Thomas precession

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Assigning the electron a spin, $\mathbf{s}$, and consequent magnetic moment $\boldsymbol{\mu}$, related by

$$\boldsymbol{\mu} = \frac{ge}{2mc}\mathbf{s}$$

introduced by Uhlenbeck and Goudsmit in 1926, explains the anomalous Zeeman effect if $g = 2$ and produces the correct multiplet splitting of spectral lines when an atom is in a magnetic field if $g = 1$. But it does not solve both problems at once. The next year, Thomas showed that the conflict is resolved by a relativistic effect.

1 Incorrect classical treatment

First, we work the problem incorrectly, as was done before Thomas' insight. The problem resides in the use of an incorrect equation of motion. Let an electron of magnetic moment $\boldsymbol{\mu}$ move in a magnetic field $\mathbf{B}$. Since the electron is moving, it sees a magnetic field $\mathbf{B}'$, so the rate of change of its angular momentum $\mathbf{s}$ is (incorrectly) given by the Newtonian formula

$$\left(\frac{d\mathbf{s}}{dt}\right)_{electron\,frame} = \boldsymbol{\mu} \times \mathbf{B}'$$

corresponding to an interaction energy of

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}'$$

We show below that the magnetic field in the moving frame is given by

$$\begin{aligned} \mathbf{B}' &= \gamma\left(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{B}\right) \\ &\approx \mathbf{B} - \boldsymbol{\beta} \times \mathbf{E} \end{aligned}$$

where in the second line we take a nonrelativistic approximation, neglecting terms of order β^2 and higher. Dropping the β^2 terms is not the problem, however.

Now, the electron is in the electric field of the nucleus, which for single outer electron atoms may be written as the gradient of a spherically symmetric potential,

$$\mathbf{E} = -\frac{dV(r)}{dr}\frac{\mathbf{r}}{r}$$

With the orbital angular momentum of the electron given by

$$\mathbf{L} = m\mathbf{v} \times \mathbf{r}$$

we can write the energy as

$$
\begin{aligned}
U &= -\boldsymbol{\mu}\cdot\mathbf{B}+\boldsymbol{\mu}\cdot(\boldsymbol{\beta}\times\mathbf{E}) \\
&= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B}+\frac{ge}{2mc^2}\mathbf{s}\cdot(\mathbf{v}\times\mathbf{E}) \\
&= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B}-\frac{ge}{2mc^2}\mathbf{s}\cdot(\mathbf{v}\times\mathbf{r})\frac{1}{r}\frac{dV(r)}{dr} \\
&= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B}+\frac{ge}{2m^2c^2}\mathbf{s}\cdot\mathbf{L}\frac{1}{r}\frac{dV(r)}{dr}
\end{aligned}
$$

so that the second term becomes a spin-orbit interaction. With $g=2$, the first term is correct but the second term gives a spin-orbit interaction that is double the actual value.

2 Relativistic treatment

The problem with this calculation is that the equation of motion above holds only in an inertial frame of reference. However, the true inertial frame of the electron is a rotating one. Let the rest frame of the nucleus be the laboratory inertial frame, $\mathcal{O}$. Then consider two frames of reference for the electron, one at time t when the electron has velocity $\boldsymbol{\beta}$, and one a moment later at time $t+\delta t$, when the electron velocity is $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$. We can write a boost that takes us to the electron rest frame at either of these times by using the appropriate velocity:

$$
\begin{aligned}
(x')^\alpha &= [A_{boost}(\boldsymbol{\beta})]^\alpha{}_\beta\, x^\beta \\
(x'')^\alpha &= [A_{boost}(\boldsymbol{\beta}+\delta\boldsymbol{\beta})]^\alpha{}_\beta\, x^\beta
\end{aligned}
$$

We can relate $\mathcal{O}''$ to $\mathcal{O}'$ by combining these

$$
\begin{aligned}
(x'')^\alpha &= [A_{boost}(\boldsymbol{\beta}+\boldsymbol{\delta\beta})]^\alpha{}_\beta\, x^\beta \\
&= [A_{boost}(\boldsymbol{\beta}+\boldsymbol{\delta\beta})]^\alpha{}_\beta \left[A^{-1}_{boost}(\boldsymbol{\beta})\right]^\beta{}_\mu (x')^\mu \\
&= [A_{boost}(\boldsymbol{\beta}+\boldsymbol{\delta\beta})]^\alpha{}_\beta \left[A_{boost}(-\boldsymbol{\beta})\right]^\beta{}_\mu (x')^\mu
\end{aligned}
$$

The essential observation here is that, since the two velocity vectors, $\boldsymbol{\beta}, \boldsymbol{\beta}+\delta\boldsymbol{\beta}$ are in different directions, the transformation from one to the other involves a rotation as well as a boost, but the force does not apply any torque, so we need to remove the rotational part of the transformation.

