The unification of Newtonian dynamics and electrodynamics

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We are now in a position to write both Newtonian dynamics and Maxwell electrodynamics in the common framework of special relativity. As we have anticipated, Newton's second law requires modification, but Maxwell's equations do not. Still, both must be written in terms of 4-dimensional, spacetime notation. This notation lets us see immediately which equations possess Lorentz symmetry.

1 Relativistic kinematics and dynamics

1.1 Addition of 4-velocities

The ordinary 3-velocity no longer adds as a vector, because it does not transform linearly under boosts. To see this, consider an infinitesimal boost in the x-direction,

$$\begin{aligned} cdt' &= \gamma\left(cdt - \beta dx\right) \\ dx' &= \gamma\left(dx - \beta cdt\right) \\ dy' &= dy \\ dz' &= dz \end{aligned}$$

If a particle moves with velocity

$$u = \frac{dx}{dt}$$

in the initial frame, then in the primed frame it moves with velocity

$$\begin{aligned} \frac{dx'}{dt'} &= \frac{\gamma\left(dx - \beta cdt\right)}{\gamma\left(dt - \frac{1}{c}\beta dx\right)} \\ &= \frac{\frac{dx}{dt} - \beta c}{1 - \frac{1}{c}\beta\frac{dx}{dt}} \\ &= \frac{u - v}{1 - \frac{uv}{c^2}} \end{aligned}$$

so the velocities do not add

$$u' \neq u - v$$

We can see this in the vector case as well. Using the differential of the general transform

$$\begin{aligned} cdt' &= \gamma\left(cdt - \boldsymbol{\beta}\cdot d\mathbf{x}\right) \\ d\mathbf{x}' &= d\mathbf{x} + \frac{\gamma - 1}{\beta^2}\left(\boldsymbol{\beta}\cdot d\mathbf{x}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}cdt \end{aligned}$$

we find

$$\mathbf{u}' = \frac{d\mathbf{x}'}{dt'}$$

$$
\begin{aligned}
&= \frac{d\mathbf{x} + \frac{\gamma-1}{\beta^2}\left(\boldsymbol{\beta}\cdot d\mathbf{x}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}c\,dt}{\gamma\left(dt - \frac{1}{c}\boldsymbol{\beta}\cdot d\mathbf{x}\right)} \\
&= \frac{\frac{d\mathbf{x}}{dt} + \frac{\gamma-1}{\beta^2}\left(\boldsymbol{\beta}\cdot\frac{d\mathbf{x}}{dt}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}c}{\gamma\left(1 - \frac{1}{c}\boldsymbol{\beta}\cdot\frac{d\mathbf{x}}{dt}\right)} \\
&= \frac{\mathbf{u} + \gamma\left(\hat{\mathbf{v}}\cdot\mathbf{u}\right)\hat{\mathbf{v}} - \left(\hat{\mathbf{v}}\cdot\mathbf{u}\right)\hat{\mathbf{v}} - \gamma\mathbf{v}}{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)} \\
&= \frac{\mathbf{u} - \left(\hat{\mathbf{v}}\cdot\mathbf{u}\right)\hat{\mathbf{v}}}{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)} - \frac{\mathbf{v} - \left(\hat{\mathbf{v}}\cdot\mathbf{u}\right)\hat{\mathbf{v}}}{1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}} \\
&= \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}} + \frac{\mathbf{u}_{\perp}}{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)}
\end{aligned}
$$

so that not only does the component of $\mathbf{u}$ along $\mathbf{v}$ not simply add, the perpendicular component is modified nonlinearly as well.

Nonetheless, spacetime is a vector space, and we should be able to add vectors. The problem here is that $\mathbf{u}$ does not transform as a contravariant vector because the time derivative is taken with respect to the coordinate time instead of the proper time. If we use the time, τ, which all inertial observers agree on, then it is natural to define the 4-velocity as

$$
u^{\alpha} = \frac{dx^{\alpha}}{d\tau}
$$

This is easily seen to be a 4-vector. Since $d\tau = d\tau'$, we have

$$
\begin{aligned}
u'^{\alpha} &= \frac{dx'^{\alpha}}{d\tau'} \\
&= \frac{dx'^{\alpha}}{d\tau} \\
&= \frac{d}{d\tau}\left(\sum_{\beta} M^{\alpha}{}_{\beta} x^{\beta}\right) \\
&= \sum_{\beta} M^{\alpha}{}_{\beta}\frac{dx^{\beta}}{d\tau} \\
&= \sum_{\beta} M^{\alpha}{}_{\beta} U^{\beta}
\end{aligned}
$$

and this is exactly the transformation law for a 4-vector. Since the transformation is linear, linear combinations of 4-vectors will also transform as 4-vectors.

To see that this gives the correct result, we write the sum of two 4-vectors in terms of their time and space components. Using the relationship between time and proper time,

$$
d\tau = dt\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}
$$

derived above, and the notation $x^{\alpha} = (ct, \mathbf{x})$ we have for a single 4-vector,

$$
\begin{aligned}
u^{\alpha} &= \frac{dx^{\alpha}}{d\tau} \\
&= \frac{dx^{\alpha}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\,dt} \\
&= \gamma\left(c\frac{dt}{dt}, \frac{d\mathbf{x}}{dt}\right) \\
&= \gamma\left(c, \mathbf{v}\right)
\end{aligned}
$$

Every 4-velocity can be written in this way for some 3-velocity, $\mathbf{v}$. Now recall that the importance of this transformation law is that the magnitude of 4-vectors is Lorentz invariant. This means that $\|u^\alpha\|^2$ must be some number that all inertial observers agree on. We can check this directly:

$$\begin{aligned}
\|u^\alpha\|^2 &= -\left(U^0\right)^2 + \left(U^1\right)^2 + \left(U^2\right)^2 + \left(U^3\right)^2 \\
&= -\gamma^2 c^2 + \gamma^2 \mathbf{v}\cdot\mathbf{v} \\
&= -\frac{1}{1-\beta^2}\left(c^2 - \mathbf{v}\cdot\mathbf{v}\right) \\
&= -\frac{c^2}{1-\beta^2}\left(1 - \frac{v^2}{c^2}\right) \\
&= -c^2
\end{aligned}$$

The 4-velocity is therefore a normalized 4-vector.

