Superposition of electromagnetic waves

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So far we have looked at properties of monochromatic plane waves. A more complete picture is found by looking at superpositions of many frequencies. Many of the important features emerge by considering the one-dimensional case, with each mode described by

$$u(x,t) = Ae^{i(kx-\omega t)}$$

If we assume that $u(x,t)$ satisfies some form of second order wave equation (though not necessarily $\Box u = 0$), then we expect some necessary relationship of the form

$$\omega = \omega(k)$$

in order to solve the equation.

1 General superposition

A superposition of plane waves may be accomplished by integrating over a range of different wavelengths and frequencies. The amplitude, A, may be different for the different modes, so the general superposition that still satisfies the wave equation will have the form

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk \int\limits_{-\infty}^{\infty} d\omega A(k,\omega) e^{i(kx-\omega t)} \delta(\omega - \omega(k))$$

where the delta function insures that the wave equation is satisfied. Integrating over frequency, we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk A(k,\omega(k)) e^{i(kx-\omega(k)t)}$$

With the understanding that $\omega = \omega(k)$ and $A(k) = A(k,\omega(k))$, we write more concisely

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk A(k) e^{i(kx-\omega t)}$$

1.1 Initial conditions

The solution above is correct if the wave equation is linear in time derivatives like the Schrödinger equation. If the wave equation is second order in time, then the condition $\omega = \omega(k)$ will be the solution to a quadratic equation, and we will get both positive and negative solutions for the frequency. In such cases, the solution includes two terms, which may be written as

$$u(x,t) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk \left[A(k) e^{i(kx-\omega t)} + A^*(k) e^{-i(kx-\omega t)} \right]$$

and this gives enough freedom to satisfy the two initial conditions, $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$:

$$\begin{aligned} u(x,0) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left[A(k)e^{ikx} + A^*(k)e^{-ikx}\right] \\ \frac{\partial u}{\partial t}(x,0) &= -\frac{1}{2}\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\omega dk\left[A(k)e^{ikx} - A^*(k)e^{-ikx}\right] \end{aligned}$$

Inverting these Fourier transforms gives the real and imaginary parts of $A(k)$ in terms of the initial conditions,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, u(x,0)e^{-ik'x} &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dk\left[A(k)e^{ikx}e^{-ik'x} + A^*(k)e^{-ikx}e^{-ik'x}\right] \\ &= \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left[A(k)2\pi\delta(k-k') + A^*(k)2\pi\delta(k+k')\right] \\ &= \frac{1}{2}\left(A(k') + A^*(-k')\right) \\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\frac{\partial u}{\partial t}(x,0)e^{-ik'x} &= -\frac{1}{2}\frac{i}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty}\omega(k)\,dk\left[A(k)e^{i\left(k-k'\right)x} - A^*(k)e^{-i\left(k+k'\right)x}\right] \\ &= -\frac{i}{2}\int_{-\infty}^{\infty}\omega(k)\,dk\left[A(k)\delta(k-k') - A^*(k)\delta(k+k')\right] \\ &= -\frac{i}{2}\omega(k')\left[A(k') - A^*(-k')\right] \end{aligned}$$

Solving for $A(k')$, and dropping the primes,

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\left(u(x,0) - \frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x,0)\right)e^{-ikx}$$

so $A(k)$ is fully determined by the pair of initial conditions.

2 Phase and group velocity

While the phase of a single wave mode propagates with velocity $v_p = \frac{\omega}{k}$, a superposition over many frequencies behaves differently.

For simplicity consider the first order solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk A(k)e^{i(kx-\omega t)}$$

where we take $A(k)$ to be a smooth distribution of frequencies peaked around some value k_0. Such a superposition is sometimes called a wave packet. For this sort of superposition, we can expand the frequency in a Taylor series as

$$\omega(k) = \omega_0 + \frac{\partial\omega}{\partial k}(k-k_0) + \ldots$$

We compute $u(x,t)$ to this linear order.

