\sqrt{\mu\varepsilon}} \\ &= \frac{\pi}{\sqrt{\mu\varepsilon}}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2}\end{aligned}$$

for the various modes.

If we want to design a waveguide with only one allowed mode, we want the lowest values for m, n. If $a > b$, the smallest value of ω_{mn} occurs for $n = 0$ and $m = 1$,

$$\omega_{10} = \frac{\pi}{a\sqrt{\mu\epsilon}}$$

For this mode, we have the fields,

$$H_z = \psi_{10} \quad = \quad H_0 \cos \frac{\pi x}{a}$$

and therefore, for waves moving in the $+z$ direction,

$$\begin{aligned}
\mathbf{B}_t &= +\frac{ik}{\gamma^2} \boldsymbol{\nabla}_t B_z \\
&= \frac{ika^2\mu}{\pi^2} \boldsymbol{\nabla}_t \left(H_0 \cos \frac{\pi x}{a} \right) \\
&= \mathbf{i} \left(-\frac{ika\mu}{\pi} H_0 \sin \frac{\pi x}{a} \right) e^{ikz - i\omega t}
\end{aligned}$$

From the impedence equation,

$$\mathbf{H}_t = +\frac{k}{\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}_t$$

so crossing with $\hat{\mathbf{k}}$

$$\begin{aligned}
\mathbf{E}_t &= -\frac{\mu\omega}{k} \hat{\mathbf{k}} \times \mathbf{H}_t \\
&= -\frac{\mu\omega}{k} \left(-\frac{ika}{\pi} H_0 \sin \frac{\pi x}{a} \right) e^{ikz - i\omega t} \hat{\mathbf{j}} \\
&= \frac{i\omega a\mu}{\pi} H_0 \sin \frac{\pi x}{a} e^{ikz - i\omega t} \hat{\mathbf{j}}
\end{aligned}$$