Let the initial instantaneous rest frame of the electron, $\mathcal{O}'$, move with speed β in the x-direction, so that the boost back to the lab frame is

$$
[A_{boost}(-\boldsymbol{\beta})]^\beta{}_\mu = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
$$

Now let the velocity, $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$, defining the frame $\mathcal{O}''$, lie in the xy-plane. Starting from our general formula for a boost,

$$
\begin{aligned}
t' &= \gamma(ct-\boldsymbol{\beta}\cdot\mathbf{x}) \\
\mathbf{x}' &= \mathbf{x}+\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}-\gamma\boldsymbol{\beta}ct
\end{aligned}
$$

we substitute $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$ for $\boldsymbol{\beta}$ and expand to keep only terms linear in $\delta\boldsymbol{\beta}=\delta\beta_1\mathbf{i}+\delta\beta_2\mathbf{j}$. We need

$$
\gamma'' = \frac{1}{\sqrt{1-(\boldsymbol{\beta}+\delta\boldsymbol{\beta})^2}}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{1-\beta^2-2\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}} \\
&= \frac{1}{\sqrt{1-\beta^2}\sqrt{1-\frac{2\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}{1-\beta^2}}} \\
&= \gamma\frac{1}{\sqrt{1-2\gamma^2\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}} \\
&\approx \gamma\left(1+\gamma^2\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}\right) \\
&= \gamma+\gamma^3\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}
\end{aligned}
$$

Then for the time component,

$$
\begin{aligned}
t'' &= \gamma''\left(ct-(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\cdot\mathbf{x}\right) \\
&= \left(\gamma+\gamma^3\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}\right)\left(ct-\beta x-\delta\boldsymbol{\beta}\cdot\mathbf{x}\right) \\
&= \gamma\left(ct-\beta x-\delta\beta_1 x-\delta\beta_2 y\right)+\gamma^3\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}\left(ct-\beta x\right) \\
&= \gamma\left(ct-\beta x-\delta\beta_1 x-\delta\beta_2 y\right)+\gamma^3\beta\delta\beta_1\left(ct-\beta x\right) \\
&= \gamma\left(1+\gamma^2\beta\delta\beta_1\right)ct-\left(\beta+\delta\beta_1+\gamma^2\beta\delta\beta_1\beta\right)\gamma x-\left(\delta\beta_2\gamma\right)y \\
&= \left(\gamma+\gamma^3\beta\delta\beta_1\right)ct-\left(\gamma\beta+\gamma^3\delta\beta_1\right)x-\left(\delta\beta_2\gamma\right)y
\end{aligned}
$$