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkA(k)\, e^{i(kx-\omega t)} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkA(k)\, e^{i\left(kx-\left(\omega_0 t+\frac{\partial\omega}{\partial k}(k-k_0)t\right)\right)} \\ &= \frac{1}{\sqrt{2\pi}} e^{i\left(k_0\frac{\partial\omega}{\partial k}-\omega_0\right)t}\int_{-\infty}^{\infty} dkA(k)\, e^{ik\left(x-\frac{\partial\omega}{\partial k}t\right)} \\ &= \frac{1}{\sqrt{2\pi}} e^{i\tilde{\omega}_0 t}\int_{-\infty}^{\infty} dkA(k)\, e^{ik\left(x-\frac{\partial\omega}{\partial k}t\right)} \end{aligned}$$

Let $x' = x - \frac{\partial\omega}{\partial k}t$. Then

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkA(k)\, e^{i(kx-\omega t)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkA(k)\, e^{ikx'} \\ &= u(x',0) \end{aligned}$$

so the solution is

$$\begin{aligned} u(x,t) &= e^{i\tilde{\omega}_0 t} u(x',0) \\ &= e^{i\tilde{\omega}_0 t} u\left(x-\frac{\partial\omega}{\partial k}t, 0\right) \end{aligned}$$

and aside from an overall phase, the waveform at time t is given by the initial waveform displaced by $\frac{\partial\omega}{\partial k}t$. Thus, the wave packet moves to the right with velocity

$$v_g = \frac{\partial\omega}{\partial k}$$

This is called the group velocity.

Examples:

For light waves in a medium, we know that the wave number and frequency are related by $k = \sqrt{\mu\varepsilon}\omega$, so the phase velocity is

$$\begin{aligned} v_{phase} &= \frac{\omega}{k} \\ &= \frac{1}{\sqrt{\mu\varepsilon}} \\ &= \frac{c}{n} \end{aligned}$$

If the dielectric constant (and hence the index of refraction) is independent of frequency, then with $\omega = \frac{kc}{n}$ the group velocity is the same as the phase velocity

$$v_{group} = \frac{\partial\omega}{\partial k} = \frac{c}{n}$$

This is always the case in free space, with $v_{ph} = v_g = c$.

However, in most materials, the index of refraction often depends on wavelength we may expand

$$
\begin{aligned}
\frac{\omega}{k} &= \frac{c}{n(k)} \\
\omega &= \frac{ck}{n(k)} \\
\frac{\partial\omega}{\partial k} &= \frac{c}{n(k)} - \frac{ck}{n^2(k)}\frac{dn}{d\omega}\frac{\partial\omega}{\partial k} \\
\frac{\partial\omega}{\partial k}\left(1 + \frac{ck}{n^2}\frac{dn}{d\omega}\right) &= \frac{c}{n} \\
\frac{\partial\omega}{\partial k} &= \frac{c}{n\left(1 + \frac{ck}{n^2}\frac{dn}{d\omega}\right)} \\
\frac{\partial\omega}{\partial k} &= \frac{c}{n + \frac{ck}{n}\frac{dn}{d\omega}} \\
\frac{\partial\omega}{\partial k} &= \frac{c}{n + \omega\frac{dn}{d\omega}}
\end{aligned}
$$

The wave and group velocities now differ, and depending on the sign of $\frac{dn}{d\omega}$, the group velocity may even exceed the speed of light. However, as Jackson points out, in the cases where this happens group velocity is not a useful concept.

As a final example, consider solutions to the Klein-Gordon equation, which is the relativistic form of the Schrödinger equation,

$$\Box\psi = \frac{m^2c^2}{\hbar^2}\psi$$

This has plane wave solutions of the form

$$\psi = Ae^{i(kx-\omega t)}$$

where k, ω are related to energy and momentum by

$$
\begin{aligned}
E &= \hbar\omega \\
p &= \hbar k
\end{aligned}
$$

Using the relativistic energy relation,

$$E = \sqrt{p^2c^2 + m^2c^4}$$

and substituting, we find $\omega(k)$,

$$
\begin{aligned}
\hbar\omega &= \sqrt{\hbar^2k^2c^2 + m^2c^4} \\
\omega &= \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}
\end{aligned}
$$

