Symmetry and tensors

Rotations and tensors

A rotation of a 3-vector is accomplished by an orthogonal transformation. Represented as a matrix, $\mathbb{A}$, we replace each vector, $\mathbf{v}$, by a rotated vector, $\mathbf{v}'$, given by multiplying by $\mathbb{A}$,

$$\mathbf{v}' = \mathbb{A}\mathbf{v}$$

In index notation,

$$v'_m = \sum_n A_{mn} v_n$$

Since a rotation must preserve lengths of vectors, we require

$$\mathbf{v}'^2 = \sum_m v'_m v'_m = \sum_m v_m v_m = \mathbf{v}^2$$

Therefore,

$$\begin{aligned}
\sum_m v_m v_m &= \sum_m v'_m v'_m \\
&= \sum_m \left(\sum_n A_{mn} v_n \right) \left(\sum_k A_{mk} v_k \right) \\
&= \sum_{k,n} \left(\sum_m A_{mn} A_{mk} \right) v_n v_k
\end{aligned}$$

Since x_n is arbitrary, this is true if and only if

$$\sum_m A_{mn} A_{mk} = \delta_{nk}$$

which we can rewrite using the transpose, $A^t_{mn} = A_{nm}$, as

$$\sum_m A^t_{nm} A_{mk} = \delta_{nk}$$

In matrix notation, this is

$$\mathbb{A}^t \mathbb{A} = \mathbb{I}$$

where $\mathbb{I}$ is the identity matrix. This is equivalent to $\mathbb{A}^t = \mathbb{A}^{-1}$.

Multi-index objects such as matrices, M_{mn}, or the Levi-Civita tensor, ε_{ijk}, have definite transformation properties under rotations. We call an object a (rotational) tensor if each index transforms in the same way as a vector. An object with no indices, that is, a function, does not transform at all and is called a scalar. A matrix M_{mn} is a (second rank) tensor if and only if, when we rotate vectors $\mathbf{v}$ to $\mathbf{v}'$, its new components are given by

$$M'_{mn} = \sum_{jk} A_{mj} A_{nk} M_{jk}$$

This is what we expect if we imagine M_{mn} to be built out of vectors as $M_{mn} = u_m v_n$, for example. In the same way, we see that the Levi-Civita tensor transforms as

$$\varepsilon'_{ijk} = \sum_{lmn} A_{il} A_{jm} A_{kn} \varepsilon_{lmn}$$

Recall that ε_{ijk}, because it is totally antisymmetric, is completely determined by only one of its components, say, ε_{123}. Another way to say this is that every totally antisymmetric 3-index object, A_{ijk}, is proportional to ε_{ijk}. In particular, this applies to ε'_{ijk}, so we have

$$\varepsilon'_{ijk} = \lambda\varepsilon_{ijk}$$

We can find λ by looking at one component,

$$\begin{aligned}\varepsilon'_{123} &= \lambda\varepsilon_{123} \sum_{lmn} A_{il}A_{jm}A_{kn}\varepsilon_{lmn} \\ &= \sum_{lmn} A_{1l}A_{2m}A_{3n}\varepsilon_{lmn}\end{aligned}$$

and this last expression is exactly the determinant of the matrix $\mathbb{A}$. Since the determinant of a product of matrices equals the product of the determinants, and the determinant of the transpose of a matrix equals the determinant of the matrix, we know that

$$\begin{aligned}\det\left(\mathbb{A}^t\mathbb{A}\right) &= \det(\mathbb{I}) \\ \det\mathbb{A}^t\det\mathbb{A} &= 1 \\ (\det\mathbb{A})^2 &= 1 \\ \det\mathbb{A} &= \pm 1\end{aligned}$$

If $\det\mathbb{A} = +1$, then the rotation is called a proper rotation, and we see that the Levi-Civita tensor is unchanged by a proper rotation. This is a very special property: only two matrices (or combinations of them) are rotationally invariant. They are the Levi-Civita tensor, ε_{klm}, and the Kronecker delta, δ_{mn}.

Exercise: Prove that the Kronecker delta is rotationally invariant.

Discrete symmetries

We next turn to discrete symmetries, that is, symmetries which have a finite number of eigenvalues. Discrete symmetries are useful in classifying solutions, eliminating integrals, or checking calculations, so it is helpful to know how each of our physical fields transforms under them. We discuss parity and time reversal.

