Scattering

April 1, 2014

Scattering of light depends on the size of the scatterer relative to the wavelength of the light. For wavelengths much smaller than the scattering object, geometric optics gives an adequate description. At larger wavelengths, corrections to geometric optics may be found. At the other extreme, it is possible to do a treatment in terms of lowest-order multipoles. Between these extremes, a full multipole treatment is required.

1 Scattering at long wavelengths

We well be especially interested in the differential cross section. In the radiation zone, the time-averaged radiated power per unit area is given by the Poynting vector, $\mathbf{S} = \frac{dP}{dA}\hat{\mathbf{r}}$. Multiplying by r^2 converts the area element $dA = r^2 d\Omega$ into the solid angle $d\Omega$, so the time-averaged radiated power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{1}{2}\left|r^2 \mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}^*)\right|$$

For electric dipole or quadrupole fields in the radiation zone we found that $\mathbf{H}^* = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\mathbf{n}\times\mathbf{E}^*\right)$, so this becomes

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{2}\left|r^2\mathbf{n}\cdot\left(\mathbf{E}\times\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\mathbf{n}\times\mathbf{E}^*\right)\right)\right| \\
&= \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}r^2\left|\mathbf{E}\right|^2 \\
&= \frac{1}{2Z_0}r^2\left|\mathbf{E}\right|^2
\end{aligned}$$

It may be that we desire the cross-section for a particular polarization, in which case we consider only that component of the electric field,

$$\frac{dP}{d\Omega} = \frac{1}{2Z_0}r^2\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}\right|^2$$

In scattering experiments, a target is struck by many incoming waves, so we express the outgoing power as a probability for scattering in a given solid angle. To do this, we normalize by the incident power per unit area. The result is the *differential cross-section*

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &\equiv \frac{1}{dP_{incident}/dA}\frac{dP_{scattered}}{d\Omega} \\
&= \frac{\frac{1}{2Z_0}r^2\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{scattered}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{inc}\right|^2}
\end{aligned}$$

Because we divide by the incident power per unit *area*, the differential cross-section and the total cross-section have units of area. The total cross-section, σ, may be thought of as the effective cross-sectional area of the target.

Now, for long wavelength or small scatterers, we may assume an incoming polarized plane wave,

$$\begin{aligned}\mathbf{E}_{sc} &= \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{H}_{sc} &= \frac{1}{Z_0}\mathbf{n}_0\times\mathbf{E}_{sc}\end{aligned}$$

which induces electric and magnetic dipole moments, $\mathbf{p},\mathbf{m}$, in the scatterer. Then the scattered radiation is the resulting dipole radiation,

$$\begin{aligned}\mathbf{E}_{sc} &= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}-\frac{1}{c}\mathbf{n}\times\mathbf{m}\right] \\ \mathbf{H}_{sc} &= \frac{1}{Z_0}\mathbf{n}\times\mathbf{E}_{sc}\end{aligned}$$

Now, substituting into the differential cross section, we have

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0) &= \frac{\frac{1}{2Z_0}r^2\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{scattered}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}\right|^2} \\ &= \frac{r^2\left|\boldsymbol{\varepsilon}^*\cdot\frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}-\frac{1}{c}\mathbf{n}\times\mathbf{m}\right]\right|^2}{E_0^2} \\ &= \frac{k^4}{(4\pi\epsilon_0E_0)^2}\left|\left[\boldsymbol{\varepsilon}^*\cdot(\mathbf{p}-(\mathbf{n}\cdot\mathbf{p})\,\mathbf{n})-\frac{1}{c}\boldsymbol{\varepsilon}^*\cdot(\mathbf{n}\times\mathbf{m})\right]\right|^2 \\ &= \frac{k^4}{(4\pi\epsilon_0E_0)^2}\left|\left[\boldsymbol{\varepsilon}^*\cdot\mathbf{p}-(\mathbf{n}\cdot\mathbf{p})(\boldsymbol{\varepsilon}^*\cdot\mathbf{n})-\frac{1}{c}\mathbf{m}\cdot(\boldsymbol{\varepsilon}^*\times\mathbf{n})\right]\right|^2 \\ &= \frac{k^4}{(4\pi\epsilon_0E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}+\frac{1}{c}\mathbf{m}\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\right|^2\end{aligned}$$

We have therefore reduced the scattering problem to finding the induced polarization and magnetization. These induced properties generically involve the direction, $\mathbf{n}_0$, and polarization, $\boldsymbol{\varepsilon}_0$ of the incident light. Notice that the differential cross section depends on k^4. This dependence, called Rayleigh's law, will be the case unless both dipole moments vanish - quadrupole radiation will depend on k^6, and so on.

2 Example 1: Scattering by a small dielectric sphere

2.1 The outgoing electric field

For a dielectric sphere much smaller than the wavelength, we may treat the electric field as momentarily constant across the sphere.

Recall the solution for a dielectric sphere in a constant field. We start with a pair of series solutions for the potential inside and out,

$$\begin{aligned}\Phi_{in} &= \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) \\ \Phi_{out} &= -E_0 r\cos\theta+\sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\theta)\end{aligned}$$

where the first term in Φ_{out} gives the constant field at "large" distances from the sphere. The remaining terms incorporate the boundary conditions at the origin and infinity. Then, equating the tangential $\mathbf{E}$ and normal $\mathbf{D}$ fields we equate like coefficients to find the solution

$$\begin{aligned}\Phi_{in} &= -\frac{3\epsilon_0}{\epsilon+2\epsilon_0}E_0 r\cos\theta \\ \Phi_{out} &= -E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^2}\cos\theta\end{aligned}$$

with the field outside being

$$\begin{aligned}\mathbf{E}_{out} &= -\nabla\Phi_{out} \\ &= -\hat{\mathbf{n}}\frac{\partial}{\partial r}\Phi_{out} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\Phi_{out}\right) \\ &= -\hat{\mathbf{n}}\frac{\partial}{\partial r}\left(-E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^2}\cos\theta\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(-E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^2}\cos\theta\right) \\ &= E_0\hat{\mathbf{k}} + \frac{2\epsilon-2\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\hat{\mathbf{n}}\cos\theta + \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\right)\hat{\boldsymbol{\theta}}\sin\theta \\ &= E_0\hat{\mathbf{k}} + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(2\hat{\mathbf{n}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \\ &= E_0\hat{\mathbf{k}} + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(3\hat{\mathbf{n}}\cos\theta - \left(\hat{\mathbf{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)\right) \\ &= E_0\hat{\mathbf{k}} + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(3\hat{\mathbf{n}}\cos\theta - \hat{\mathbf{k}}\right)\end{aligned}$$

where we used

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \hat{\mathbf{n}}\sin\theta - \hat{\mathbf{k}}\cos\theta \\ \hat{\mathbf{k}} &= \hat{\mathbf{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\end{aligned}$$

Notice that the potential inside is just proportional to $z = r\cos\theta$, so the induced electric field inside is parallel to the applied field, but changed in magnitude by $\frac{3\epsilon_0}{\epsilon+2\epsilon_0}$.