The phase velocity is

$$
\begin{aligned}
v_{phase} &= \frac{\omega}{k} \\
&= \frac{1}{k}\sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}} \\
&= \sqrt{c^2 + \frac{m^2c^4}{\hbar^2k^2}} \\
&= c\sqrt{1 + \frac{m^2c^2}{\hbar^2k^2}} \\
&> c
\end{aligned}
$$

hence always greater than the speed of light, while

$$\begin{aligned} v_{group} &= \frac{\partial\omega}{\partial k} \\ &= \frac{1}{2}\frac{1}{\omega}2kc^2 \\ &= \frac{k}{\omega}c^2 \\ &= \left(\frac{c}{v_p}\right)c \end{aligned}$$

which is always less than the speed of light. The product of the two satisfies,

$$v_g v_{ph} = c^2$$

3 Propagation of a Gaussian wave packet

Now consider the time evolution of an initially Gaussian pulse

$$u(x,0) = e^{-x^2/2L^2}e^{ik_0x}$$

Jackson takes the real part of this, and starts the packet at rest, but let's suppose instead that it has a constant initial "velocity"

$$\frac{\partial u}{\partial t}(x,0) = \omega_0$$

We first use this to find $A(k)$, then use $A(k)$ to find $u(x,t)$.

3.1 Solving for the mode amplitude

We have the solution for $A(k)$,

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\left(u(x,0) - \frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x,0)\right)e^{-ikx} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\left(e^{-x^2/2L^2}e^{ik_0x} - \frac{i\omega_0}{\omega(k)}\right)e^{-ikx} \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}dxe^{-x^2/2L^2}e^{-ikx}e^{ik_0x} - \frac{i\omega_0}{\omega(k)}\int_{-\infty}^{\infty}dxe^{-ikx}\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}dxe^{-x^2/2L^2}e^{-ikx}e^{ik_0x} - \frac{i\omega_0}{\omega(k)}\delta(k)\right) \end{aligned}$$

To do the Gaussian integral, we complete the square:

$$\begin{aligned} \int_{-\infty}^{\infty}dxe^{-x^2/2L^2}e^{-ikx}e^{ik_0x} &= \int_{-\infty}^{\infty}dxe^{-x^2/2L^2-i(k-k_0)x} \\ &= \\ &= \int_{-\infty}^{\infty}dxe^{-\frac{1}{2L^2}\left(x^2-2iL^2(k-k_0)x\right)} \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} dx e^{-\frac{1}{2L^2}\left(x^2-2iL^2(k-k_0)x+\left[iL^2(k-k_0)\right]^2\right)+\frac{1}{2L^2}\left[iL^2(k-k_0)\right]^2} \\
&= e^{\frac{1}{2L^2}\left[iL^2(k-k_0)\right]^2}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2L^2}\left(x-iL^2(k-k_0)\right)^2}
\end{aligned}$$

Let $x' = x - iL^2(k-k_0)$, so we have simply

$$\int_{-\infty}^{\infty} dx e^{-x^2/2L^2} e^{-ikx} e^{ik_0x} = e^{\frac{1}{2L^2}\left[iL^2(k-k_0)\right]^2}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2L^2}x'^2}$$

Integrate this by squaring and changing to polar coordinates,

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} dx e^{-\frac{1}{2L^2}x'^2} \\
I^2 &= \int_{-\infty}^{\infty} dx e^{-\frac{1}{2L^2}x'^2}\int_{-\infty}^{\infty} dy e^{-\frac{1}{2L^2}y'^2} \\
&= \int_0^{\infty} \rho d\rho \int_0^{2\pi} d\varphi e^{-\frac{1}{2L^2}\rho^2} \\
&= 2\pi\int_0^{\infty} \rho d\rho e^{-\frac{1}{2L^2}\rho^2} \\
&= \pi\int_0^{\infty} d\xi e^{-\frac{1}{2L^2}\xi} \\
&= 2L^2\pi \\
I &= \sqrt{2\pi}L
\end{aligned}$$

Therefore,

$$\begin{aligned}
A(k) &= \frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi}Le^{\frac{1}{2L^2}\left[iL^2(k-k_0)\right]^2} - \frac{i\omega_0}{\omega(k)}2\pi\delta(k)\right) \\
&= Le^{-\frac{L^2}{2}(k-k_0)^2} - \sqrt{2\pi}\frac{i\omega_0}{\omega(k)}\delta(k)
\end{aligned}$$

3.2 Dispersion relation

To continue, we need the form of $\omega(k)$. Consider the case of high-frequency waves in plasma, for which we found

$$ck = \sqrt{\omega^2 - \omega_P^2}$$

Solving for ω,

$$\begin{aligned}
\omega^2 &= \omega_P^2 + c^2k^2 \\
\omega &= \sqrt{\omega_P^2 + c^2k^2}
\end{aligned}$$