Now suppose a particle with 4-velocity

$$\begin{aligned}
u^\alpha &= \gamma_u \left(c, \mathbf{u}\right) \\
&= c\left(\cosh\zeta_u, \mathbf{n}\sinh\zeta_u\right)
\end{aligned}$$

where

$$\begin{aligned}
\beta &\equiv \tanh\zeta \\
\gamma &\equiv \cosh\zeta \\
\gamma\beta &\equiv \sinh\zeta
\end{aligned}$$

in a given initial frame, is viewed from a second frame of reference moving with 3-velocity $\mathbf{v} = v\hat{\mathbf{i}}$. Then a Lorentz transformation of u^α gives the velocity in the new frame,

$$\begin{aligned}
c\cosh\zeta' &= u_0\cosh\zeta_v - u_1\sinh\zeta_v \\
&= c\left(\cosh\zeta_u\cosh\zeta_v - \sinh\zeta_u\sinh\zeta_v\right) \\
&= c\cosh\left(\zeta_u - \zeta_v\right) \\
c\sinh\zeta' &= c\left(-\cosh\zeta_u\sinh\zeta_v + \sinh\zeta_u\cosh\zeta_v\right) \\
&= c\sinh\left(\zeta_u - \zeta_v\right) \\
u_2' &= 0 \\
u_3' &= 0
\end{aligned}$$

so the rapidities simply add,

$$\zeta' = \zeta_u - \zeta_v$$

In terms of the velocities,

$$\begin{aligned}
\beta' &= \tanh\zeta' \\
&= \tanh\left(\zeta_u - \zeta_v\right) \\
&= \frac{\sinh\zeta_u\cosh\zeta_v - \cosh\zeta_u\sinh\zeta_v}{\cosh\zeta_u\cosh\zeta_v - \sinh\zeta_u\sinh\zeta_v} \\
&= \frac{\tanh\zeta_u - \tanh\zeta_v}{1 - \tanh\zeta_u\tanh\zeta_v} \\
&= \frac{\beta_u - \beta_v}{1 - \beta_u\beta_v}
\end{aligned}$$

Dividing both the numerator and the denominator by $\cosh\zeta_u \cosh\zeta_v$ gives

$$\begin{aligned} \beta' &= \frac{\tanh\zeta_u - \tanh\zeta_v}{1 - \tanh\zeta_u \tanh\zeta_v} \\ &= \frac{\beta_u - \beta_v}{1 - \beta_u\beta_v} \end{aligned}$$

which is the same as the previous result.

1.2 Relativistic energy and momentum

We next need to generalize the energy and momentum of a particle. The momentum should add as a 4-vector and should reduce to $\mathbf{p} = m\mathbf{v}$ in the $c \longrightarrow \infty$ limit. To be linear in the 3-velocity, we take it linear in the 4-velocity,

$$p^\alpha = mu^\alpha$$

where we might have

$$\begin{aligned} m &= m(\mathbf{v}) \\ \lim_{\mathbf{v}\longrightarrow 0} m(\mathbf{v}) &= m_{Newton} \end{aligned}$$

and in order for this to transform as a 4-vector, its norm must be Lorentz invariant. Computing,

$$\begin{aligned} \|p^\alpha\|^2 &= \|mu^\alpha\|^2 \\ &= m^2 \|u^\alpha\|^2 \\ &= -m^2c^2 \end{aligned}$$

Since we already know that c^2 is invariant, the mass m must also be invariant in order for p^α to be a 4-vector.

Now look at the various expressions for the 4-velocity:

$$\begin{aligned} u^\alpha &= \gamma(c, \mathbf{u}) \\ &= c(\cosh\zeta, \mathbf{n}\sinh\zeta) \end{aligned}$$

We have similar expressions for the 4-momentum,

$$\begin{aligned} p^\alpha &= mu^\alpha \\ &= (m\gamma c, m\gamma\mathbf{u}) \\ &= (mc\cosh\zeta_u, mc\mathbf{n}\sinh\zeta_u) \end{aligned}$$

From this we identify the *relativistic 3-momentum,*

$$\mathbf{p} = \gamma m\mathbf{u}$$

and notice that it has the correct limit,

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \mathbf{p} &= \lim_{\beta\longrightarrow 0} \frac{m\mathbf{u}}{\sqrt{1-\beta^2}} \\ &= m\mathbf{u} \end{aligned}$$

The meaning of the remaining component follows from the same limit, but we need to keep the first order term:

$$\begin{aligned} p^0c &= m\gamma c^2 \\ \lim_{\beta\longrightarrow 0} p^0c &= \lim_{\beta\longrightarrow 0} \frac{mc^2}{\sqrt{1-\beta^2}} \\ &= \lim_{\beta\longrightarrow 0} mc^2\left(1 + \frac{1}{2}\beta^2 + \dots\right) \\ &= mc^2 + \frac{1}{2}m\mathbf{v}^2 + \dots \end{aligned}$$

We immediately recognize the classical kinetic energy, together with the constant mass-energy. We therefore define the *relativistic energy*,

$$E = p^0 c = m\gamma c^2$$

We may also write the energy and momentum in terms of the rapidity,

$$\begin{aligned} p^\alpha &= \left(\frac{E}{c}, \mathbf{p}\right) \\ &= (m\gamma c, m\gamma \mathbf{u}) \\ &= (mc\cosh\zeta, mc\mathbf{n}\sinh\zeta) \end{aligned}$$

The invariant mass gives the important relationship between relativistic energy, momentum and mass:

$$\begin{aligned} -m^2c^2 &= \|p^\alpha\|^2 \\ &= -\left(\frac{E}{c}\right)^2 + \mathbf{p}^2 \\ E &= \sqrt{\mathbf{p}^2c^2 + m^2c^4} \end{aligned}$$

Though we take the positive square root here, the negative root also turns out to be important in quantum field theory.

We now generalize Newton's second law,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

by replacing the vectors on each side with 4-vectors,

$$F^\alpha = m\frac{du^\alpha}{d\tau}$$

where the acceleration 4-vector is defined as the proper time rate of change of the 4-velocity. Notice that it is orthogonal to the 4-velocity,

$$\begin{aligned} u_\alpha \frac{du^\alpha}{d\tau} &= \eta_{\alpha\beta}u^\beta\frac{du^\alpha}{d\tau} \\ &= \frac{1}{2}\frac{d\left(\eta_{\alpha\beta}u^\beta u^\alpha\right)}{d\tau} \\ &= \frac{1}{2}\frac{d\left(-c^2\right)}{d\tau} \\ &= 0 \end{aligned}$$

This means that the force is also orthogonal to the velocity,

$$u_\alpha F^\alpha = 0$$

and since, in the rest frame of a particle, $u^\alpha = (c, \mathbf{0})$, we must have

$$\begin{aligned} 0 &= u_\alpha F^\alpha \\ &= -cF^0 \end{aligned}$$

In this frame, therefore, the 4-force may be written in terms of the 3-force,

$$[F^\alpha]_{rest\,frame} = \left(0, \vec{\mathbf{F}}\right)$$

From this, we can boost to find the 4-force in any inertial frame.

Since the mass is invariant, this may also be written as

$$F^\alpha = \frac{dp^\alpha}{d\tau}$$

and, as a vector equation, may be summed over many particles to give the total force and total momentum. For an isolated system (by definition, one with no net external force acting on it), we have conservation of total momentum,

$$\begin{aligned} \frac{dp^\alpha_{total}}{d\tau} &= 0 \\ p^\alpha_{total,initial} &= p^\alpha_{total,final} \end{aligned}$$

This relationship is used for collision problems.