We know that a dipole $\mathbf{p} = p\hat{\mathbf{k}}$ at the origin produces an electric field

$$\begin{aligned}\mathbf{E} &= \frac{3\hat{\mathbf{n}}\left(\hat{\mathbf{n}}\cdot\mathbf{p}\right) - \mathbf{p}}{4\pi\epsilon_0 r^3} \\ &= \frac{p}{4\pi\epsilon_0 r^3}\left(3\hat{\mathbf{n}}\cos\theta - \hat{\mathbf{k}}\right)\end{aligned}$$

Comparing this dipole field to the non-constant part of the exterior field,

$$\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(3\hat{\mathbf{n}}\cos\theta - \hat{\mathbf{k}}\right) = \frac{p}{4\pi\epsilon_0 r^3}\left(3\hat{\mathbf{n}}\cos\theta - \hat{\mathbf{k}}\right)$$

we see that we can identify the dipole strength as

$$\mathbf{p} = 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{k}}$$

2.2 Differential cross section

Now return to the differential cross-section. Since there is no magnetic dipole moment, we set $\mathbf{m} = 0$, leaving

$$\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}, \mathbf{n}_0, \boldsymbol{\varepsilon}_0\right) = \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2$$

From here there are several cases, depending on the polarizations of the incoming and outgoing waves.

We have computed the dipole moment assuming the electric field is in the z-direction, but to find the angular distribution it is easier to let the incoming wave propagate in the z-direction. Rotating the coordinate system so that the incoming wave moves in the $\hat{\mathbf{k}}$-direction with polarization in the $\hat{\boldsymbol{\varepsilon}}_0$-direction, we have

$$\begin{aligned}\mathbf{E} &= E_0\hat{\boldsymbol{\varepsilon}}_0 e^{ikz-i\omega t} \\ \mathbf{p} &= 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\boldsymbol{\varepsilon}}_0\end{aligned}$$

where the polarization vector, $\hat{\boldsymbol{\varepsilon}}_0$, may be in any combination of the x and y directions. We consider these two cases separately:

Case 1: Polarization in the x direction In this case, we have

$$\boldsymbol{\varepsilon}_0 = \hat{\mathbf{i}}$$

For the outgoing wave, the possible linear polarizations are in the directions orthogonal to the outward radial unit vector, $\mathbf{n}$. It is convenient to write out the unit vectors for spherical coordinates:

$$\begin{aligned}\mathbf{n} &= \hat{\mathbf{i}}\sin\theta\cos\varphi+\hat{\mathbf{j}}\sin\theta\sin\varphi+\hat{\mathbf{k}}\cos\theta \\ \hat{\boldsymbol{\theta}} &= \hat{\mathbf{i}}\cos\theta\cos\varphi+\hat{\mathbf{j}}\cos\theta\sin\varphi-\hat{\mathbf{k}}\sin\theta \\ \hat{\boldsymbol{\varphi}} &= -\hat{\mathbf{i}}\sin\varphi+\hat{\mathbf{j}}\cos\varphi\end{aligned}$$

It is easy to check that these are orthonormal. The outward moving, scattered wave may have polarization in any combination of the $\hat{\boldsymbol{\theta}}$ and the $\hat{\boldsymbol{\varphi}}$ directions. We can now compute the differential cross-section. If the measured polarization is in the $\hat{\boldsymbol{\theta}}$ direction,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}},\mathbf{n}_0=\hat{\mathbf{k}},\boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right) &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2 \\ &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\boldsymbol{\theta}}\cdot 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{i}}\right|^2 \\ &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left(4\pi\epsilon_0 E_0 a^3\right)^2\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left|\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{i}}\right|^2 \\ &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left(4\pi\epsilon_0 E_0 a^3\right)^2\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left|\cos\theta\cos\varphi\right|^2 \\ &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4a^6\cos^2\theta\cos^2\varphi\end{aligned}$$

Notice that k^4a^6 has units of area. For polarization in the $\hat{\boldsymbol{\varphi}}$ direction, we have

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}},\mathbf{n}_0=\hat{\mathbf{k}},\boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right) &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2 \\ &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varphi}}\cdot 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{i}}\right|^2 \\ &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4a^6\left|\hat{\boldsymbol{\varphi}}\cdot\hat{\mathbf{i}}\right|^2 \\ &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4a^6\sin^2\varphi\end{aligned}$$

Case 2: Polarization in the y direction Changing the incident polarization to the y-direction gives us two more cases. For outgoing polarization in the $\hat{\boldsymbol{\theta}}$ direction,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right) &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2 \\
&= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\boldsymbol{\theta}}\cdot 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{j}}\right|^2 \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\left|\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{j}}\right|^2 \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\theta\sin^2\varphi
\end{aligned}$$

Finally, for outgoing polarization in the $\hat{\boldsymbol{\varphi}}$ direction, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right) &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2 \\
&= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varphi}}\cdot 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{j}}\right|^2 \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\left|\hat{\boldsymbol{\varphi}}\cdot\hat{\mathbf{j}}\right|^2 \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\varphi
\end{aligned}$$

2.3 Unpolarized incoming wave

When the incoming light is unpolarized, we *average* over the possible incoming polarizations. For outgoing polarization in the $\hat{\boldsymbol{\theta}}$ direction, this gives

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}\right) &= \frac{1}{2}\left[\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right)+\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\theta\cos^2\varphi+\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\theta\sin^2\varphi\right] \\
&= \frac{1}{2}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\theta
\end{aligned}$$

which depends only on θ, while for the outgoing polarization in the $\hat{\boldsymbol{\varphi}}$ direction,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}\right) &= \frac{1}{2}\left[\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right)+\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\sin^2\varphi+\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\varphi\right] \\
&= \frac{1}{2}k^4 a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2
\end{aligned}$$

which has no angular dependence.

2.4 Final polarization not measured

As a final possibility, suppose we have unpolarized light coming in, and we do not measure the outgoing polarization. Then the result is the *sum* of the results for the outgoing radiation,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\mathbf{n}_0=\hat{\mathbf{k}}\right) &= \frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}},\mathbf{n}_0=\hat{\mathbf{k}}\right)+\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}},\mathbf{n}_0=\hat{\mathbf{k}}\right) \\ &= \frac{1}{2}k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left(1+\cos^2\theta\right)\end{aligned}$$

2.5 Total cross-section

The total light scattered gives an estimate of the size of the scatterer. In the present case, for unpolarized light, we integrate over all angles,

$$\begin{aligned}\sigma &= \int d\sigma \\ &= \int_0^{\pi}\int_0^{2\pi}\frac{1}{2}k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left(1+\cos^2\theta\right)d\Omega\end{aligned}$$

The integral is

$$\int\int\left(1+\cos^2\theta\right)d\Omega=\frac{16\pi}{3}$$

so the total cross section is

$$\sigma=\frac{8}{3}\pi a^2\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2k^4a^4$$

which in this case is the cross-sectional area, πa^2, times a dimensionless factor depending on the ratio $\frac{a}{\lambda}$.