Now we let $\omega_P \gg kc$ so that we may expand the square root,

$$\begin{aligned}\omega &\approx \omega_P\left(1+\frac{c^2}{2\omega_P^2}k^2\right)\\ &\approx \omega_P\left(1+\frac{1}{2}l_P^2k^2\right)\end{aligned}$$

where we define the plasma wavelength, $l_P = \frac{c}{\omega_P}$. This gives our dispersion relation. Notice that this form also works for a Klein-Gordon wave when the momentum due to mass is much greater than the wave momentum, $mc \gg \hbar k$, where we found

$$\begin{aligned}\omega &= \sqrt{\frac{m^2c^4}{\hbar^2}+k^2c^2}\\ &= \frac{mc^2}{\hbar}\sqrt{1+\frac{\hbar^2k^2c^2}{m^2c^4}}\\ &\approx \frac{mc^2}{\hbar}\left(1+\frac{1}{2}\frac{\hbar^2}{m^2c^2}k^2\right)\end{aligned}$$

We continue below with a wave packet in plasma.

3.3 Time evolution of the waveform

With this dispersion relation, $\omega = \omega_P\left(1+\frac{1}{2}l_P^2k^2\right)$, we can solve for $u(x,t)$,

$$\begin{aligned}u(x,t) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left[A(k)\,e^{i(kx-\omega t)}+A^*(k)\,e^{-i(kx-\omega t)}\right]\\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left(Le^{-\frac{L^2}{2}(k-k_0)^2}-\sqrt{2\pi}\frac{i\omega_0}{\omega(k)}\delta(k)\right)e^{i(kx-\omega t)}\\ &\quad+\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left(Le^{-\frac{L^2}{2}(k-k_0)^2}+\sqrt{2\pi}\frac{i\omega_0}{\omega(k)}\delta(k)\right)e^{-i(kx-\omega t)}\end{aligned}$$

The two velocity terms are trivial,

$$\begin{aligned}u_1(x,t) &\equiv \frac{i\omega_0}{2}\int_{-\infty}^{\infty} dk\frac{1}{\omega(k)}\delta(k)\left(e^{-i(kx-\omega t)}-e^{i(kx-\omega t)}\right)\\ &= \frac{\omega_0}{2i\omega(0)}\left(e^{-i\omega t}-e^{i\omega t}\right)\\ &= -\frac{\omega_0\sin\omega t}{\omega(0)}\\ &= -\frac{\omega_0}{\omega_P}\sin\omega t\end{aligned}$$

For the Gaussian terms, we have

$$u_2(x,t) \equiv \frac{1}{2}\frac{L}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dke^{-\frac{L^2}{2}(k-k_0)^2}\left(e^{i(kx-\omega t)}+e^{-i(kx-\omega t)}\right)$$

$$= \frac{L}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2}(k-k_0)^2} e^{i(kx-\omega t)}$$

$$= \frac{L}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2}(k-k_0)^2 + ikx - i\omega_P\left(1+\frac{1}{2}l_P^2 k^2\right)t}$$

Expanding the exponent allows us to complete the square,

$$\begin{aligned}
-\frac{L^2}{2}(k-k_0)^2 + ikx - i\omega_P\left(1+\frac{1}{2}l_P^2k^2\right)t &= -\frac{L^2}{2}k^2 + L^2k_0k - \frac{1}{2}L^2k_0^2 + ikx - i\omega_P t - \frac{i}{2}\omega_P l_P^2 k^2 t \\
&= -\left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)k^2 + \left(L^2k_0 + ix\right)k - \left(\frac{1}{2}L^2k_0^2 + i\omega_P t\right) \\
&= -\left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)\left(k^2 + 2\left(\frac{L^2k_0 + ix}{L^2 + il_P^2\omega_P t}\right)k\right) - \left(\frac{1}{2}L^2k_0^2 + i\omega_P t\right) \\
&= -\left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)\left(k + \frac{L^2k_0 + ix}{L^2 + il_P^2\omega_P t}\right)^2 \\
&\quad + \left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)\left(\frac{L^2k_0 + ix}{L^2 + il_P^2\omega_P t}\right)^2 - \left(\frac{1}{2}L^2k_0^2 + i\omega_P t\right)
\end{aligned}$$