2 Covariance of electrodynamics

Just as we modified Newton's second law to write a covariant equation,

$$F^\alpha = \frac{dP^\alpha}{d\tau}$$

we must rewrite the Lorentz force law, Maxwell equations and continuity equation in a manifestly covariant form.

2.1 Lorentz force law

We begin with the Lorentz force law,

$$\begin{aligned} \mathbf{F} &= q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\ &= \frac{q}{c}\left(c\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \end{aligned}$$

We may replace the left side with the relativistic momentum, $F^\alpha = \frac{dp^\alpha}{d\tau}$, making the right side into a 4-vector. To determine the covariant form of the electric and magnetic fields, we need to know that the electric charge is invariant. Jackson discusses a number of experiments that test this to parts in 10^{19}, so the velocity-independence of electric charge is well established. Furthermore, the right side of the equation is linear in the velocity, and so must be linear in the 4-velocity of the charged particle, leaving us with an equation of the form

$$F^\alpha = \frac{q}{c} F^\alpha{}_\beta u^\beta$$

where our goal is to find the nature of the unknown tensor $F^\alpha{}_\beta$. It is equivalent and more convenient to write this as a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor,

$$F^\alpha = \frac{q}{c} F^{\alpha\beta} u_\beta$$

called the Faraday tensor. From the Lorentz force law, we see that $F^{\alpha\beta}$ must be linear in the electric and magnetic fields. This is sufficient to find the full form of the Faraday tensor.

Equating the spatial components, $\alpha = i = 1, 2, 3$, in the limit $\gamma \approx 1$ we have

$$\begin{aligned} F^i &= \frac{q}{c} F^{\alpha\beta} u_\beta \\ \frac{q}{c}\left(c\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)^i &= \frac{q}{c}\left(F^{i0} u_0 + F^{ij} u_j\right) \\ \frac{q}{c}\left(cE^i + \varepsilon^{ijk} v_j B_k\right) &= \frac{q}{c}\left(-cF^{i0} + F^{ij} v_j\right) \end{aligned}$$

This gives us two equations,

$$\begin{aligned} F^{i0} &= -E^i \\ F^{ij} v_j &= \varepsilon^{ijk} v_j B_k \end{aligned}$$

Since the second equation holds for all $v_j \ll c$, we have $F^{ij} = \varepsilon^{ijk} B_k$. For the remaining components, we let $\alpha = 0$,

$$\begin{aligned} \frac{1}{c} \gamma \frac{dE}{dt} &= \frac{q}{c} \gamma \left(-cF^{00} + F^{0i} \right) \\ \frac{dE}{dt} &= -qcF^{00} + qv_i F^{i0} \end{aligned}$$

Since the Newtonian rate of change of energy is $q\mathbf{E} \cdot \mathbf{v}$, we require $F^{00} = 0$ and $F^{i0} = E^i$. This determines the Faraday tensor to be

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

We can derive this independently from the Maxwell equations.

2.2 Levi-Civita tensor

There is one more tensor we require. Just as we have the Levi-Civita tensor, ε_{ijk}, in 3-dimensions, there is a Levi-Civita tensor in 4-dimensions,

$$\varepsilon_{\alpha\beta\mu\nu} = \begin{cases} +1 & Even\, permutations\, of\; 0,1,2,3 \\ -1 & Odd\, permutations\, of\; 0,1,2,3 \\ 0 & Otherwise \end{cases}$$

Notice that if one index is fixed to be zero, then we recover the 3-dimensional Levi-Civita tensor, $\varepsilon_{0ijk} = \varepsilon_{ijk}$. Using this, we can form a second field tensor dual to the Faraday tensor,

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu}$$

The components are easy to check. For $\alpha = 0, \beta = i$,

$$\mathcal{F}_{0i} = \frac{1}{2} \varepsilon_{0i\mu\nu} F^{\mu\nu}$$

The remaining indices must be spatial,

$$\begin{aligned} \mathcal{F}_{0i} &= \frac{1}{2} \varepsilon_{0ijk} F^{jk} \\ &= \frac{1}{2} \varepsilon_{ijk} \varepsilon^{jkm} B_m \\ &= \frac{1}{2} 2\delta_i^m B_m \\ &= B_i \end{aligned}$$

Antisymmetry tells us that $\mathcal{F}_{i0} = -B_i$, so all that remains are the spatial components,

$$\mathcal{F}_{ij} = \frac{1}{2} \varepsilon_{ij\mu\nu} F^{\mu\nu}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\varepsilon_{ij0k}F^{0k} + \varepsilon_{ijk0}F^{k0}\right) \\
&= \frac{1}{2}\left(\varepsilon_{0ijk}F^{0k} - \varepsilon_{0ijk}F^{k0}\right) \\
&= \frac{1}{2}\left(\varepsilon_{0ijk}F^{0k} + \varepsilon_{0ijk}F^{0k}\right) \\
&= \varepsilon_{ijk}F^{0k} \\
&= \varepsilon_{ijk}E^{k}
\end{aligned}
$$

Therefore,

$$
\mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & E^3 & -E^2 \\ -B^2 & -E^3 & 0 & E^1 \\ -B^3 & E^2 & -E^1 & 0 \end{pmatrix}
$$

Raising the α, β indices with $\eta^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ changes the sign of the $0i$ and $i0$ components only, so

$$
\begin{aligned}
\mathcal{F}^{\alpha\beta} &= \eta^{\alpha\mu}\eta^{\beta\nu}\mathcal{F}_{\mu\nu} \\
&= \eta^{\alpha\mu}\mathcal{F}_{\mu\nu}\eta^{\beta\nu} \\
&= [\eta]\,[\mathcal{F}]\,\left[\eta^{t}\right] \\
&= [\eta]\,[\mathcal{F}]\,[\eta] \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -B^1 & -B^3 & -B^3 \\ B^1 & 0 & -E^3 & E^2 \\ B^2 & E^3 & 0 & -E^1 \\ B^3 & -E^2 & E^1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}
\end{aligned}
$$

Therefore, we get $\mathcal{F}^{\alpha\beta}$ from $F^{\alpha\beta}$ by the replacements $\mathbf{E} \rightarrow -\mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{E}$.

2.3 Continuity equation

The continuity equation is an easy place to start, and it gives us the 4-vector form of the source current. We have

$$
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0
$$

and we know that the gradient operator transforms as a covariant vector,

$$
\begin{aligned}
\partial_\alpha &= \frac{\partial}{\partial x^\alpha} \\
&= \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)
\end{aligned}
$$

From this we can recognize the form of the continuity equation as a vanishing divergence,

$$
\begin{aligned}
0 &= \partial_\alpha J^\alpha \\
&= \frac{1}{c}\frac{\partial J^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}
\end{aligned}
$$

We immediately identify

$$J^\alpha = (\rho c, \mathbf{J})$$

so that the charge density times c, together with the current density, form a 4-vector.