3 Example 2: Scattering by a small, perfectly conducting sphere

For a perfectly conducting sphere, the boundary conditions change. In problem 1 you are asked to work out this case, so here we simply state the results for the electric and magnetic dipole strengths:

$$\begin{aligned}\mathbf{p} &= 4\pi\epsilon_0a^3\mathbf{E}_{inc} \\ \mathbf{m} &= -2\pi a^3\mathbf{H}_{inc}\end{aligned}$$

For linear polarization, $\mathbf{E}_{inc}$ and $\mathbf{H}_{inc}$ are orthogonal and orthogonal to the direction of propagation,

$$\begin{aligned}\mathbf{E}_{inc} &= E_{inc}\hat{\boldsymbol{\varepsilon}}_0e^{i\mathbf{n}_0\cdot\mathbf{x}-i\omega t} \\ \mathbf{H}_{inc} &= \frac{1}{\mu_0c}\mathbf{n}_0\times\mathbf{E}_{inc}\end{aligned}$$

so the dipole strengths are perpendicular as well. We can immediately write the differential cross-section,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0\right) &= \frac{k^4}{\left(4\pi\epsilon_0E_{inc}\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}+\frac{1}{c}\mathbf{m}\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2 \\ &= \frac{k^4}{\left(4\pi\epsilon_0E_{inc}\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\left(4\pi\epsilon_0a^3\mathbf{E}_{inc}\right)-2\pi a^3\mathbf{H}_{inc}\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2 \\ &= \frac{k^4}{\left(4\pi\epsilon_0E_{inc}\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\left(4\pi\epsilon_0a^3\mathbf{E}_{inc}\right)-\frac{2}{c}\pi a^3\frac{1}{\mu_0c}\left(\mathbf{n}_0\times\mathbf{E}_{inc}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{k^4}{(4\pi\epsilon_0)^2}\left|\boldsymbol{\varepsilon}^* \cdot \left(4\pi\epsilon_0 a^3 \hat{\boldsymbol{\varepsilon}}_0\right) - 2\pi a^3 \epsilon_0 \left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \boldsymbol{\varepsilon}^*\right)\right|^2 \\
&= k^4 a^6 \left|\boldsymbol{\varepsilon}^* \cdot \hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \boldsymbol{\varepsilon}^*\right)\right|^2
\end{aligned}
$$

Now compute the differential cross-section in the case of unpolarized incident light (*average* over incident polarizations) but outgoing polarization parallel to the plane of incidence,

$$
\boldsymbol{\varepsilon}_\parallel = \frac{1}{\sin\theta}\left(\mathbf{n}_0 \times \mathbf{n}\right) \times \mathbf{n}
$$

or outgoing polarization perpendicular to the plane of incidence,

$$
\boldsymbol{\varepsilon}_\perp = \frac{1}{\sin\theta}\mathbf{n}_0 \times \mathbf{n}
$$

For the parallel case, working through the outrageous cross-products,

$$
\begin{aligned}
\frac{d\sigma_\parallel}{d\Omega} &= \frac{1}{2}k^4 a^6 \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel}, \hat{\boldsymbol{\varepsilon}}_{0\perp}} \left|\boldsymbol{\varepsilon}_\parallel^* \cdot \hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \boldsymbol{\varepsilon}_\parallel^*\right)\right|^2 \\
&= \frac{1}{2}k^4 a^6 \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel}, \hat{\boldsymbol{\varepsilon}}_{0\perp}} \left|\frac{1}{\sin\theta}\left(\left(\mathbf{n}_0 \times \mathbf{n}\right) \times \mathbf{n}\right) \cdot \hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \left(\frac{1}{\sin\theta}\left(\mathbf{n}_0 \times \mathbf{n}\right) \times \mathbf{n}\right)\right)\right|^2 \\
&= \frac{1}{2}\frac{k^4 a^6}{\sin^2\theta} \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel}, \hat{\boldsymbol{\varepsilon}}_{0\perp}} \left|\left(\left(\mathbf{n}_0 \times \mathbf{n}\right) \times \mathbf{n}\right) \cdot \hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \left(\left(\mathbf{n}_0 \times \mathbf{n}\right) \times \mathbf{n}\right)\right)\right|^2 \\
&= \frac{1}{2}\frac{k^4 a^6}{\sin^2\theta} \left|\left(\left(\mathbf{n}_0 \times \mathbf{n}\right) \times \mathbf{n}\right) \cdot \hat{\boldsymbol{\varepsilon}}_{0\parallel} - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_{0\parallel}\right) \cdot \left(\mathbf{n}_0 \times \mathbf{n}\right)\right|^2 \\
&= \frac{1}{2}k^4 a^6 \left|\cos\theta - \frac{1}{2}\right|^2
\end{aligned}
$$

For the perpendicular case,

$$
\begin{aligned}
\frac{d\sigma_\perp}{d\Omega} &= \frac{1}{2}k^4 a^6 \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel}, \hat{\boldsymbol{\varepsilon}}_{0\perp}} \left|\boldsymbol{\varepsilon}_\perp^* \cdot \hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \boldsymbol{\varepsilon}_\perp^*\right)\right|^2 \\
&= \frac{1}{2}k^4 a^6 \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel}, \hat{\boldsymbol{\varepsilon}}_{0\perp}} \left|\frac{1}{\sin\theta}\left(\mathbf{n}_0 \times \mathbf{n}\right) \cdot \hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_0\right) \cdot \left(\mathbf{n} \times \left(\frac{1}{\sin\theta}\mathbf{n}_0 \times \mathbf{n}\right)\right)\right|^2 \\
&= \frac{1}{2}k^4 a^6 \left|\frac{1}{\sin\theta}\left(\mathbf{n}_0 \times \mathbf{n}\right) \cdot \hat{\boldsymbol{\varepsilon}}_{0\perp} - \frac{1}{2}\left(\mathbf{n}_0 \times \hat{\boldsymbol{\varepsilon}}_{0\perp}\right) \cdot \left(\mathbf{n} \times \left(\frac{1}{\sin\theta}\mathbf{n}_0 \times \mathbf{n}\right)\right)\right|^2 \\
&= \frac{1}{2}k^4 a^6 \left|1 - \frac{1}{2}\cos\theta\right|^2
\end{aligned}
$$