while for the spatial components,

$$
\begin{aligned}
\mathbf{x}'' &= \mathbf{x}+\frac{\gamma''-1}{(\boldsymbol{\beta}+\delta\boldsymbol{\beta})^2}\left(\left((\boldsymbol{\beta}+\delta\boldsymbol{\beta})\cdot\mathbf{x}\right)(\boldsymbol{\beta}+\delta\boldsymbol{\beta})-\gamma''(\boldsymbol{\beta}+\delta\boldsymbol{\beta})ct\right. \\
&= \mathbf{x}+\left(\frac{\gamma''-1}{(\boldsymbol{\beta}+\delta\boldsymbol{\beta})^2}\left(\beta x+\delta\beta_1 x+\delta\beta_2 y\right)-\gamma'' ct\right)\boldsymbol{\beta} \\
&\quad+\left(\frac{\gamma''-1}{(\boldsymbol{\beta}+\delta\boldsymbol{\beta})^2}\left(\beta x+\delta\beta_1 x+\delta\beta_2 y\right)-\gamma'' ct\right)\delta\boldsymbol{\beta} \\
&= \mathbf{x}+\left(\frac{\gamma+\gamma^3\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}-1}{\beta^2\left(1+\frac{2}{\beta}\delta\beta_1\right)}\left(\beta x+\delta\beta_1 x+\delta\beta_2 y\right)-\left(\gamma+\gamma^3\beta\delta\beta_1\right)ct\right)\boldsymbol{\beta} \\
&\quad+\left(\frac{\gamma-1}{\beta}x-\gamma ct\right)\delta\boldsymbol{\beta} \\
&= \mathbf{x}+\left(\frac{\gamma-1}{\beta}x-\gamma ct\right)\delta\boldsymbol{\beta} \\
&\quad+\left(\frac{1}{\beta^2}\left(\gamma+\gamma^3\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}-1\right)\left(1-\frac{2}{\beta}\delta\beta_1\right)\left(\beta x+\delta\beta_1 x+\delta\beta_2 y\right)-\left(\gamma+\gamma^3\beta\delta\beta_1\right)ct\right)\boldsymbol{\beta} \\
&= \mathbf{x}+\left(\frac{\gamma-1}{\beta}x-\gamma ct\right)\delta\boldsymbol{\beta}+\left(\frac{1}{\beta^2}\left(\gamma-1+\gamma^3\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}\right)\left(\beta x-\delta\beta_1 x+\delta\beta_2 y\right)-\left(\gamma+\gamma^3\beta\delta\beta_1\right)ct\right)\boldsymbol{\beta} \\
&= \mathbf{x}+\left(\frac{\gamma-1}{\beta}x-\gamma ct\right)\delta\boldsymbol{\beta}+\left(\frac{\gamma-1}{\beta^2}\left(\beta x-\delta\beta_1 x+\delta\beta_2 y\right)+\gamma^3\delta\beta_1 x-\left(\gamma+\gamma^3\beta\delta\beta_1\right)ct\right)\boldsymbol{\beta}
\end{aligned}
$$

In components,

$$
\begin{aligned}
x' &= \left(1+\frac{\gamma-1}{\beta}\delta\beta_1+(\gamma-1)-\frac{\gamma-1}{\beta}\delta\beta_1+\gamma^3\beta\delta\beta_1\right)x+\left(\frac{\gamma-1}{\beta}\delta\beta_2\right)y+\left(-\left(\gamma+\gamma^3\beta\delta\beta_1\right)\beta-\gamma\delta\beta_1\right)ct \\
&= \left(\gamma+\gamma^3\beta\delta\beta_1\right)x+\left(\frac{\gamma-1}{\beta}\delta\beta_2\right)y-\left(\gamma\beta+\gamma^3\delta\beta_1\right)ct
\end{aligned}
$$

$$
\begin{aligned}
y' &= y + \frac{\gamma - 1}{\beta}\delta\beta_2 x - \gamma\delta\beta_2 ct \\
z' &= z
\end{aligned}
$$

Now write this as a matrix equation,

$$
\begin{pmatrix} ct'' \\ x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \gamma + \gamma^3\beta\delta\beta_1 & -\gamma\beta - \gamma^3\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\beta\gamma - \gamma^3\delta\beta_1 & \gamma + \gamma^3\beta\delta\beta_1 & \frac{\gamma-1}{\beta}\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & \frac{\gamma-1}{\beta}\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}
$$

so that we have

$$
\begin{aligned}
x^\alpha(t+\delta t) &= \left[A_{boost}(\boldsymbol{\beta} + \boldsymbol{\delta\beta})\right]^\alpha_{\ \ \beta} \left[A_{boost}(-\boldsymbol{\beta})\right]^\beta_{\ \ \mu} x^\mu(t) \\
&= \begin{pmatrix} \gamma + \gamma^3\beta\delta\beta_1 & -\gamma\beta - \gamma^3\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\beta\gamma - \gamma^3\delta\beta_1 & \gamma + \gamma^3\beta\delta\beta_1 & \frac{\gamma-1}{\beta}\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & \frac{\gamma-1}{\beta}\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} \gamma\left(\gamma + \gamma^3\beta\delta\beta_1\right) - \gamma\beta\left(\gamma\beta + \gamma^3\delta\beta_1\right) & \gamma\beta\left(\gamma + \gamma^3\beta\delta\beta_1\right) - \gamma\left(\gamma\beta + \gamma^3\delta\beta_1\right) & -\gamma\delta\beta_2 & 0 \\ \gamma\left(-\beta\gamma - \gamma^3\delta\beta_1\right) + \gamma\beta\left(\gamma + \gamma^3\beta\delta\beta_1\right) & \gamma\beta\left(-\beta\gamma - \gamma^3\delta\beta_1\right) + \gamma\left(\gamma + \gamma^3\beta\delta\beta_1\right) & \frac{\gamma-1}{\beta}\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_2 + \gamma\beta\left(\frac{\gamma-1}{\beta}\delta\beta_2\right) & -\gamma^2\beta\delta\beta_2 + \gamma\left(\frac{\gamma-1}{\beta}\delta\beta_2\right) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu(t)
\end{aligned}
$$