Shifting the integration variable, $k' \equiv k + \frac{L^2k_0 + ix}{L^2 + il_P^2\omega_P t}$, and defining

$$c(x,t) \equiv \frac{1}{2}\left(L^2 + il_P^2\omega_P t\right)\left(\frac{L^2k_0 + ix}{L^2 + il_P^2\omega_P t}\right)^2 - \left(\frac{1}{2}L^2k_0^2 + i\omega_P t\right)$$

now gives a Gaussian integral,

$$u_2(x,t) = \frac{L}{\sqrt{2\pi}} Re\, e^{c(x,t)} \int_{-\infty}^{\infty} dk' e^{-\left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)k'^2}$$

The integral just depends on the coefficient in the exponent,

$$\int_{-\infty}^{\infty} dk' \exp\left(-\left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)k'^2\right) = \sqrt{\frac{2\pi}{L^2 + il_P^2\omega_P t}}$$

Now simplify the constant term,

$$\begin{aligned}
c(x,t) &= \frac{1}{2}\left[\left(L^2 + il_P^2\omega_P t\right)\left(\frac{L^2k_0 + ix}{L^2 + il_P^2\omega_P t}\right)^2 - \left(L^2k_0^2 + 2i\omega_P t\right)\right] \\
&= \frac{1}{2}\left[\frac{\left(L^2k_0 + ix\right)^2}{L^2 + il_P^2\omega_P t} - \left(L^2k_0^2 + 2i\omega_P t\right)\right] \\
&= \frac{1}{2}\left(\frac{\left(L^2k_0 + ix\right)^2 - \left(L^2k_0^2 + 2i\omega_P t\right)\left(L^2 + il_P^2\omega_P t\right)}{L^2 + il_P^2\omega_P t}\right)
\end{aligned}$$

We need to separate the real and imaginary parts. Multiplying the numerator and the denominator by $L^2 - il_P^2\omega_P t$,

$$
\begin{aligned}
c(x,t) &= \frac{1}{2}\frac{L^2 - il_p^2\omega_P t}{L^4 + l_p^4\omega_P^2 t^2}\left(\left(L^2 k_0 + ix\right)^2 - \left(L^2 k_0^2 + 2i\omega_P t\right)\left(L^2 + il_P^2\omega_P t\right)\right) \\
&= \frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(L^2 - il_p^2\omega_P t\right)\left(k_0^2 L^4 + 2ik_0 L^2 x - x^2 - L^4 k_0^2 - 2iL^2\omega_P t - iL^2 k_0^2 l_p^2 \omega_P t + 2l_P^2\omega_P^2 t^2\right) \\
&= \frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(L^2 - il_p^2\omega_P t\right)\left(-x^2 + 2l_P^2\omega_P^2 t^2 + 2iL^2\left(k_0 x - \left(1 + \frac{1}{2}k_0^2 l_p^2\right)\omega_P t\right)\right)
\end{aligned}
$$

Noting that $\omega(k_0) = \left(1 + \frac{1}{2}k_0^2 l_p^2\right)\omega_P$, and continuing the expansion,

$$
\begin{aligned}
c(x,t) &= \frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(L^2 - il_p^2\omega_P t\right)\left(-x^2 + 2l_P^2\omega_P^2 t^2 + 2iL^2\left(k_0 x - \omega(k_0)t\right)\right) \\
&= \frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(L^2\left(-x^2 + 2l_P^2\omega_P^2 t^2\right) + 2l_p^2\omega_P t L^2\left(k_0 x - \omega(k_0)t\right)\right) \\
&\quad + \frac{1}{2}\frac{i}{L^4 + l_p^4\omega_P^2 t^2}\left(2L^4\left(k_0 x - \omega(k_0)t\right) - l_p^2\omega_P t\left(-x^2 + 2l_P^2\omega_P^2 t^2\right)\right)
\end{aligned}
$$