Parity

If $\det\mathbb{A} = -1$, then $\mathbb{A}$ is called an improper rotation, and it may be written as the product of a proper rotation and the parity operator, $\mathbb{P}$, with components

$$P_{mn} = -\delta_{mn}$$

Equivalently, we may define

$$\begin{aligned}\mathbb{P}\mathbf{x} &= -\mathbf{x} \\ \mathbb{P}t &= t\end{aligned}$$

Then vectors which behave in the same way,

$$\mathbb{P}\mathbf{v} = -\mathbf{v}$$

are called *proper vectors*. Proper vectors are odd under parity.

Notice that parity turns a right-handed coordinate system (i.e., one for which $\mathbf{i}\times\mathbf{j} = \mathbf{k}$) into a left-handed system,

$$\begin{aligned}(-\mathbf{i})\times(-\mathbf{j}) &= -(-\mathbf{k}) \\ \mathbf{i}'\times\mathbf{j}' &= -\mathbf{k}'\end{aligned}$$

Combinations of vectors may have different properties under parity. For example, suppose we take the cross product of two vectors,

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}$$

Since

$$\begin{aligned} \mathbb{P}\mathbf{u} &= -\mathbf{u} \\ \mathbb{P}\mathbf{v} &= -\mathbf{v} \end{aligned}$$

we see that $\mathbf{w}$ is *even* under parity even though it looks like a vector. The dot product also gives an even scalar,

$$\begin{aligned} \mathbb{P}(\mathbf{u} \cdot \mathbf{v}) &= (-\mathbf{u}) \cdot (-\mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{v} \end{aligned}$$

There are various names for even- and odd-parity vectors. Odd parity vectors are either simply called vectors, or are called polar vectors. Even parity vectors are called axial vectors or pseudovectors.

Time reversal

A second important discrete symmetry is time reversal, $t \to -t$. Any tensor is said to be even or odd under time reversal if it stays the same (even) or changes sign (odd) when the sign of t is reversed. We define

$$\begin{aligned} \mathbb{T}\mathbf{x} &= \mathbf{x} \\ \mathbb{T}t &= -t \end{aligned}$$

Examples

It is not hard to find the parity and time reversal. Starting with Newton's second law,

$$\mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2}$$

The position vector $\mathbf{x}$ is a vector, mass is a scalar, and time is unaffected by parity, so the parity operator applied to force,

$$\mathbb{P}\mathbf{F} = m\frac{d^2(\mathbb{P}\mathbf{x})}{d(\mathbb{P}t)^2} = -m\frac{d^2\mathbf{x}}{dt^2} = -\mathbf{F}$$

shows that it is also odd under parity and therefore a proper vector. Replacing $t \to -t$ changes the derivatives,

$$\frac{d^2}{dt^2} = \frac{d}{dt}\frac{d}{dt} \to \left(-\frac{d}{dt}\right)\left(-\frac{d}{dt}\right) = \frac{d^2}{dt^2}$$

while leaving the position unchanged, so the acceleration and force are even under time reversal. The velocity and momentum,

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{x}}{dt}$$

however, are different. Like $\mathbf{x}$ they are vectors (odd under parity), but because of the single time derivative, both $\mathbf{p}$ and $\mathbf{v}$ are odd under time reversal.

Angular momentum is a cross product,

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}$$

Applying the parity operator,

$$\begin{aligned} \mathbb{P}\mathbf{L} &= (\mathbb{P}\mathbf{x}) \times (\mathbb{P}\mathbf{p}) \\ &= (-\mathbf{x}) \times (-\mathbf{p}) \\ &= \mathbf{L} \end{aligned}$$

so $\mathbf{L}$ is a pseudovector. Under time reversal, $\mathbf{L}$ is odd, since

$$\begin{aligned}\mathbb{T}\mathbf{L} &= (\mathbb{T}\mathbf{x}) \times (\mathbb{T}\mathbf{p}) \\ &= \mathbf{x} \times (-\mathbf{p}) \\ &= -\mathbf{L}\end{aligned}$$

The discrete transformations of electromagnetic quantities are also straightforward to find. Starting from the Lorentz force law,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

the parity operator gives

$$\begin{aligned}\mathbb{P}\mathbf{F} &= q\left((\mathbb{P}\mathbf{E}) + (\mathbb{P}\mathbf{v}) \times (\mathbb{P}\mathbf{B})\right) \\ -\mathbf{F} &= q\left((\mathbb{P}\mathbf{E}) + (-\mathbf{v}) \times (\mathbb{P}\mathbf{B})\right)\end{aligned}$$

and substituting the original expression for the force into this,

$$-q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = q\left((\mathbb{P}\mathbf{E}) + (-\mathbf{v}) \times (\mathbb{P}\mathbf{B})\right)$$

Comparing like terms shows that

$$\begin{aligned}\mathbb{P}\mathbf{E} &= -\mathbf{E} \\ \mathbb{P}\mathbf{B} &= \mathbf{B}\end{aligned}$$

so that $\mathbf{E}$ is a (proper) vector and $\mathbf{B}$ is a pseudovector.