If the final polarization is unmeasured, we sum these, giving

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega} \\
&= \frac{1}{2}k^4 a^6 \left|\cos\theta - \frac{1}{2}\right|^2 + \frac{1}{2}k^4 a^6 \left|1 - \frac{1}{2}\cos\theta\right|^2 \\
&= k^4 a^6 \left[\frac{1}{2}\left(\cos^2\theta - \cos\theta + \frac{1}{4}\right) + \frac{1}{2}\left(1 - \cos\theta + \frac{1}{4}\cos^2\theta\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= k^4 a^6 \left[\frac{1}{2}\cos^2\theta + \frac{1}{8}\cos^2\theta - \frac{1}{2}\cos\theta - \frac{1}{2}\cos\theta + \frac{1}{8} + \frac{1}{2}\right] \\
&= k^4 a^6 \left[\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}\right]
\end{aligned}
$$

We define the *polarization* of the scattered radiation to be the difference between the parallel and perpendicular cross-sections, normalized by the total differential cross-section,

$$
\begin{aligned}
\Pi &\equiv \frac{1}{\frac{d\sigma}{d\Omega}}\left[\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}\right] \\
&= \frac{1}{k^4 a^6\left[\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}\right]}\left[\frac{1}{2}k^4 a^6\left|1 - \frac{1}{2}\cos\theta\right|^2 - \frac{1}{2}k^4 a^6\left|\cos\theta - \frac{1}{2}\right|^2\right] \\
&= \frac{1}{\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}}\left[\frac{1}{2}\left(1 - \cos\theta + \frac{1}{4}\cos^2\theta\right) - \frac{1}{2}\left(\cos^2\theta - \cos\theta + \frac{1}{4}\right)\right] \\
&= \frac{1}{\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}}\left[\frac{3}{8} - \frac{3}{8}\cos^2\theta\right] \\
&= \frac{3\sin^2\theta}{5\cos^2\theta - 8\cos\theta + 5}
\end{aligned}
$$

4 Collections of scatterers

When light travels through a medium, it encounters many scatterers, so the scattering is a superposition of the results of many scatterings. Suppose there are scatterers located at positions, $\mathbf{x}_i$. Then since the fields vary as $e^{i\mathbf{k}\cdot\mathbf{x}}$ there will be factors

$$
\mathbf{E}(\mathbf{x}_i), \mathbf{B}(\mathbf{x}_i) \sim e^{ik\mathbf{n}_0\cdot\mathbf{x}_i}
$$

associated with the corresponding induced dipole moments,

$$
\mathbf{p}_i, \mathbf{m}_i \sim e^{ik\mathbf{n}_0\cdot\mathbf{x}_i}
$$

Recalling that the total differential cross-section depends on $\mathbf{E}\times\mathbf{B}^*$, we will have a sum over conjugate pairs of phase factors:

$$
\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\sum_{i,j}\left[\boldsymbol{\varepsilon}^*\cdot\mathbf{p}_i + \frac{1}{c}\mathbf{m}_i\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\right]e^{ik\mathbf{n}_0\cdot\mathbf{x}_i}e^{-ik\mathbf{n}\cdot\mathbf{x}_i}\right|^2
$$

Define

$$
ik\mathbf{q}\cdot\mathbf{x}_i \equiv ik(\mathbf{n}_0 - \mathbf{n})\cdot\mathbf{x}_i
$$

and assume all the scatterers are identical, so that

$$
\begin{aligned}
\mathbf{p}_i &= \mathbf{p} \\
\mathbf{m}_i &= \mathbf{m}
\end{aligned}
$$

Then the sum applies only to the phase factor, giving an overall factor of

$$
\begin{aligned}
\mathscr{F}(\mathbf{q}) &= \left|\sum_i e^{ik\mathbf{q}\cdot\mathbf{x}_i}\right|^2 \\
&= \sum_i e^{ik\mathbf{q}\cdot\mathbf{x}_i}\sum_j e^{-ik\mathbf{q}\cdot\mathbf{x}_j} \\
&= \sum_{i,j} e^{ik\mathbf{q}\cdot(\mathbf{x}_i - \mathbf{x}_j)}
\end{aligned}
$$

There are two important limiting cases of this. When the scatterers are randomly distributed, as in a gas, the different phases tend to cancel, so only the diagonal terms contribute,

$$\begin{aligned}\mathscr{F}(\mathbf{q}) &= \sum_{i=j} e^{ik\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j)} \\ &= \sum_{i=j} 1 \\ &= N\end{aligned}$$

where N is the total number of scatterers. The second limiting case is when the scatterers form some sort of regular lattice. For a perfect lattice, the same thing happens, with the effect of a scatterer at $\mathbf{x}_i$ cancelling the effect of another scatterer at $-\mathbf{x}_i$. The wave progresses only in the forward direction (picture a clear crystal of pure quartz, for example). Scatterings do occur as a result of thermal vibrations which make the lattice imperfect. Jackson gives and explicit example of an exact result in eq.(10.20).

5 Perturbation theory of scattering

5.1 General formalism

Scattering can occur in a medium with spatially varying or time varying properties. If these variations are small, they may be treated perturbatively.

The analysis starts by assuming that

$$\begin{aligned}\mathbf{D} &\neq \epsilon_0\mathbf{E} \\ \mathbf{H} &\neq \frac{1}{\mu_0}\mathbf{B}\end{aligned}$$

so we treat all four fields as independent. Let ϵ_0, μ_0 be the *unperturbed values* of the dielectric constant and permeability (and *not* the vacuum values). Then we can rewrite Maxwell's equations in terms of the small differences

$$\begin{aligned}&\mathbf{D} - \epsilon_0\mathbf{E} \\ &\mathbf{B} - \mu_0\mathbf{H}\end{aligned}$$

Starting from

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{D} &= 0 \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} &= 0\end{aligned}$$

we have

$$\begin{aligned}0 &= \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} \\ &= \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times\mathbf{D} - \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E}) + \frac{\partial\mathbf{B}}{\partial t}\end{aligned}$$

and taking the curl,

$$0 = \boldsymbol{\nabla}\times\left(\frac{1}{\epsilon_0}\boldsymbol{\nabla}\times\mathbf{D} - \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E}) + \frac{\partial\mathbf{B}}{\partial t}\right)$$

$$
\begin{aligned}
&= \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{D}) - \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E})] + \boldsymbol{\nabla}\times\frac{\partial\mathbf{B}}{\partial t} \\
&= \frac{1}{\epsilon_0}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \frac{1}{\epsilon_0}\nabla^2\mathbf{D} - \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E})) + \mu_0\boldsymbol{\nabla}\times\frac{\partial\mathbf{H}}{\partial t} + \boldsymbol{\nabla}\times\frac{\partial}{\partial t}(\mathbf{B}-\mu_0\mathbf{H}) \\
&= -\frac{1}{\epsilon_0}\nabla^2\mathbf{D} + \mu_0\frac{\partial^2\mathbf{D}}{\partial t^2} - \frac{1}{\epsilon_0}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E})) + \boldsymbol{\nabla}\times\frac{\partial}{\partial t}(\mathbf{B}-\mu_0\mathbf{H})
\end{aligned}
$$

so that

$$
\nabla^2\mathbf{D} - \mu_0\epsilon_0\frac{\partial^2\mathbf{D}}{\partial t^2} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E})) + \epsilon_0\frac{\partial}{\partial t}[\boldsymbol{\nabla}\times(\mathbf{B}-\mu_0\mathbf{H})]
$$

where the terms on the right are small. With harmonic time dependence, and setting $\mu_0\epsilon_0\omega^2 = k^2$, this becomes

$$
\left(\nabla^2 + k^2\right)\mathbf{D} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E})) - i\omega\epsilon_0[\boldsymbol{\nabla}\times(\mathbf{B}-\mu_0\mathbf{H})]
$$