Simplifying,

$$
\begin{aligned}
\gamma\left(\gamma + \gamma^3\beta\delta\beta_1\right) - \gamma\beta\left(\gamma\beta + \gamma^3\delta\beta_1\right) &= \gamma^2 + \gamma^4\beta\delta\beta_1 - \gamma^2\beta^2 - \gamma^4\beta\delta\beta_1 \\
&= \gamma^2 - \gamma^2\beta^2 \\
&= 1 \\
\gamma\beta\left(\gamma + \gamma^3\beta\delta\beta_1\right) - \gamma\left(\gamma\beta + \gamma^3\delta\beta_1\right) &= \gamma^2\beta + \gamma^4\beta^2\delta\beta_1 - \gamma^2\beta - \gamma^4\delta\beta_1 \\
&= \gamma^4\left(\beta^2 - 1\right)\delta\beta_1 \\
&= -\gamma^2\delta\beta_1 \\
\beta\gamma\left(-\beta\gamma - \gamma^3\delta\beta_1\right) + \gamma\left(\gamma + \gamma^3\beta\delta\beta_1\right) &= -\beta^2\gamma^2 - \beta\gamma^4\delta\beta_1 + \gamma^2 + \gamma^4\beta\delta\beta_1 \\
&= \gamma^2 - \beta^2\gamma^2 \\
&= 1 \\
-\gamma^2\delta\beta_2 + \gamma\beta\left(\frac{\gamma-1}{\beta}\delta\beta_2\right) &= -\gamma^2\delta\beta_2 + \left(\gamma^2 - \gamma\right)\delta\beta_2 \\
&= -\gamma\delta\beta_2 \\
-\gamma^2\beta\delta\beta_2 + \gamma\left(\frac{\gamma-1}{\beta}\delta\beta_2\right) &= \frac{1}{\beta}\left(-\gamma^2\beta^2 + \left(\gamma^2 - \gamma\right)\right)\delta\beta_2 \\
&= \frac{1}{\beta}\left(\gamma^2\left(1 - \beta^2\right) - \gamma\right)\delta\beta_2 \\
&= \frac{1-\gamma}{\beta}\delta\beta_2
\end{aligned}
$$

Therefore,

$$
x^\alpha(t+\delta t) = \begin{pmatrix} 1 & -\gamma^2\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_1 & 1 & \frac{\gamma-1}{\beta}\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & \frac{1-\gamma}{\beta}\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\alpha_{\ \ \mu} x^\mu(t)
$$

The most useful way to think about this matrix is as the product of an infinitesimal boost and an infinitesimal rotation,

$$\begin{pmatrix} 1 & -\gamma^2\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_1 & 1 & \frac{\gamma-1}{\beta}\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & \frac{1-\gamma}{\beta}\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(1 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\gamma-1}{\beta}\right)\delta\beta_2 & 0 \\ 0 & -\left(\frac{\gamma-1}{\beta}\right)\delta\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)$$

$$\times \left(1 + \begin{pmatrix} 0 & -\gamma^2\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_1 & 0 & 0 & 0 \\ -\gamma\delta\beta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)$$

Recall that for a rotation in the xy plane,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so if θ is infinitesimal the transformation matrix is

$$\begin{pmatrix} 1 & -\theta & \\ \theta & 1 & \\ & & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so the first factor is a rotation by an angle

$$\delta\theta = -\left(\frac{\gamma - 1}{\beta}\right)\delta\beta_2$$

around the z axis. We can get both angle and axis by writing this as

$$\begin{aligned} \delta\boldsymbol{\theta} &= -\left(\frac{\gamma-1}{\beta^2}\right)\boldsymbol{\beta}\times\delta\boldsymbol{\beta} \\ &= -\left(\frac{\gamma-1}{\left(\frac{\gamma^2-1}{\gamma^2}\right)}\right)\boldsymbol{\beta}\times\delta\boldsymbol{\beta} \\ &= -\left(\frac{\gamma^2(\gamma-1)}{\gamma^2-1}\right)\boldsymbol{\beta}\times\delta\boldsymbol{\beta} \\ &= -\left(\frac{\gamma^2}{\gamma+1}\right)\boldsymbol{\beta}\times\delta\boldsymbol{\beta} \end{aligned}$$

The second factor in $x^\alpha(t+\delta t)$ is a pure infinitesimal boost.