For time reversal,

$$\begin{aligned}\mathbb{T}\mathbf{F} &= q\left((\mathbb{T}\mathbf{E}) + (\mathbb{T}\mathbf{v}) \times (\mathbb{T}\mathbf{B})\right) \\ \mathbf{F} &= q\left((\mathbb{T}\mathbf{E}) + (-\mathbf{v}) \times (\mathbb{T}\mathbf{B})\right) \\ q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) &= q\left((\mathbb{T}\mathbf{E}) + (-\mathbf{v}) \times (\mathbb{T}\mathbf{B})\right)\end{aligned}$$

so comparing terms we have

$$\begin{aligned}\mathbb{T}\mathbf{E} &= \mathbf{E} \\ \mathbb{T}\mathbf{B} &= -\mathbf{B}\end{aligned}$$

The electric field is even and the magnetic field odd.

Finally, the Maxwell equations give the parity and time reversal properties of the charge and current densities. First, notice that the del operator is odd under parity and even under time reversal:

$$\begin{aligned}\mathbb{P}\boldsymbol{\nabla} &= \frac{\partial}{\partial\left(\mathbb{P}x^i\right)} = -\boldsymbol{\nabla} \\ \mathbb{T}\boldsymbol{\nabla} &= \boldsymbol{\nabla}\end{aligned}$$

Then we have

$$\begin{aligned}\mathbb{P}\rho &= \mathbb{P}\left(\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E}\right) \\ &= \epsilon_0\left(\mathbb{P}\boldsymbol{\nabla}\right) \cdot \left(\mathbb{P}\mathbf{E}\right) \\ &= \epsilon_0\left(-\boldsymbol{\nabla}\right) \cdot \left(-\mathbf{E}\right) \\ &= \rho\end{aligned}$$

and

$$\begin{aligned}\mathbb{T}\rho &= \mathbb{T}\left(\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E}\right) \\ &= \epsilon_0\left(\mathbb{T}\boldsymbol{\nabla}\right) \cdot \left(\mathbb{T}\mathbf{E}\right) \\ &= \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} \\ &= \rho\end{aligned}$$

so the charge density is even under both parity and time reversal. For the current density,

$$\begin{aligned}
\mathbb{P}\mathbf{J} &= \frac{1}{\mu_0}\mathbb{P}\left(\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= \frac{1}{\mu_0}\left(\left(\mathbb{P}\boldsymbol{\nabla}\right)\times\left(\mathbb{P}\mathbf{B}\right)-\frac{1}{c^2}\frac{\partial\left(\mathbb{P}\mathbf{E}\right)}{\partial\left(\mathbb{P}t\right)}\right) \\
&= \frac{1}{\mu_0}\left(\left(-\boldsymbol{\nabla}\right)\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\left(-\mathbf{E}\right)}{\partial t}\right) \\
&= -\frac{1}{\mu_0}\left(\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= -\mathbf{J}
\end{aligned}$$

and for time reversal,

$$\begin{aligned}
\mathbb{T}\mathbf{J} &= \frac{1}{\mu_0}\mathbb{T}\left(\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= \frac{1}{\mu_0}\left(\left(\mathbb{T}\boldsymbol{\nabla}\right)\times\left(\mathbb{T}\mathbf{B}\right)-\frac{1}{c^2}\frac{\partial\left(\mathbb{T}\mathbf{E}\right)}{\partial\left(\mathbb{T}t\right)}\right) \\
&= \frac{1}{\mu_0}\left(\boldsymbol{\nabla}\times\left(-\mathbf{B}\right)-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial\left(-t\right)}\right) \\
&= -\frac{1}{\mu_0}\left(\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= -\mathbf{J}
\end{aligned}$$

so current density is odd under both. This is consistent with $\mathbf{J}=\rho\mathbf{v}$ since $\mathbf{v}$ is odd under both time reversal and parity.