Time Dependent Green's Function

January 24, 2014

1 Solving the wave equation

We have established from the basic equations of electrodynamics that the potentials and gauge condition all satisfy equations of the form

$$\Box\Psi = -4\pi f(\mathbf{x}, t)$$

where the d'Alembertian, $\Box$, is the wave operator, given by

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

and ∇^2 is the usual 3-dimensional Laplacian. We wish to solve this equation in the absence of boundaries. This means the boundary conditions reduce to regularity at the origin and vanishing at infinity.

There are three steps to solving. First we solve the homogeneous equation,

$$\Box\psi = 0$$

Then, we use homogeneous solutions together with a boundary condition to solve for the empty space Green's function,

$$\Box G(\mathbf{x}, t; \mathbf{x}', t') = -4\pi\delta^3(\mathbf{x} - \mathbf{x}')\,\delta(t - t')$$

Finally, we construct the full solution for Ψ by integrating the source with the Green's function,

$$\Psi = \int G(\mathbf{x}, t; \mathbf{x}', t')\, f(\mathbf{x}', t')\, d^3x'dt'$$

It is immediate that applying the d'Alembertion to Ψ, interchanging primed integrations with the unprimed derivatives of the d'Alembertion, then integrating over the resulting delta functions reproduces the equation we wish to solve.

We vary this procedure in one significant way, by performing a Fourier transformation on the time coordinate. This is a practical consideration, since it immediately puts things in terms of electromagnetic waves, a topic we will consider in detail. The Fourier transformation is the replacement in the original equation of Ψ and f by integrals over frequency,

$$\begin{aligned}
\Psi(\mathbf{x}, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(\mathbf{x}, \omega)\, e^{-i\omega t}d\omega \\
f(\mathbf{x}, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\mathbf{x}, \omega)\, e^{-i\omega t}d\omega
\end{aligned}$$

Substituting these into the wave equation we have

$$\begin{aligned}
\Box\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\Psi\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\right) &= -4\pi\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}f\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\right) \\
\int\limits_{-\infty}^{\infty}\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\left(\Psi\left(\mathbf{x},\omega\right)e^{-i\omega t}\right)d\omega &= -4\pi\int\limits_{-\infty}^{\infty}f\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega \\
\int\limits_{-\infty}^{\infty}\left(-\Psi\left(\mathbf{x},\omega\right)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}e^{-i\omega t}+e^{-i\omega t}\nabla^2\Psi\left(\mathbf{x},\omega\right)\right)d\omega &= -4\pi\int\limits_{-\infty}^{\infty}f\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega \\
\int\limits_{-\infty}^{\infty}\left(\frac{\omega^2}{c^2}\Psi\left(\mathbf{x},\omega\right)+\nabla^2\Psi\left(\mathbf{x},\omega\right)\right)e^{-i\omega t}d\omega &= -4\pi\int\limits_{-\infty}^{\infty}f\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega
\end{aligned}$$

We see that the result is just the Fourier transform of the equation

$$\left(\nabla^2+k^2\right)\Psi\left(\mathbf{x},\omega\right)=-4\pi f\left(\mathbf{x},\omega\right)$$

where we define $k=\frac{\omega}{c}$. It is the completeness relation for Fourier transforms that guarantees we can invert the transform. See my notes on *Complex analysis* for a proof of the completeness relation.

The resulting equation is the *Helmholz equation*.

1.1 The homogeneous equation

With the sole boundary at infinity and no sources, the equation

$$\Box\psi=0$$

is not hard to solve. We consider the Fourier transform,

$$\left(\nabla^2+k^2\right)\psi\left(\mathbf{x},\omega\right)=0$$

Now, since there are no sources or boundaries, the system is spherically symmetric. As a result, there can be no angular dependence in ψ and the equation reduces to

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+k^2\psi &= 0 \\
\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+k^2\psi &= 0 \\
\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\psi\right)+k^2\psi &= 0 \\
\frac{\partial^2}{\partial r^2}\left(r\psi\right)+k^2\left(r\psi\right) &= 0
\end{aligned}$$

and this last is just the one dimensional wave equation for the product, $r\psi$, with solution

$$r\psi=Ae^{ikr}+Be^{-ikr}$$

and finally

$$\psi=\frac{Ae^{ikr}}{r}+\frac{Be^{-ikr}}{r}$$

which holds everywhere except $r=0$. In fact, we already know that the $\frac{1}{r}$ gives us the electrostatic Green's function, and the same will happen here. Specifically, we compute

$$\begin{aligned}
\left(\nabla^2+k^2\right)\frac{e^{\pm ikr}}{r} &= \boldsymbol{\nabla}\cdot\left(\frac{\pm ike^{\pm ikr}}{r}\hat{\mathbf{r}}+e^{\pm ikr}\boldsymbol{\nabla}\frac{1}{r}\right)+\frac{k^2e^{\pm ikr}}{r} \\
&= \left(\boldsymbol{\nabla}\cdot\left(\frac{\pm ike^{\pm ikr}}{r}\hat{\mathbf{r}}\right)+\boldsymbol{\nabla}e^{\pm ikr}\cdot\boldsymbol{\nabla}\frac{1}{r}+e^{\pm ikr}\nabla^2\frac{1}{r}\right)+\frac{k^2e^{\pm ikr}}{r} \\
&= -\frac{k^2e^{\pm ikr}}{r}-\frac{\pm ike^{\pm ikr}}{r^2}-\frac{\pm ike^{\pm ikr}}{r^2}+e^{\pm ikr}\nabla^2\frac{1}{r}+\frac{k^2e^{\pm ikr}}{r} \\
&= e^{\pm ikr}\nabla^2\frac{1}{r} \\
&= -4\pi e^{\pm ikr}\delta^3\left(\mathbf{x}\right)
\end{aligned}$$