2.4 The vector potential

Using the Lorentz gauge, we have found the following wave equations for the scalar and vector potentials,

$$\begin{aligned} \Box\phi &= -4\pi\rho \\ \Box\mathbf{A} &= -\frac{4\pi}{c}\mathbf{J} \end{aligned}$$

Since

$$J^\alpha = (\rho c, \mathbf{J})$$

is already known to be a 4-vector, and the d'Alembertian, $\Box = \partial_\alpha \partial^\alpha$, is shown to be a Lorentz-invariant operator, the potentials must also form a 4-vector,

$$A^\alpha = (\phi, \mathbf{A})$$

making the four equations into a single, 4-vector equation,

$$\Box A^\alpha = -\frac{4\pi}{c} J^\alpha$$

2.5 The inhomogeneous Maxwell equations

Now consider the Maxwell equations,

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \end{aligned}$$

Now, since J^α is a 4-vector, we know that the inhomogeneous Maxwell equations go together,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \end{aligned}$$

because ρ and $\mathbf{J}$ belong to the same 4-vector. The right side is linear in the four derivatives, ∂_α, and linear in the fields. The object which we have that is linear in the fields is $F^{\alpha\beta}$, so we must combine ∂_α and $F^{\alpha\beta}$ in such a way as to give $\frac{4\pi}{c} J^\alpha$. Since $F^{\alpha 0} = \left(0, E^i\right)$ the divergence may be written as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{4\pi}{c}\rho c \\ \partial_\beta F^{0\beta} &= \frac{4\pi}{c} J^0 \end{aligned}$$

and consistency under Lorentz transformations demands that we extend this to $\partial_\beta F^{\alpha\beta} = \frac{4\pi}{c} J^\alpha$. Now setting $\alpha = i$ to look at the spatial components,

$$\partial_\beta F^{i\beta} = \frac{4\pi}{c} J^i$$

$$\begin{aligned}
\partial_0 F^{i0} + \partial_j F^{ij} &= \frac{4\pi}{c} J^i \\
-\frac{1}{c}\frac{\partial}{\partial t} E^i + \partial_j \varepsilon^{ijk} B_k &= \frac{4\pi}{c} J^i \\
-\frac{1}{c}\frac{\partial}{\partial t} E^i + [\boldsymbol{\nabla} \times \mathbf{B}]^i &= \frac{4\pi}{c} J^i
\end{aligned}$$

This is exactly Ampère's law, so the two inhomogeneous Maxwell equations may be written as

$$\partial_\beta F^{\alpha\beta} = \frac{4\pi}{c} J^\alpha$$

In this form, the equations are *manifestly Lorentz covariant.* This simply means that we can see directly from the form of the equation that its form is preserved by Lorentz transformations – our index conventions guarantee this. To see it explicitly, recall that raised indices transform with $\Lambda^\alpha_{\ \beta}$ and lowered indices with the inverse, $\bar{\Lambda}^\alpha_{\ \beta}$, so that in a new inertial frame of reference,

$$\begin{aligned}
\partial_\beta &\rightarrow \tilde{\partial}_\beta = \bar{\Lambda}^\alpha_{\ \beta} \tilde{\partial}_\alpha \\
F^{\alpha\beta} &\rightarrow \tilde{F}^{\alpha\beta} = \Lambda^\alpha_{\ \mu} \Lambda^\beta_{\ \nu} F^{\mu\nu} \\
J^\alpha &\rightarrow \tilde{J}^\alpha = \Lambda^\alpha_{\ \mu} J^\mu
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{\partial}_\beta \tilde{F}^{\alpha\beta} - \frac{4\pi}{c} \tilde{J}^\alpha &= \bar{\Lambda}^\rho_{\ \beta} \partial_\rho \left(\Lambda^\alpha_{\ \mu} \Lambda^\beta_{\ \nu} F^{\mu\nu} \right) - \frac{4\pi}{c} \Lambda^\alpha_{\ \mu} J^\mu \\
&= \Lambda^\alpha_{\ \mu} \partial_\rho \left(\bar{\Lambda}^\rho_{\ \beta} \Lambda^\beta_{\ \nu} F^{\mu\nu} \right) - \frac{4\pi}{c} \Lambda^\alpha_{\ \mu} \tilde{J} \\
&= \Lambda^\alpha_{\ \mu} \partial_\rho \left(\delta^\rho_\nu F^{\mu\nu} \right) - \frac{4\pi}{c} \Lambda^\alpha_{\ \mu} J^\mu \\
&= \Lambda^\alpha_{\ \mu} \left(\partial_\nu F^{\mu\nu} - \frac{4\pi}{c} J^\mu \right) \\
&= 0
\end{aligned}$$

and the equation takes the same form in the boosted or rotated frame of reference.

2.6 Conservation of charge

We can derive the conservation of charge from the antisymmetry of $F^{\alpha\beta}$,

$$F^{\alpha\beta} = -F^{\beta\alpha}$$

Suppose we take two derivatives and contract,

$$\partial_\alpha \partial_\beta F^{\alpha\beta} = -\partial_\alpha \partial_\beta F^{\beta\alpha}$$

We cna show that this expression vanishes identically. This follows because mixed partial derivatives commute,

$$\begin{aligned}
\partial_\alpha \partial_\beta &= \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \\
&= \frac{\partial^2}{\partial x^\beta \partial x^\alpha} \\
&= \partial_\beta \partial_\alpha
\end{aligned}$$

so we may write the right hand side as

$$-\partial_\alpha \partial_\beta F^{\beta\alpha} = -\partial_\beta \partial_\alpha F^{\beta\alpha}$$

and renaming the dummy indices this is

$$-\partial_\alpha \partial_\beta F^{\beta\alpha} = -\partial_\alpha \partial_\beta F^{\alpha\beta}$$

Returning this to the original expression,

$$\begin{aligned} \partial_\alpha \partial_\beta F^{\alpha\beta} &= -\partial_\alpha \partial_\beta F^{\alpha\beta} \\ 2\partial_\alpha \partial_\beta F^{\alpha\beta} &= 0 \end{aligned}$$

so this double derivative vanishes identically by symmetry. Applying this to our relativistic expression

$$\begin{aligned} \partial_\beta F^{\alpha\beta} &= \frac{4\pi}{c} J^\alpha \\ \partial_\alpha \partial_\beta F^{\alpha\beta} &= \frac{4\pi}{c} \partial_\alpha J^\alpha \end{aligned}$$

we have the continuity equation,

$$\partial_\alpha J^\alpha = 0$$

showing that the Maxwell equations imply the conservation of charge.