5.2 Born approximation

We can now perform a systematic perturbation theory, allowing small corrections to the permittivity and permeability,

$$
\begin{aligned}
\epsilon &= \epsilon^{(0)} + \epsilon^{(1)}(\mathbf{x}) + \epsilon^{(2)}(\mathbf{x}) + \ldots \\
\mu &= \mu^{(0)} + \mu^{(1)}(\mathbf{x}) + \mu^{(2)}(\mathbf{x}) + \ldots
\end{aligned}
$$

where $\epsilon^{(0)} = \epsilon_0, \mu^{(0)} = \mu_0$, and setting the fields equal to

$$
\begin{aligned}
\mathbf{D} &= \mathbf{D}^{(0)} + \mathbf{D}^{(1)} + \ldots \\
\mathbf{E} &= \mathbf{E}^{(0)} + \mathbf{E}^{(1)} + \ldots \\
\mathbf{D} &= \epsilon\mathbf{E} \\
&= \left(\epsilon^{(0)} + \epsilon^{(1)} + \epsilon^{(2)} + \ldots\right)\left(\mathbf{E}^{(0)} + \mathbf{E}^{(1)} + \ldots\right) \\
&= \epsilon^{(0)}\mathbf{E}^{(0)} + \left[\epsilon^{(0)}\mathbf{E}^{(1)} + \epsilon^{(1)}\mathbf{E}^{(0)}\right] + \left[\epsilon^{(1)}\mathbf{E}^{(1)} + \epsilon^{(2)}\mathbf{E}^{(0)} + \epsilon^{(0)}\mathbf{E}^{(2)}\right] + \ldots \\
\mathbf{B} &= \mathbf{B}^{(0)} + \mathbf{B}^{(1)} + \ldots \\
\mathbf{H} &= \mathbf{H}^{(0)} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)} + \ldots \\
\mathbf{B} &= \mu\mathbf{H} \\
&= \left(\mu^{(0)} + \mu^{(1)} + \mu^{(2)} + \ldots\right)\left(\mathbf{H}^{(0)} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)} + \ldots\right) \\
&= \mu^{(0)}\mathbf{H}^{(0)} + \left[\mu^{(1)}\mathbf{H}^{(0)} + \mu^{(0)}\mathbf{H}^{(1)}\right] + \left[\mu^{(2)}\mathbf{H}^{(0)} + \mu^{(1)}\mathbf{H}^{(1)} + \mu^{(0)}\mathbf{H}^{(2)}\right] + \ldots
\end{aligned}
$$

This means that the differences on the right side of the wave equation are of higher order than the field on the left.

$$
\begin{aligned}
\mathbf{D} - \epsilon_0\mathbf{E} &= \mathbf{D} - \epsilon_0\left[\mathbf{E}^{(0)} + \mathbf{E}^{(1)} + \mathbf{E}^{(2)} + \ldots\right] \\
&= \left[\epsilon^{(1)}\mathbf{E}^{(0)}\right] + \left[\epsilon^{(1)}\mathbf{E}^{(1)} + \epsilon^{(2)}\mathbf{E}^{(0)}\right] + \ldots \\
\mathbf{B} - \mu_0\mathbf{H} &= \mathbf{B} - \mu_0\left[\mathbf{H}^{(0)} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)} + \ldots\right] \\
&= \mu^{(0)}\mathbf{H}^{(0)} + \left[\mu^{(1)}\mathbf{H}^{(0)} + \mu^{(0)}\mathbf{H}^{(1)}\right] + \left[\mu^{(2)}\mathbf{H}^{(0)} + \mu^{(1)}\mathbf{H}^{(1)} + \mu^{(0)}\mathbf{H}^{(2)}\right] + \ldots \\
&\quad -\mu_0\left[\mathbf{H}^{(0)} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)} + \ldots\right] \\
&= \left[\mu^{(1)}\mathbf{H}^{(0)}\right] + \left[\mu^{(2)}\mathbf{H}^{(0)} + \mu^{(1)}\mathbf{H}^{(1)}\right] + \ldots
\end{aligned}
$$

Notice that the small source terms only depend on lower order fields.

As a first approximation, we put in the zeroth order approximations for $\mathbf{D}$ and $\mathbf{B}$,

$$\begin{aligned}\left(\nabla^2+k^2\right)\mathbf{D}^{(0)} &= -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left(\mathbf{D}-\epsilon_0\mathbf{E}\right)^{(0)}\right)-i\omega\epsilon_0\left[\boldsymbol{\nabla}\times\left(\mathbf{B}-\mu_0\mathbf{H}\right)^{(0)}\right]\\ &= 0\end{aligned}$$

and we see that $\mathbf{D}^{(0)}$ is the plane wave solution. At the next order, we have

$$\left(\nabla^2+k^2\right)\mathbf{D}^{(1)} = -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left(\epsilon^{(1)}\mathbf{E}^{(0)}\right)\right)-i\omega\epsilon_0\left[\boldsymbol{\nabla}\times\left(\mu^{(1)}\mathbf{H}^{(0)}\right)\right]$$

The solution to this equation is the first Born approximation, and as we show below it gives the first approximation to the scattered wave. Notice again that the wave equation for $\mathbf{D}^{(1)}$ has sources that only depend on the lowest order solutions for the fields, $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$.

We may continue to arbitrary order, to find corrections. Thus, at second order, we have

$$\left(\nabla^2+k^2\right)\mathbf{D}^{(2)} = -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left[\epsilon^{(1)}\mathbf{E}^{(1)}+\epsilon^{(2)}\mathbf{E}^{(0)}\right]\right)-i\omega\epsilon_0\left(\boldsymbol{\nabla}\times\left[\mu^{(2)}\mathbf{H}^{(0)}+\mu^{(1)}\mathbf{H}^{(1)}\right]\right)$$

As long as we have a sensible perturbative series for the permittivity and permeability, we may continue this process to arbitrary order.