Notice that $e^{\pm ikr}\delta^3\left(\mathbf{x}\right)$ is the same as $\delta^3\left(\mathbf{x}\right)$ since the Dirac delta forces $r=0$. Therefore,

$$\left(\nabla^2+k^2\right)\frac{e^{\pm ikr}}{r} = -4\pi\delta^3\left(\mathbf{x}\right)$$

1.2 The Green's function

Now we need the Fourier transform of the equation for the Green function,

$$\Box G\left(\mathbf{x},t;\mathbf{x}',t'\right)=-4\pi\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right)$$

Because we have spherical symmetry, $G\left(\mathbf{x},t;\mathbf{x}',t'\right)$ can depend only on

$$\begin{aligned}
R &= \left|\mathbf{x}-\mathbf{x}'\right| \\
\tau &= \left|t-t'\right|
\end{aligned}$$

and we transform τ,

$$G\left(R,\tau\right)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}G\left(R,\omega\right)e^{-i\omega\tau}d\omega$$

We also need the Fourier transform of the δ-function source. Using the completeness relation for the Fourier modes,

$$\int\limits_{-\infty}^{\infty}e^{-i\omega t}d\omega = 2\pi\delta\left(t\right)$$

and substituting, the equation becomes

$$\begin{aligned}
\Box\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}G\left(R,\omega\right)e^{-i\omega\tau}d\omega\right) &= -4\pi\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\omega\tau}d\omega \\
\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left(\nabla^2+k^2\right)G\left(R,\omega\right)e^{-i\omega\tau}d\omega &= -4\pi\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\omega\tau}d\omega \\
\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\left(\nabla^2+k^2\right)G\left(R,\omega\right)+\frac{4\pi}{\sqrt{2\pi}}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right]e^{-i\omega\tau}d\omega &= 0
\end{aligned}$$

Inverting the transform, we need

$$\left(\nabla^2 + k^2\right) G\left(R,\omega\right) = -4\pi\left[\frac{1}{\sqrt{2\pi}}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right]$$

and this is exactly the result we have for the homogeneous solution if we set

$$G_\pm\left(R,\omega\right) = \frac{1}{\sqrt{2\pi}}\frac{e^{\pm ikr}}{r}$$

We now just need to take the Fourier transform,

$$\begin{aligned} G_\pm\left(R,\tau\right) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} G\left(R,\omega\right)e^{-i\omega\tau}d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{\pm ikR}}{R}e^{-i\omega\tau}d\omega \\ &= \frac{1}{2\pi R}\int\limits_{-\infty}^{\infty} e^{-i\omega\tau\pm ikR}d\omega \\ &= \frac{1}{2\pi R}\int\limits_{-\infty}^{\infty} e^{-i\omega\left(\tau\mp\frac{R}{c}\right)}d\omega \\ &= \frac{1}{R}\delta\left(\tau\mp\frac{R}{c}\right) \end{aligned}$$

where we have used $k = \frac{\omega}{c}$. Replacing R and τ, we have the *retarded and advanced Green functions*,

$$G_\pm\left(\mathbf{x},t;\mathbf{x}',t'\right) = \frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left(\left|t-t'\right|\mp\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}\right)$$

The retarded Green function is

$$\begin{aligned} G_+\left(\mathbf{x},t;\mathbf{x}',t'\right) &= \frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left(\left|t-t'\right|-\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}\right) \\ &= \frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left(t-\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}-t'\right) \\ &= \frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left(t'-\left(t-\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}\right)\right) \end{aligned}$$

This includes a time delay by the amount of time it takes a wave to propagate from $\mathbf{x}'$ to $\mathbf{x}$.

For example, suppose we have a localized source, $\rho\left(\mathbf{x}',t'\right)$. Then

$$\Box\Phi = -\frac{1}{\epsilon_0}\rho\left(\mathbf{x},t\right)$$

has the solution

$$\begin{aligned} \Phi &= \frac{1}{4\pi\epsilon_0}\int G\left(\mathbf{x},t;\mathbf{x}',t'\right)\rho\left(\mathbf{x}',t'\right)d^3x'dt' \\ &= \frac{1}{4\pi\epsilon_0}\int\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left(t'-\left(t-\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}\right)\right)\rho\left(\mathbf{x}',t'\right)d^3x'dt' \\ &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho\left(\mathbf{x}',\left(t-\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}\right)\right)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3x'dt' \end{aligned}$$

which is just like our static solution except that the effect of $\rho(\mathbf{x}')$ on the field at $\mathbf{x}$ and time t, is not $\rho(\mathbf{x}', t)$ but instead it is what ρ was at the earlier time $t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$. The effect is delayed by the time $\frac{|\mathbf{x}-\mathbf{x}'|}{c}$ that it takes light to propagate from $\mathbf{x}'$ to $\mathbf{x}$. For simplicity, for any source $f(\mathbf{x}', t')$, we define

$$[f(\mathbf{x}', t')]_{ret} \equiv f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$