2.7 Homogeneous Maxwell equations

Now consider the remaining Maxwell equations. In these,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}$$

The roles of the electric and magnetic fields are reversed. In fact, we may get these two equations from the the sourced Maxwell equations by the replacements

$$\begin{aligned} \mathbf{E} &\longrightarrow -\mathbf{B} \\ \mathbf{B} &\longrightarrow \mathbf{E} \\ J^\alpha &\longrightarrow 0 \end{aligned}$$

and this is exactly what happens if we take the dual of the Faraday tensor,

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu}$$

which we showed above to embody exactly this replacement. We immediately see that the homogeneous Maxwell equations must be

$$\partial_\beta \mathcal{F}^{\alpha\beta} = 0$$

We check this in detail. We have

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

so for $\alpha = 0$,

$$\begin{aligned}
\partial_\beta \mathcal{F}^{0\beta} &= 0 \\
\partial_0 \mathcal{F}^{00} + \partial_i \mathcal{F}^{0i} &= 0 \\
-\partial_i B^i &= 0
\end{aligned}$$

so we recover $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. For $\alpha = i$,

$$\begin{aligned}
\partial_\beta \mathcal{F}^{i\beta} &= 0 \\
\partial_0 \mathcal{F}^{i0} + \partial_j \mathcal{F}^{ij} &= 0 \\
\partial_0 B^i + \partial_j \varepsilon^{ijk} E_k &= 0
\end{aligned}$$

to give exactly $\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$.

2.8 The homogeneous Maxwell equations and the potential

We know that the homogeneous Maxwell equations

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0
\end{aligned}$$

lead to the existence of the scalar and vector potentials. Vanishing divergence of $\mathbf{B}$ means that it may be written as a curl,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

and substituting this into the second equations gives a vanishing curl,

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} \right) = 0$$

The term in parenthesis must therefore be a gradient, $-\boldsymbol{\nabla}\phi$, so solving for the electric field,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

We would like to understand these results in a covariant way.

In three dimensions, the components of the curl of a vector are

$$\begin{aligned}
[\boldsymbol{\nabla} \times \mathbf{v}]^i &= \varepsilon^{ijk} \partial_j v_k \\
&= \frac{1}{2}\varepsilon^{ijk} \left(\partial_j v_k - \partial_k v_j \right)
\end{aligned}$$

The important part of this expression is the antisymmetric *matrix*, $\partial_j v_k - \partial_k v_j$, which in 3-dimensions *only* can be made into a vector using the Levi-Civita tensor, ε^{ijk}. The higher dimensional generalization of the curl is therefore a *matrix*,

$$\partial_\beta v_\alpha - \partial_\alpha v_\beta$$

Consider the 4-dimensional curl of the 4-vector potential, $\partial_\beta A_\alpha - \partial_\alpha A_\beta$. For the $\alpha = 0, \beta = i$ components, the 4-curl is

$$\partial_i A_0 - \partial_0 A_i = -\partial_i \phi - \partial_0 A_i = E_i$$

while the spatial components are

$$\partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B^k$$

while the diagonal elements vanish. This is exactly the Faraday tensor in doubly covariant form,

$$\begin{aligned} F_{\alpha\beta} &= \partial_\beta A_\alpha - \partial_\alpha A_\beta \\ &= A_{\alpha,\beta} - A_{\beta,\alpha} \end{aligned}$$

where in the second line we introduce the convention of writing a partial derivative with a comma before the subscript, for example, $\partial_\alpha f = f_{,\alpha}$.

It can be shown that an antisymmetric tensor such as $F_{\alpha\beta}$ can be written as the 4-curl of a vector if and only if its totally antisymmetrized derivative vanishes, i.e.,

$$F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} - F_{\beta\alpha,\mu} - F_{\mu\beta,\alpha} - F_{\alpha\mu,\beta} = 0$$

Notice that since $F_{\beta\alpha} = -F_{\alpha\beta}$, this condition is just twice the simpler expression,

$$F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} = 0$$

We prove the theorem as follows. Suppose $F_{\alpha\beta}$ is given by a curl. Then its derivative is

$$\begin{aligned} F_{\alpha\beta,\mu} &= \partial_\mu \partial_\beta A_\alpha - \partial_\mu \partial_\alpha A_\beta \\ &= A_{\alpha,\beta\mu} - A_{\beta,\alpha\mu} \end{aligned}$$

Notice that the partial derivatives are symmetric, so these terms will vanish under antisymmetrization. Expicitly,

$$\begin{aligned} F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} &= (A_{\alpha,\beta\mu} - A_{\beta,\alpha\mu}) + (A_{\beta,\mu\alpha} - A_{\mu,\beta\alpha}) + (A_{\mu,\alpha\beta} - A_{\alpha,\mu\beta}) \\ &= A_{\alpha,\beta\mu} - A_{\alpha,\mu\beta} + A_{\beta,\mu\alpha} - A_{\beta,\alpha\mu} + A_{\mu,\alpha\beta} - A_{\mu,\beta\alpha} \\ &= 0 \end{aligned}$$

because of the equality of mixed partials, e.g., $A_{\alpha,\beta\mu} = A_{\alpha,\mu\beta}$. Therefore, if $F_{\alpha\beta}$ is the 4-curl of a potential, then this condition is satisfied.

For the converse, begin with the condition

$$F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} = 0$$

and contract with the Levi-Civita tensor,

$$\begin{aligned} 0 &= \varepsilon^{\alpha\beta\mu\nu} \left(F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} \right) \\ &= 3\varepsilon^{\alpha\beta\mu\nu} \partial_\mu F_{\alpha\beta} \\ &= 6\partial_\mu \left(\frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \right) \\ &= 6\partial_\mu \mathcal{F}^{\mu\nu} \end{aligned}$$

so the condition implies the homogeneous Maxwell equations, which in turn imply the existence of the 4-potential.

2.9 Summary: The Maxwell equations in covariant form

The Lorentz force law may be written as

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} u_\beta$$

where u^α is the 4-velocity of a particle charge q and p^α is its 4-momentum. The Faraday tensor is then found to be

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

with dual $\mathcal{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$. The charge density and current density together form a 4-vector,

$$J^\alpha = (\rho c, \mathbf{J})$$

as do the scalar and vector potentials,

$$A^\alpha = (\phi, \mathbf{A})$$

Using these, we can replace all four Maxwell equations with only two manifestly Lorentz covariant equations,

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= \frac{4\pi}{c}J^\beta \\ \partial_\alpha \mathcal{F}^{\alpha\beta} &= 0 \end{aligned}$$