5.3 First Born approximation for light scattering

For light scattering in gasses, higher order Born approximations are not generally appropriate because we do not really know ϵ or μ. Instead, we can make use of the first Born approximation by supposing

$$\begin{aligned}\epsilon &= \epsilon_0+\delta\epsilon\left(\mathbf{x}\right)\\ \mu &= \mu_0+\delta\mu\left(\mathbf{x}\right)\end{aligned}$$

where ϵ_0 and μ_0 are constant (but not necessarily the vacuum values) and that there are small, position and/or time dependent fluctuations in addition. First Born approximation immediately gives

$$\left(\nabla^2+k^2\right)\mathbf{D}^{(0)}=0$$

for the incoming plane wave, and the first order correction

$$\begin{aligned}\left(\nabla^2+k^2\right)\mathbf{D}^{(1)} &= -\mu_0\epsilon_0\boldsymbol{J}\\ \mu_0\epsilon_0\boldsymbol{J} &= \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)+i\omega\epsilon_0\left[\boldsymbol{\nabla}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right]\end{aligned}$$

This has solution

$$\mathbf{D}^{(1)}=\frac{1}{4\pi}\int \mathrm{d}^3x\frac{e^{ik\left|\mathbf{x}-\mathbf{x}'\right|}\boldsymbol{J}}{\left|\mathbf{x}-\mathbf{x}'\right|}$$

We now assume that the fluctuations in ϵ and μ are localized in space and look at the scattered wave $\mathbf{D}^{(1)}$ in the radiation zone,

$$\begin{aligned}\mathbf{D} &= \mathbf{D}^{(0)}+\mathbf{D}^{(1)}\\ &= \mathbf{D}^{(0)}+\frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\boldsymbol{J}\end{aligned}$$

The integral of $\boldsymbol{J}$ may be simplified by integration by parts by recalling the identity

$$\boldsymbol{\nabla}\times\left(f\mathbf{v}\right)=\boldsymbol{\nabla}f\times\mathbf{v}+f\boldsymbol{\nabla}\times\left(\mathbf{v}\right)$$

Applying this to the curl terms in the electric part of $\boldsymbol{J}$,

$$\begin{aligned}
\mathbf{D}_1^{(1)} &= \frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\boldsymbol{\nabla}'\times\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right) \\
&= \frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'\boldsymbol{\nabla}'\times\left[e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right] \\
&\quad -\frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'\left[\boldsymbol{\nabla}' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\times\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right]
\end{aligned}$$

The first term on the right may be integrated, giving a result that depends on the fields at large distance, where the perturbation, $\delta\epsilon$, vanishes. Then

$$\begin{aligned}
\mathbf{D}_1^{(1)} &= -\frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'\left[-ike^{-ik\mathbf{n}\cdot\mathbf{x}'}\mathbf{n}\times\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right] \\
&= \frac{ike^{ikr}}{4\pi r}\mathbf{n}\times\int \mathrm{d}^3x'\left[\left(e^{-ik\mathbf{n}\cdot\mathbf{x}'}\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right] \\
&= \frac{ike^{ikr}}{4\pi r}\mathbf{n}\times\int \mathrm{d}^3x'\left[\left(-\boldsymbol{\nabla}' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\right)\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right] \\
&= -\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\mathbf{n}\times\left(\mathbf{n}\times\int \mathrm{d}^3x'\left[e^{-ik\mathbf{n}\cdot\mathbf{x}'}\delta\epsilon\mathbf{E}^{(0)}\right]\right)
\end{aligned}$$

where we have done a second integration by parts and again dropped the surface term. The electric dipole term is the lowest order term when we expand $e^{-ik\mathbf{n}\cdot\mathbf{x}'} = 1 - ik\mathbf{n}\cdot\mathbf{x}' + \ldots$, so the electric dipole field is

$$\mathbf{D}_1^{(1)} = -\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\int \mathrm{d}^3x'\left[\mathbf{n}\times\left(\mathbf{n}\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right]$$

Comparing with the general electric dipole expression,

$$\mathbf{D}_{sc} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\right]$$

we see that we have an effective electric dipole source,

$$\mathbf{p} = \int \mathrm{d}^3x'\delta\epsilon\left(\mathbf{x}'\right)\mathbf{E}^{(0)}\left(\mathbf{x}'\right)$$

where $\mathbf{E}^{(0)}$ is the incident wave.

The second term in $\boldsymbol{J}$ works the same way,

$$\begin{aligned}
\mathbf{D}_2^{(1)} &= \frac{i\omega\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\boldsymbol{\nabla}'\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right] \\
&= -\frac{i\omega\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[-ik\mathbf{n}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right] \\
&= -\frac{k^2\epsilon_0}{4\pi c}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{n}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right]
\end{aligned}$$

and this expression has the same form as the magnetic dipole field,

$$\mathbf{D}_{sc} = \frac{k^2\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\left[-\frac{1}{c}\mathbf{n}\times\mathbf{m}\right]$$

if we identify

$$\mathbf{m} = \int \mathrm{d}^3x'\left(\delta\mu\mathbf{H}^{(0)}\right)$$

The total scattering amplitude is $\mathbf{D}_{sc} = \mathbf{D}^{(1)} = \mathbf{D}_1^{(1)} + \mathbf{D}_2^{(1)}$, and the form is exactly the same as the form of the electric field for electric and magnetic dipole radiation,

$$\mathbf{D}_{sc} = \frac{k^2\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \frac{1}{c}\mathbf{n}\times\mathbf{m}\right]$$

If we define

$$\mathbf{D}_{sc} = \frac{e^{ikr}}{r}\mathbf{A}_{sc}$$

then the differential cross-section follows immediately as

$$\frac{d\sigma}{d\Omega} = \frac{\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}\right|^2}{\left|\mathbf{D}^{(0)}\right|^2}$$

If the initial wave has the form

$$\begin{aligned}\mathbf{D}^{(0)} &= \boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}\\ \mathbf{B}^{(0)} &= \sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{n}_0\times\mathbf{D}^{(0)}\end{aligned}$$

then the general multipole expression for the scattered wave becomes

$$\begin{aligned}\mathbf{A}_{sc} &= -\frac{k^2}{4\pi}\mathbf{n}\times\left(\mathbf{n}\times\int \mathrm{d}^3x'\left[e^{-ik\mathbf{n}\cdot\mathbf{x}'}\delta\epsilon\mathbf{E}^{(0)}\right]\right) - \frac{k^2\epsilon_0}{4\pi c}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{n}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right]\\ &= \frac{k^2}{4\pi}D_0\int \mathrm{d}^3x' e^{ik\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}(\mathbf{n}\times\boldsymbol{\varepsilon}_0)\times\mathbf{n} - \frac{\delta\mu}{\mu_0}\left[\mathbf{n}\times(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\right]\right]\end{aligned}$$

so that $\frac{d\sigma}{d\Omega}$ is the square of

$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} = \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{ik\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\left[(\mathbf{n}\times\boldsymbol{\varepsilon}_0)\times\mathbf{n}\right] - \frac{\delta\mu}{\mu_0}\boldsymbol{\varepsilon}^*\cdot\left[\mathbf{n}\times(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\right]\right]$$