The rotation transformation results in an angular velocity, the Thomas precession, given by

$$\begin{aligned} \boldsymbol{\omega}_T &= -\frac{d\boldsymbol{\theta}}{dt} \\ &= -\frac{1}{\gamma}\frac{d\boldsymbol{\theta}}{d\tau} \\ &= \left(\frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{d\tau} \\ &= -\frac{\gamma}{\gamma+1}\frac{\mathbf{a}\times\mathbf{v}}{c^2} \end{aligned}$$

In the case of our atom, the (non-relativistic) acceleration is produced by the potential $V(r)$,

$$\begin{aligned}\mathbf{a} &= \frac{e\mathbf{E}}{m} \\ &= -\frac{e}{m}\frac{dV(r)}{dr}\frac{\mathbf{r}}{r}\end{aligned}$$

so that

$$\begin{aligned}\boldsymbol{\omega}_T &= -\frac{\gamma}{\gamma+1}\frac{\mathbf{a}\times\mathbf{v}}{c^2} \\ &= \frac{e}{m}\frac{1}{r}\frac{dV(r)}{dr}\frac{1}{2}\frac{\mathbf{r}\times\mathbf{v}}{c^2} \\ &= \frac{e}{m}\frac{1}{r}\frac{dV(r)}{dr}\frac{1}{2}\frac{\mathbf{L}}{mc^2} \\ &= \frac{e}{2m^2c^2}\frac{1}{r}\frac{dV(r)}{dr}\mathbf{L}\end{aligned}$$

Now return to the equation of motion for the electron. Now, since we have a rotating frame of reference, we must replace

$$\frac{d}{dt} \to \frac{d}{dt} + \boldsymbol{\omega}_T\times$$

This means that instead of the equation of motion

$$\left(\frac{d\mathbf{s}}{dt}\right)_{electron\,frame} = \boldsymbol{\mu}\times(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E})$$

we have

$$\begin{aligned}\left(\frac{d}{dt}+\boldsymbol{\omega}_T\times\right)\mathbf{s} &= \boldsymbol{\mu}\times(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}) \\ \frac{d\mathbf{s}}{dt} &= \boldsymbol{\mu}\times(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}) - \boldsymbol{\omega}_T\times\mathbf{s} \\ &= \frac{ge}{2mc}\mathbf{s}\times\mathbf{B} - \frac{ge}{2mc}\mathbf{s}\times(\boldsymbol{\beta}\times\mathbf{E}) + \mathbf{s}\times\boldsymbol{\omega}_T \\ &= \frac{ge}{2mc}\mathbf{s}\times\mathbf{B} - \mathbf{s}\times\left(\frac{ge}{2mc}(\boldsymbol{\beta}\times\mathbf{E}) - \boldsymbol{\omega}_T\right)\end{aligned}$$

and then, replacing $\mathbf{s}\times\frac{ge}{2mc}(\boldsymbol{\beta}\times\mathbf{E}) \to \mathbf{s}\times\left(\frac{ge}{2mc}(\boldsymbol{\beta}\times\mathbf{E})-\boldsymbol{\omega}_T\right)$, the energy is modified from

$$U = -\boldsymbol{\mu}\cdot\mathbf{B} + \boldsymbol{\mu}\cdot(\boldsymbol{\beta}\times\mathbf{E})$$

to

$$\begin{aligned}U &= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B} + \mathbf{s}\cdot\left(\frac{ge}{2mc}\boldsymbol{\beta}\times\mathbf{E} - \boldsymbol{\omega}_T\right) \\ &= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B} + \frac{ge}{2m^2c^2}\mathbf{s}\cdot\mathbf{L}\frac{1}{r}\frac{dV(r)}{dr} - \mathbf{s}\cdot\boldsymbol{\omega}_T \\ &= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B} + \frac{ge}{2m^2c^2}\mathbf{s}\cdot\mathbf{L}\frac{1}{r}\frac{dV(r)}{dr} - \frac{e}{2m^2c^2}\mathbf{s}\cdot\mathbf{L}\frac{1}{r}\frac{dV(r)}{dr} \\ &= -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B} + \frac{(g-1)e}{2m^2c^2}\mathbf{s}\cdot\mathbf{L}\frac{1}{r}\frac{dV(r)}{dr}\end{aligned}$$

The coefficient of the spin-orbit interaction energy is changed from $g=2$ to $g-1=1$, giving the required factor of 2.