The second of these is the necessary and sufficient condition for the existence of the 4-potential, such that

$$F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha}$$

3 Lorentz transformation of the Maxwell equations

3.1 The transformations of the fields

Now that we have written the Maxwell equations in covariant form, we know exactly how they transform under Lorentz transformations. Consider a boost in the x-direction, from $\mathcal{O}$ to $\tilde{\mathcal{O}}$ given by the transformation matrix

$$M^\alpha_{\ \beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, since the Faraday tensor is a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor, it transforms as

$$\begin{aligned} \tilde{F}^{\alpha\beta} &= M^\alpha_{\ \mu} M^\alpha_{\ \nu} F^{\mu\nu} \\ &= M^\alpha_{\ \mu} F^{\mu\nu} M^\alpha_{\ \nu} \\ &= M^\alpha_{\ \mu} F^{\mu\nu} \left[M^t\right]_\nu^{\ \alpha} \end{aligned}$$

where the rearrangement may now be written as the matrix product,

$$\tilde{F} = MFM^t$$

We find,

$$\begin{aligned} \tilde{F}^{\alpha\beta} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma\beta E^1 & \gamma E^1 & E^2 & E^3 \\ -\gamma E^1 & \gamma\beta E^1 & -B^3 & B^2 \\ -\gamma E^2 - \gamma\beta B^3 & \gamma\beta E^2 + \gamma B^3 & 0 & -B^1 \\ -\gamma E^3 + \gamma\beta B^2 & \gamma\beta E^3 - \gamma B^2 & B^1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \left(\gamma^2 - \gamma^2\beta^2\right)E^1 & \gamma E^2 + \gamma\beta B^3 & \gamma E^3 - \gamma\beta B^2 \\ \left(\gamma^2\beta^2 - \gamma^2\right)E^1 & 0 & -\gamma\beta E^2 - \gamma B^3 & -\gamma\beta E^3 + \gamma B^2 \\ -\gamma E^2 - \gamma\beta B^3 & \gamma\beta E^2 + \gamma B^3 & 0 & -B^1 \\ -\gamma E^3 + \gamma\beta B^2 & \gamma\beta E^3 - \gamma B^2 & B^1 & 0 \end{pmatrix} \end{aligned}$$

$$
= \begin{pmatrix} 0 & E^1 & \gamma\left(E^2+\beta B^3\right) & \gamma\left(E^3-\beta B^2\right) \\ -E^1 & 0 & -\gamma\left(B^3+\beta E^2\right) & \gamma\left(B^2-\beta E^3\right) \\ -\gamma\left(E^2+\beta B^3\right) & \gamma\left(B^3+\beta E^2\right) & 0 & -B^1 \\ -\gamma\left(E^3-\beta B^2\right) & -\gamma\left(B^2-\beta E^3\right) & B^1 & 0 \end{pmatrix}
$$

Comparing components with

$$
\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & \tilde{E}^1 & \tilde{E}^2 & \tilde{E}^3 \\ -\tilde{E}^1 & 0 & -\tilde{B}^3 & \tilde{B}^2 \\ -\tilde{E}^2 & \tilde{B}^3 & 0 & -\tilde{B}^1 \\ -\tilde{E}^3 & -\tilde{B}^2 & \tilde{B}^1 & 0 \end{pmatrix}
$$

we see that

$$
\begin{aligned}
\tilde{E}^1 &= E^1 \\
\tilde{E}^2 &= \gamma\left(E^2-\beta B^3\right) \\
\tilde{E}^3 &= \gamma\left(E^3+\beta B^2\right) \\
\tilde{B}^1 &= B^1 \\
\tilde{B}^2 &= \gamma\left(B^2+\beta E^3\right) \\
\tilde{B}^3 &= \gamma\left(B^3-\beta E^2\right)
\end{aligned}
$$

which we can write vectorially in terms of the components of $\mathbf{E}$ and $\mathbf{B}$ parallel and perpenducular to $\boldsymbol{\beta}$,

$$
\begin{aligned}
\mathbf{E}_{\parallel} &= \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
\mathbf{E}_{\perp} &= \mathbf{E}-\frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}
\end{aligned}
$$

and similarly for $\mathbf{B}$. In terms of these, the transformation of the fields becomes

$$
\begin{aligned}
\tilde{\mathbf{E}}_{\parallel} &= \mathbf{E}_{\parallel} \\
\tilde{\mathbf{E}}_{\perp} &= \gamma\mathbf{E}_{\perp}-\gamma\boldsymbol{\beta}\times\mathbf{B}_{\perp} \\
\tilde{\mathbf{B}}_{\parallel} &= \mathbf{B}_{\parallel} \\
\tilde{\mathbf{B}}_{\perp} &= \gamma\mathbf{B}_{\perp}-\gamma\boldsymbol{\beta}\times\mathbf{E}_{\perp}
\end{aligned}
$$

and we may reconstruct the full vectors,

$$
\begin{aligned}
\tilde{\mathbf{E}} &= \tilde{\mathbf{E}}_{\parallel}+\tilde{\mathbf{E}}_{\perp} \\
&= \mathbf{E}_{\parallel}+\gamma\mathbf{E}_{\perp}-\gamma\boldsymbol{\beta}\times\mathbf{B}_{\perp} \\
&= \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}+\gamma\left(\mathbf{E}-\frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}\right)-\gamma\boldsymbol{\beta}\times\left(\mathbf{B}-\frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\boldsymbol{\beta}\right) \\
&= \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}+\gamma\mathbf{E}-\frac{1}{\beta^2}\gamma\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}-\gamma\boldsymbol{\beta}\times\mathbf{B} \\
&= \gamma\left(\mathbf{E}-\boldsymbol{\beta}\times\mathbf{B}\right)-\frac{\gamma-1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
&= \gamma\left(\mathbf{E}-\boldsymbol{\beta}\times\mathbf{B}\right)-\frac{\gamma-1}{1-\frac{1}{\gamma^2}}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
&= \gamma\left(\mathbf{E}-\boldsymbol{\beta}\times\mathbf{B}\right)-\frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
\tilde{\mathbf{B}} &= \mathbf{B}_{\parallel}+\gamma\mathbf{B}_{\perp}-\gamma\boldsymbol{\beta}\times\mathbf{E}_{\perp}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{B})\,\boldsymbol{\beta} + \gamma\left(\mathbf{B} - \frac{1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{B})\,\boldsymbol{\beta}\right) - \gamma\boldsymbol{\beta}\times\left(\mathbf{E} - \frac{1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{E})\,\boldsymbol{\beta}\right) \\
&= \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\,\boldsymbol{\beta}
\end{aligned}
$$

so the complete transformation laws are

$$
\begin{aligned}
\tilde{\mathbf{E}} &= \gamma(\mathbf{E} - \boldsymbol{\beta}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{E})\,\boldsymbol{\beta} \\
\tilde{\mathbf{B}} &= \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\,\boldsymbol{\beta}
\end{aligned}
$$