Rewriting

$$\begin{aligned}\boldsymbol{\varepsilon}^*\cdot\left[(\mathbf{n}\times\boldsymbol{\varepsilon}_0)\times\mathbf{n}\right] &= \boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\\ \boldsymbol{\varepsilon}^*\cdot\left[\mathbf{n}\times(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\right] &= (\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\cdot(\boldsymbol{\varepsilon}^*\times\mathbf{n})\\ &= -(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\end{aligned}$$

this becomes

$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} = \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 + \frac{\delta\mu}{\mu_0}(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\right]$$

where $\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n})$. A simple approximation is to then take $\delta\epsilon$ constant inside a sphere of radius a, and set $\delta\mu = 0$. Then the integral becomes

$$\begin{aligned}\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} &= \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right]\\ &= \frac{k^2}{4\pi}\frac{\delta\epsilon}{\epsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\\ &= \frac{k^2}{4\pi}\frac{\delta\epsilon}{\epsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)\int r'^2\mathrm{d}r'\mathrm{d}\varphi'\mathrm{d}(\cos\theta')\, e^{i\mathbf{q}\cdot\mathbf{x}'}\end{aligned}$$

Taking $\mathbf{n}_0$ in the z-direction,

$$\begin{aligned}
\int r'^2 \mathrm{d}r' \mathrm{d}\varphi' \mathrm{d}\left(\cos\theta'\right) e^{i\mathbf{q}\cdot\mathbf{x}'} &= \int r'^2 \mathrm{d}r' \mathrm{d}\varphi' \mathrm{d}\left(\cos\theta'\right) e^{iqr'\cos\theta'} \\
&= 2\pi \int r'^2 \mathrm{d}r' \mathrm{d}\left(\cos\theta'\right) e^{iqr'\cos\theta'} \\
&= 2\pi \int_0^a r'^2 \mathrm{d}r' \int_{-1}^1 \mathrm{d}x e^{iqr'x} \\
&= 2\pi \int_0^a r'^2 \mathrm{d}r' \left[\frac{e^{iqr'x}}{iqr'}\right]_{-1}^{+1} \\
&= 2\pi \int_0^a r'^2 \mathrm{d}r' \left[\frac{e^{iqr'} - e^{-iqr'}}{iqr'}\right] \\
&= 4\pi \int_0^a r'^2 \mathrm{d}r' \left[\frac{\sin qr'}{qr'}\right] \\
&= \frac{4\pi}{q} \int_0^a \mathrm{d}r' r' \sin qr' \\
&= \frac{4\pi}{q} \left(-\frac{d}{dq} \int_0^a \mathrm{d}r' \cos qr'\right) \\
&= \frac{4\pi}{q} \left(-\frac{d}{dq} \left[\frac{\sin qr'}{q}\right]_0^a\right) \\
&= \frac{4\pi}{q} \frac{d}{dq} \left(-\frac{\sin qa}{q}\right) \\
&= \frac{4\pi}{q^3} \left(\sin qa - qa\cos qa\right)
\end{aligned}$$

In the long wavelength limit, $k, q \to 0$, and this becomes

$$\begin{aligned}
4\pi \lim_{q\to 0} \left(\frac{\sin qa - qa\cos qa}{q^3}\right) &= 4\pi \lim_{q\to 0} \left(\frac{qa - \frac{1}{3!}q^3a^3 - qa\left(1 - \frac{1}{2}q^2a^2\right) + O\left(q^4\right)}{q^3}\right) \\
&= 4\pi \lim_{q\to 0} \left(\frac{-\frac{1}{3!}q^3a^3 + \frac{1}{2}q^3a^3 + O\left(q^4\right)}{q^3}\right) \\
&= 4\pi \lim_{q\to 0} \left(-\frac{1}{3!}a^3 + \frac{1}{2}a^3 + O\left(q^3\right)\right) \\
&= \frac{4}{3}\pi a^3
\end{aligned}$$

The differential cross section is therefore

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left|\frac{k^2}{4\pi}\frac{\delta\epsilon}{\epsilon_0}\left(\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right)\frac{4}{3}\pi a^3\right|^2 \\
&= k^4 a^6 \left|\frac{\delta\epsilon}{3\epsilon_0}\right|^2 \left|\left(\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right)\right|^2
\end{aligned}$$

This agrees, to first order in $\delta\epsilon$, with our result for scattering from a dielectric sphere,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \sin^2\varphi$$

when we set $\epsilon - \epsilon_0 = \delta\varepsilon$ and $\epsilon + 2\epsilon_0 = 3\epsilon_0 + \delta\epsilon$, since

$$\begin{aligned}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} &= \frac{\delta\epsilon}{3\epsilon_0+\delta\epsilon} \\ &= \frac{\delta\epsilon}{3\epsilon_0\left(1+\frac{\delta\epsilon}{3\epsilon_0}\right)} \\ &= \frac{\delta\epsilon}{3\epsilon_0}\left(1-\frac{\delta\epsilon}{3\epsilon_0}+\ldots\right) \\ &= \frac{\delta\epsilon}{3\epsilon_0}+O\left((\delta\epsilon)^2\right)\end{aligned}$$

6 The blue sky

Jackson gives two treatments of scattering in the atmosphere. The first is to approximate the atmosphere as a dilute gas with randomly distributed molecules. Then taking the molecules to have dipole moments

$$\mathbf{p}_i = \epsilon_0 \gamma_{mol} \mathbf{E}$$

where γ_{mol} is the molecular polarizability, the effective dielectric constant is

$$\delta\epsilon = \epsilon_0 \sum_i \gamma_{mol} \delta^3(\mathbf{x}-\mathbf{x}_i)$$

Then

$$\begin{aligned}\frac{\boldsymbol{\varepsilon}^* \cdot \mathbf{A}_{sc}}{D_0} &= \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right] \\ &= \frac{k^2}{4\pi}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\gamma_{mol}\sum_i\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\delta^3(\mathbf{x}-\mathbf{x}_i) \\ &= \frac{k^2}{4\pi}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\gamma_{mol}\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}\end{aligned}$$

so that

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left|\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0}\right|^2 \\ &= \frac{k^4}{16\pi^2}\left|\gamma_{mol}\right|^2\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2\left|\sum e_i^{i\mathbf{q}\cdot\mathbf{x}'}\right|^2\end{aligned}$$

When we sum over all particles, the structure function

$$\mathcal{F} = \left|\sum e_i^{i\mathbf{q}\cdot\mathbf{x}'}\right|^2$$

will be the total number of particles because of the randomness of the gas. For a single particle, we may therefore drop the phase – the average differential cross-section per particle is just

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}\left|\gamma_{mol}\right|^2\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2$$