For the real part,

$$
\begin{aligned}
Re\, c(x,t) &= \frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(L^2\left(-x^2 + 2l_P^2\omega_P^2 t^2\right) + 2l_p^2\omega_P t L^2\left(k_0 x - \omega(k_0)t\right)\right) \\
&= \frac{1}{2}\frac{L^2}{L^4 + l_p^4\omega_P^2 t^2}\left(-x^2 + 2l_p^2\omega_P k_0 x t + 2l_P^2\omega_P^2 t^2\left(1 - \frac{\omega(k_0)}{\omega_P}\right)\right) \\
&= \frac{1}{2}\frac{L^2}{L^4 + l_p^4\omega_P^2 t^2}\left(-x^2 + 2l_p^2\omega_P k_0 x t + 2l_p^2\omega_P^2\left(1 - \left(1 + \frac{1}{2}k_0^2 l_p^2\right)\right)t^2\right) \\
&= \frac{1}{2}\frac{L^2}{L^4 + l_p^4\omega_P^2 t^2}\left(-x^2 + 2l_p^2\omega_P k_0 x t - l_p^4\omega_P^2 k_0^2 t^4\right) \\
&= -\frac{1}{2}\frac{L^2}{L^4 + l_p^4\omega_P^2 t^2}\left(x - l_p^2\omega_P k_0 t\right)^2
\end{aligned}
$$

The imaginary part is just a phase,

$$
\begin{aligned}
\varphi(x,t) &\equiv Im\, c(x,t) \\
&= \frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(2L^4\left(k_0 x - \omega(k_0)t\right) - l_p^2\omega_P t\left(-x^2 + 2l_P^2\omega_P^2 t^2\right)\right)
\end{aligned}
$$

For long times, notice that

$$
\begin{aligned}
\lim_{t\to\infty}\varphi(x,t) &= \lim_{t\to\infty}\frac{1}{2}\frac{1}{L^4 + l_p^4\omega_P^2 t^2}\left(2L^4\left(k_0 x - \omega(k_0)t\right) - l_p^2\omega_P t\left(-x^2 + 2l_P^2\omega_P^2 t^2\right)\right) \\
&= \lim_{t\to\infty}\frac{1}{2}\frac{1}{l_p^4\omega_P^2 t^2}\left(-l_p^2\omega_P t\left(2l_P^2\omega_P^2 t^2\right)\right) \\
&= -\omega_P t
\end{aligned}
$$

Therefore, putting $u_2(x,t)$ back together, we have

$$
u_2(x,t) = \frac{L}{\sqrt{2\pi}} Re\, e^{c(x,t)} \int_{-\infty}^{\infty} dk' e^{-\left(\frac{L^2}{2} + \frac{i}{2}l_P^2\omega_P t\right)k'^2}
$$

$$= \frac{L}{\sqrt{2\pi}} Re \left(\sqrt{\frac{2\pi}{L^2 + i l_P^2 \omega_P t}} e^{i\varphi} \right) \exp\left(-\frac{1}{2} \frac{L^2}{L^4 + l_p^4 \omega_P^2 t^2} \left(x - l_p^2 \omega_P k_0 t \right)^2 \right)$$

This describes a Gaussian with amplitude

$$\begin{aligned} \frac{L}{\sqrt{2\pi}} Re \left(\sqrt{\frac{2\pi}{L^2 + i l_P^2 \omega_P t}} e^{i\varphi} \right) &= L\, Re \left(\left(L^2 + i l_P^2 \omega_P t \right)^{-1/2} e^{i\varphi} \right) \\ &= \frac{L}{\left(L^4 + l_P^4 \omega_P^2 t^2 \right)^{1/4}} \cos\left(\varphi - \frac{1}{2} \tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2} \right) \right) \end{aligned}$$

but the phase of the cosine is a nightmare to sort out. See the appendix if you're interested. The long time limit of the phase is immediate

$$\lim_{t\to\infty} \cos\left(\varphi - \frac{1}{2} \tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2} \right) \right) = \cos\left(\omega_P t + \frac{\pi}{4} \right)$$

so after $t \gg \frac{L^2}{l_P^2 \omega_P}$, and defining the velocity $v_0 \equiv l_p^2 \omega_P k_0$ the solution becomes

$$u_2 \left(x, t \gg \frac{L^2}{l_P^2 \omega_P} \right) = \frac{L}{l_P \sqrt{\omega_P t}} \cos\left(\omega_P t + \frac{\pi}{4} \right) \exp\left(-\frac{\left(x - v_0 t \right)^2}{2 v_0^2 t^2} \right) \cos\left(\omega_P t + \frac{\pi}{4} \right)$$

This describes a harmonic oscillation with a moving Gaussian envelope. The center of the Gaussian moves to the right at $v_0 = l_p^2 \omega_P k_0$. The standard deviation of the Gaussian is given by $\sigma = v_0 t$ so the width of the packet increases with time, while the amplitude decreases as $\frac{L}{l_P \sqrt{\omega_P t}}$.