3.2 Example: Pure electric field from a rapidly moving frame

Suppose we have a pure electric field in the original frame, with $\mathbf{B} = 0$. Then in $\tilde{\mathcal{O}}$, as the speed approaches the speed of light, $\boldsymbol{\beta} \to \hat{\boldsymbol{\beta}}$, $\frac{\gamma^2}{\gamma+1} \to \gamma$, and the electric field approaches

$$
\begin{aligned}
\tilde{\mathbf{E}} &= \gamma\mathbf{E} - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{E})\,\boldsymbol{\beta} \\
&\longrightarrow \gamma\left(\mathbf{E} - \left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\hat{\boldsymbol{\beta}}\right) \\
&= \gamma\mathbf{E}_\perp
\end{aligned}
$$

so the electric field flattens into the plane orthogonal to the motion. At the same time, the magnetic field approaches

$$
\tilde{\mathbf{B}} = -\gamma\hat{\boldsymbol{\beta}}\times\mathbf{E}
$$

which also lies in the orthogonal plane, and is perpendicular to $\mathbf{E}$. The Poynting vector of the field in the rapidly moving frame is

$$
\begin{aligned}
\tilde{\mathbf{S}} &= \frac{1}{\mu_0}\tilde{\mathbf{E}}\times\tilde{\mathbf{B}} \\
&= -\frac{\gamma^2}{\mu_0}\mathbf{E}_\perp\times\left(\hat{\boldsymbol{\beta}}\times\mathbf{E}\right) \\
&= -\frac{\gamma^2}{\mu_0}\left(\mathbf{E} - \frac{1}{\beta^2}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\hat{\boldsymbol{\beta}}\right)\times\left(\hat{\boldsymbol{\beta}}\times\mathbf{E}\right) \\
&= -\frac{\gamma^2}{\mu_0}\left(\mathbf{E}\times\left(\hat{\boldsymbol{\beta}}\times\mathbf{E}\right) - \frac{1}{\beta^2}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\hat{\boldsymbol{\beta}}\times\left(\hat{\boldsymbol{\beta}}\times\mathbf{E}\right)\right) \\
&= -\frac{\gamma^2}{\mu_0}\left(E^2\hat{\boldsymbol{\beta}} - \left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\mathbf{E} - \frac{1}{\beta^2}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\left(\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\hat{\boldsymbol{\beta}} - \beta^2\mathbf{E}\right)\right) \\
&= -\frac{\gamma^2}{\mu_0}\left(E^2\hat{\boldsymbol{\beta}} - \left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\mathbf{E} - \frac{1}{\beta^2}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)^2\hat{\boldsymbol{\beta}} + \left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\mathbf{E}\right) \\
&= -\frac{\gamma^2}{\mu_0}\left(E^2 - \frac{1}{\beta^2}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)^2\right)\hat{\boldsymbol{\beta}} \\
&= -\frac{\gamma^2 E_\perp^2}{\mu_0}\hat{\boldsymbol{\beta}}
\end{aligned}
$$

where we have used

$$
\mathbf{E}_\perp = \mathbf{E} - \frac{1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{E})\,\boldsymbol{\beta}
$$

$$
\begin{aligned}
E_{\perp}^{2} &= E^{2}-\frac{2}{\beta^{2}}(\boldsymbol{\beta}\cdot\mathbf{E})^{2}+\frac{1}{\beta^{2}}(\boldsymbol{\beta}\cdot\mathbf{E})^{2} \\
&= E^{2}-\frac{1}{\beta^{2}}(\boldsymbol{\beta}\cdot\mathbf{E})^{2}
\end{aligned}
$$

Not surprisingly, we find a strong flux opposing the observer's motion.

4 Thomas precession and the BMT equation

First consider orbital angular momentum,

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}$$

We may immediately generalize this to spacetime by defining

$$L^{\alpha\beta}\equiv\frac{1}{2}\left(x^{\alpha}p^{\beta}-x^{\beta}p^{\alpha}\right)$$

where x^{α} is the position 4-vector of a particle and p^{α} is its momentum. Consider the dual,

$$
\begin{aligned}
\mathcal{L}_{\mu\nu} &\equiv \varepsilon_{\mu\nu\alpha\beta}L^{\alpha\beta} \\
&= \varepsilon_{\mu\nu\alpha\beta}x^{\alpha}p^{\beta}
\end{aligned}
$$

and contract with the *observer's* 4-velocity,

$$
\begin{aligned}
L_{\nu} &\equiv \frac{1}{c}u^{\mu}\mathcal{L}_{\mu\nu} \\
&= \frac{1}{c}u^{\mu}\varepsilon_{\mu\nu\alpha\beta}x^{\alpha}p^{\beta}
\end{aligned}
$$

Now look at L_{ν} in the rest frame of the observer, where $u^{\mu}=(c,\mathbf{0})$,

$$
\begin{aligned}
L_{\nu} &\equiv \frac{1}{c}u^{\mu}\mathcal{L}_{\mu\nu} \\
&= \varepsilon_{0\nu\alpha\beta}x^{\alpha}p^{\beta} \\
&= \left(0,\varepsilon_{ijk}x^{j}p^{k}\right) \\
&= (0,\mathbf{L})
\end{aligned}
$$

This shows that we expect angular momentum to be described by a spacelike vector, and we'll treat the spin vector in the same way. This property, of reducing to a purely spacelike vector in the rest frame, may be characterized by orthogonality to the 4-velocity, $u^{\alpha}L_{\alpha}=0$.

We are now in a position to consider the fully relativistic evolution of spin angular momentum. We know that the equation of motion of the 3-dimensional spin vector, $\mathbf{s}$, in the rest frame $\tilde{\mathcal{O}}$ of the particle is

$$\frac{d\mathbf{s}}{d\tilde{t}}=\frac{ge}{2mc}\mathbf{s}\times\tilde{\mathbf{B}}$$

and want to generalize this to a covariant expression. We begin by generalizing the spin 3-vector to a spin 4-vector, which we assume reduces to

$$\tilde{s}^{\alpha}=(0,\mathbf{s})$$

in the rest frame $\tilde{\mathcal{O}}$, or equivalently,

$$\tilde{u}_{\alpha}\tilde{s}^{\alpha}=0$$

and this relation is invariant.

To generalize the rest frame equation, we will clearly want the proper time derivative of the 4-vector spin

$$\frac{ds^\alpha}{d\tau}$$

expressed in terms of tensors. Rather than trying to transform the known expression, we try to write the most general covariant expression we can that reduces to the known non-relativistic expression. The only tensors relevant to the problem are

$$F^{\alpha\beta}, u^\alpha, s^\alpha, \frac{du^\alpha}{d\tau}$$

Notice that the 4-acceleration, $a^\alpha = \frac{du^\alpha}{d\tau}$, has a part expressible in terms of the first two but may also depend on non-electromagnetic forces, so we need to consider it separately. Our goal is to write $\frac{ds^\alpha}{d\tau}$ in terms of these, i.e., we need to construct a 4-vector. Looking at the classical limit in the rest frame, we expect that the correct expression is at most linear in the fields $F^{\alpha\beta}$, and each term is linear in the spin vector. Then there are only three possibilities:

$$F^{\alpha\beta} s_\beta$$
$$(s_\mu F^{\mu\nu} u_\nu) u^\alpha$$
$$\left(s_\beta \frac{du^\beta}{d\tau}\right) u^\alpha$$

(Notice that $\left(u^\beta s_\beta\right) u^\alpha = 0$, $F^{\alpha\beta} u_\beta$ is independent of s^α, and $(F^{\mu\nu} u_\mu u_\nu) u^\alpha = 0$ and since the acceleration is already linear in $F^{\alpha\beta}$ we do not consider terms involving $F^{\alpha\beta} a_\beta$).