Now, for a dilute gas,

$$\epsilon_r \approx 1 + N\gamma_{mol}$$

where ϵ_r is the relative dielectric constant and N is the number density of molecules. Therefore,

$$|\gamma_{mol}|^2 = \frac{|\epsilon_r - 1|^2}{N^2}$$

Rewriting this in terms of the index of refraction,

$$\begin{aligned}
n &= \sqrt{\frac{\mu}{\mu_0}\frac{\epsilon}{\epsilon_0}} \\
&= \sqrt{\epsilon_r} \\
n - 1 &= \sqrt{\epsilon_r} - 1 \\
n - 1 &= \sqrt{1 + (\epsilon_r - 1)} - 1 \\
n - 1 &\approx 1 + \frac{1}{2}(\epsilon_r - 1) - 1 \\
2(n-1) &= \epsilon_r - 1
\end{aligned}$$

so that the differential cross section becomes

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{k^4}{16\pi^2}|\gamma_{mol}|^2 |\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2 \\
&= \frac{k^4}{16\pi^2}\frac{|\epsilon_r - 1|^2}{N^2}|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2 \\
&= \frac{k^4}{4\pi^2}\frac{|n - 1|^2}{N^2}|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2
\end{aligned}$$

For unpolarized initial and final states, we average over initial and sum over final polarizations, so the polarization factor becomes

$$\left(\frac{1}{2}\sum_{\boldsymbol{\varepsilon}_0 = \mathbf{i},\mathbf{j}}\right)\left(\sum_{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2}\right)|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2 = \frac{1}{2}\sum_{\boldsymbol{\varepsilon}_0 = \mathbf{i},\mathbf{j}}\sum_{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2}|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2$$

which we can find by dropping the magnetic dipole part of our earlier calculation for the sphere. Averaging over

$$\begin{aligned}
\boldsymbol{\varepsilon}_{\parallel} &= \frac{1}{\sin\theta}(\mathbf{n}_0 \times \mathbf{n}) \times \mathbf{n} \\
\boldsymbol{\varepsilon}_{\perp} &= \frac{1}{\sin\theta}\mathbf{n}_0 \times \mathbf{n}
\end{aligned}$$

we first find the parallel and perpendicular cases separately:

$$\begin{aligned}
\frac{d\sigma_{\parallel}}{d\Omega} &= \left(\frac{k^4}{4\pi^2}\frac{|n-1|^2}{N^2}\right)\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\boldsymbol{\varepsilon}_{\parallel}^* \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2 \\
\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\boldsymbol{\varepsilon}_{\parallel}^* \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2 &= \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\frac{1}{\sin\theta}((\mathbf{n}_0 \times \mathbf{n}) \times \mathbf{n}) \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2 \\
&= \frac{1}{2}\frac{1}{\sin^2\theta}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}|((\mathbf{n}_0 \times \mathbf{n}) \times \mathbf{n}) \cdot \hat{\boldsymbol{\varepsilon}}_0|^2 \\
&= \frac{1}{2}\frac{1}{\sin^2\theta}\left|((\mathbf{n}_0 \times \mathbf{n}) \times \mathbf{n}) \cdot \hat{\boldsymbol{\varepsilon}}_{0\parallel}\right|^2 \\
&= \frac{1}{2}\cos^2\theta
\end{aligned}$$

and

$$
\begin{aligned}
\frac{d\sigma_\perp}{d\Omega} &= \left(\frac{k^4}{4\pi^2}\frac{|n-1|^2}{N^2}\right)\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}|\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_0|^2 \\
\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}|\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_0|^2 &= \frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\frac{1}{\sin\theta}(\mathbf{n}_0\times\mathbf{n})\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2 \\
&= \frac{1}{2}\left|\frac{1}{\sin\theta}(\mathbf{n}_0\times\mathbf{n})\cdot\hat{\boldsymbol{\varepsilon}}_{0\perp}\right|^2 \\
&= \frac{1}{2}
\end{aligned}
$$

If the final polarization is unmeasured, we sum these, giving

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma_\parallel}{d\Omega}+\frac{d\sigma_\perp}{d\Omega} \\
&= \left(\frac{k^4}{4\pi^2}\frac{|n-1|^2}{N^2}\right)\frac{1}{2}\left(1+\cos^2\theta\right)
\end{aligned}
$$

The total cross section is now given by integrating over angles

$$
\begin{aligned}
\sigma &= \frac{k^4}{8\pi^2}\frac{|n-1|^2}{N^2}\int\left(1+\cos^2\theta\right)\mathrm{d}\Omega \\
&= \frac{k^4}{8\pi^2}\frac{|n-1|^2}{N^2}2\pi\int_{-1}^{1}\left(1+x^2\right)\mathrm{d}x \\
&= \frac{k^4}{8\pi^2}\frac{|n-1|^2}{N^2}2\pi\left[x+\frac{1}{3}x^3\right]_{-1}^{1} \\
&= \frac{k^4}{8\pi^2}\frac{|n-1|^2}{N^2}2\pi\left[\frac{8}{3}\right] \\
&= \frac{2k^4}{3\pi}\frac{|n-1|^2}{N^2}
\end{aligned}
$$

This is the total scattered flux per incident flux from a single molecule. Remembering that σ is the effective area of scatterers, the total fraction of light scattered, $\left|\frac{dI}{I}\right|$, in a travel distance dx is therefore $N\sigma\mathrm{d}x$. Suppose the incident beam has intensity I_0. Then

$$
\begin{aligned}
\frac{dI}{I} &= -N\sigma\mathrm{d}x \\
I &= I_0e^{-N\sigma x}
\end{aligned}
$$

The absorption coefficient, $\alpha = N\sigma$ is therefore

$$
\alpha = \frac{2k^4}{3\pi}\frac{|n-1|^2}{N}
$$

This result describes Rayleigh scattering.

And improved version, due to Einstein, starts with the density fluctuations ΔN_j in cells of volume v, as given by the Clausius-Mossotti relation (eq.4.70),

$$
\delta\epsilon_j = \frac{(\epsilon_r-1)(\epsilon_r+2)}{3Nv}\Delta N_j
$$

Carrying out the cross-section calculation again, and summing over all cells, gives the attenuation coefficient as

$$\alpha = \frac{k^4}{6\pi N}\left|\frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3}\right|^2 \frac{\Delta N_V^2}{NV}$$

where ΔN_V^2 is the mean-square number fluctuation per unit volume,

$$\Delta N_V^2 = \left\langle N_i^2 \right\rangle - \left\langle N_i \right\rangle^2$$

The final ratio may be expressed in terms of the isothermal compressibility, $\beta_T = -\frac{1}{V}\left(\frac{dV}{dP}\right)_T$,

$$\frac{\Delta N_V^2}{NV} = NkT\beta_T$$

so that the absorption coefficient becomes

$$\alpha = \frac{kT\beta_T}{6\pi} k^4 \left|\frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3}\right|^2$$

This is called the Einstein-Smoluchowski formula.

With $NkT\beta_T = 1$ and the approximation

$$(\epsilon_r - 1)\frac{(\epsilon_r + 2)}{3} \approx 2(n-1)\cdot 1$$

as above, we recover the previous result.