3.4 The time evolution in wave number space

Finally, consider how the distribution in k-space evolves in time. Returning to the integral for $u_2 (x, t)$,

$$u_2 (x, t) = \frac{L}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2} (k - k_0)^2 + ikx - i\omega_P \left(1 + \frac{1}{2} l_P^2 k^2 \right) t}$$

we take the inverse Fourier transform,

$$\begin{aligned} u_2 (k', t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, u_2 (x, t)\, e^{-ik'x} \\ &= \frac{L}{2\pi} Re \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx\, e^{-ik'x} e^{-\frac{L^2}{2} (k - k_0)^2 + ikx - i\omega_P \left(1 + \frac{1}{2} l_P^2 k^2 \right) t} \\ &= \frac{L}{2\pi} Re \int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2} (k - k_0)^2 - i\omega_P \left(1 + \frac{1}{2} l_P^2 k^2 \right) t} \int_{-\infty}^{\infty} dx\, e^{i \left(k - k' \right) x} \\ &= L Re \int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2} (k - k_0)^2 - i\omega_P \left(1 + \frac{1}{2} l_P^2 k^2 \right) t} \delta\left(k - k' \right) \\ &= L Re e^{-\frac{L^2}{2} \left(k' - k_0 \right)^2 - i\omega_P \left(1 + \frac{1}{2} l_P^2 k'^2 \right) t} \\ &= L \cos\left(\omega_P \left(1 + \frac{1}{2} l_P^2 k'^2 \right) t \right) e^{-\frac{L^2}{2} \left(k' - k_0 \right)^2} \end{aligned}$$

Dropping the primes and identifying the frequency $\omega(k) = \omega_P\left(1 + \frac{1}{2} l_P^2 k'^2\right)$, we simply have a modulated form of the original Gaussian,

$$u_2(k', t) = L \cos(\omega t)\, e^{-\frac{L^2}{2}(k' - k_0)^2}$$

Appendix: Sorting out the ugly phase

We need a few trig identities,

$$\begin{aligned}
\cos\left(\varphi - \frac{1}{2}\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right) &= \cos\varphi \cos\left(\frac{1}{2}\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right) + \sin\varphi \sin\left(\frac{1}{2}\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right) \\
\sin\frac{x}{2} &= \sqrt{\frac{1}{2}(1 - \cos x)} \\
\cos\frac{x}{2} &= \sqrt{\frac{1}{2}(1 + \cos x)}
\end{aligned}$$

Then

$$\begin{aligned}
\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right) &= \sqrt{\frac{1}{2}\left(1 + \cos\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right)} \\
&= \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \left(\frac{l_P^2 \omega_P t}{L^2}\right)^2}}\right)} \\
\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right) &= \sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \left(\frac{l_P^2 \omega_P t}{L^2}\right)^2}}\right)} \\
\cos\left(\varphi - \frac{1}{2}\tan^{-1}\left(\frac{l_P^2 \omega_P t}{L^2}\right)\right) &= \sqrt{\frac{\sqrt{L^4 + \left(l_P^2 \omega_P t\right)^2} + L^2}{2\sqrt{L^4 + \left(l_P^2 \omega_P t\right)^2}}}\cos\varphi + \sqrt{\frac{\sqrt{L^4 + \left(l_P^2 \omega_P t\right)^2} - L^2}{2\sqrt{L^4 + \left(l_P^2 \omega_P t\right)^2}}}\sin\varphi \\
&= \frac{1}{\left(4L^4 + 4 l_P^4 \omega_P^2 t^2\right)^{1/4}}\left(\sqrt{\sqrt{L^4 + \left(l_P^2 \omega_P t\right)^2} + L^2}\cos\varphi + \sqrt{\sqrt{L^4 + \left(l_P^2 \omega_P t\right)^2} - L^2}\sin\varphi\right)
\end{aligned}$$

where φ was found previously.