We bethin with an arbitrary linear combination of these, and ask what combination gives the right behavior in the particle rest frame,

$$\frac{ds^\alpha}{d\tau} = A F^{\alpha\beta} s_\beta + B (s_\mu F^{\mu\nu} u_\nu) u^\alpha + C \left(s_\beta \frac{du^\beta}{d\tau}\right) u^\alpha$$

First, notice that since $U^\alpha S_\alpha = 0$, it follows that

$$\begin{aligned} 0 &= \frac{d}{d\tau}(u^\alpha s_\alpha) \\ &= \frac{du^\alpha}{d\tau} s_\alpha + u^\alpha \frac{ds_\alpha}{d\tau} \end{aligned}$$

so that

$$u^\alpha \frac{ds_\alpha}{d\tau} = -s_\alpha \frac{du^\alpha}{d\tau}$$

Rewrite this using the Lorentz force law, combined with other non-electromagnetic forces, F^α

$$m\frac{du^\alpha}{d\tau} = F^\alpha + \frac{e}{mc} u_\beta F^{\alpha\beta}$$

Then the contraction of the rate of change of spin with the 4-velocity is

$$\begin{aligned} u^\alpha \frac{ds_\alpha}{d\tau} &= -s_\alpha \frac{du^\alpha}{d\tau} \\ &= -\frac{1}{m} s_\alpha \left(F^\alpha + \frac{e}{c} F^{\alpha\beta} u_\beta\right) \end{aligned}$$

Take this contraction with our ansatz, substituting it into the final term as well:

$$\begin{aligned} \frac{ds^\alpha}{d\tau} &= A F^{\alpha\beta} s_\beta + B (s_\mu F^{\mu\nu} u_\nu) u^\alpha + C \left(s_\beta \frac{du^\beta}{d\tau}\right) u^\alpha \\ u_\alpha \frac{ds^\alpha}{d\tau} &= A u_\alpha F^{\alpha\beta} s_\beta + B (s_\mu F^{\mu\nu} u_\nu) u_\alpha u^\alpha + C \left(s_\beta \frac{du^\beta}{d\tau}\right) u_\alpha u^\alpha \\ -\frac{1}{m} s_\alpha \left(F^\alpha + \frac{e}{mc} u_\beta F^{\beta\alpha}\right) &= A u_\alpha F^{\alpha\beta} s_\beta - c^2 B (s_\mu F^{\mu\nu} u_\nu) - \frac{c^2}{m} C \left(s_\beta \left(F^\beta + \frac{e}{mc} u_\alpha F^{\alpha\beta}\right)\right) \end{aligned}$$

Since the non-electromagnetic force F^α is independent of the electromagnetic force, we may split this into two independent equations,

$$\begin{aligned} -\frac{1}{m}s_\alpha F^\alpha &= -\frac{c^2}{m}Cs_\beta F^\beta \\ -\frac{e}{m^2c}u_\beta F^{\beta\alpha}s_\alpha &= Au_\alpha F^{\alpha\beta}s_\beta - c^2B\left(s_\mu F^{\mu\nu}u_\nu\right) - \frac{c^2}{m}C\left(\frac{e}{mc}u_\alpha F^{\alpha\beta}s_\beta\right) \end{aligned}$$

The first of these immediately gives $C = \frac{1}{c^2}$, while the second becomes

$$\begin{aligned} 0 &= \left(A + c^2B - \frac{c^2}{m}C\frac{e}{mc} + \frac{e}{m^2c}\right)u_\beta F^{\beta\alpha}s_\alpha \\ &= \left(A + c^2B\right)u_\beta F^{\beta\alpha}s_\alpha \end{aligned}$$

With these restrictions, the equation of motion reduces to

$$\frac{ds^\alpha}{d\tau} = AF^{\alpha\beta}s_\beta - \frac{A}{c^2}\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha + \frac{1}{c^2}\left(s_\beta\frac{du^\beta}{d\tau}\right)u^\alpha$$

Now consider the limit to the rest frame, in the case of a pure magnetic field, so $\mathbf{E} = \mathbf{0}$ in $F^{\alpha\beta}$, and the 4-velocity is given by

$$u^\alpha = (c, \mathbf{0})$$

Then $s_\mu F^{\mu\nu}u_\nu = -cs_iF^{k0} = cs_iE^i = 0$ so the middle term drops out. The final term is purely timelike, so the spatial components give

$$\begin{aligned} \frac{ds^i}{d\tau} = \frac{ds^i}{dt} &= AF^{i\alpha}s_\alpha \\ &= A\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B^3 & -B^2 \\ 0 & -B^3 & 0 & B^1 \\ 0 & B^2 & -B^1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ s^1 \\ s^2 \\ s^3 \end{pmatrix} \\ &= A\begin{pmatrix} 0 \\ B^3s^2 - B^2s^3 \\ B^1s^3 - B^3s^1 \\ B^2s^1 - B^1s^2 \end{pmatrix} \end{aligned}$$

or simply

$$\frac{d\mathbf{s}}{dt} = A\left(\mathbf{s}\times\mathbf{B}\right)$$

and comparing this to

$$\frac{d\mathbf{s}}{dt} = \frac{ge}{2mc}\mathbf{s}\times\mathbf{B}$$

we see that

$$A = \frac{ge}{2mc}$$

and substituting the electromagnetic force for the acceleration, the full equation of motion for $\mathbf{s}$ becomes

$$\begin{aligned} \frac{ds^\alpha}{d\tau} &= \frac{ge}{2mc}F^{\alpha\beta}s_\beta - \frac{ge}{2mc^3}\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha + \frac{1}{c^2}\left(s_\beta\frac{du^\beta}{d\tau}\right)u^\alpha \\ &= \frac{ge}{2mc}F^{\alpha\beta}s_\beta - \frac{ge}{2mc^3}\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha + \frac{1}{c^2}\left(s_\beta\left(\frac{e}{mc}F^{\alpha\beta}u_\beta\right)\right)u^\alpha \\ &= \frac{ge}{2mc}F^{\alpha\beta}s_\beta - \frac{(g-2)\,e}{2mc^3}\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha \end{aligned}$$

This is the BMT equation. The signs differ from Jackson because we use the opposite convention for the sign of the metric, $\eta_{\alpha\beta}$.