Reflection and refraction of plane waves at an interface

Consider space filled with two uniform linear materials, one with permitivity and permeability, ϵ, μ and the other with ϵ', μ'. Let the first occupy all space below the xy-plane and the other all space above the xy-plane. The usual electromagnetic boundary conditions therefore apply at the interface at $z = 0$.

Let the incident wave move with wave vector, $\mathbf{k}$, frequency ω, and fields $\mathbf{E}, \mathbf{B}$. Denote the same quantities for the refracted wave by $\mathbf{k}', \omega', \mathbf{E}', \mathbf{B}'$, and those for the reflected wave by $\mathbf{k}'', \omega'', \mathbf{E}'', \mathbf{B}''$.

In order to avoid buildup of energy at the boundary, the space and time dependence of the incoming, reflected, and refracted waves must match,

$$\begin{aligned} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\Big|_{z=0} &= e^{i\left(\mathbf{k}'\cdot\mathbf{x}-\omega' t\right)}\Big|_{z=0} = e^{i(\mathbf{k}"\cdot\mathbf{x}-\omega" t)}\Big|_{z=0} \\ \mathbf{k}\cdot\mathbf{x}-\omega t &= \mathbf{k}'\cdot\mathbf{x}-\omega' t = \mathbf{k}"\cdot\mathbf{x}-\omega" t \end{aligned}$$

These relations have to hold at all times and at all x and y. Therefore,

$$\omega = \omega' = \omega''$$

and

$$k_x x + k_y y = k_x^{'} x + k_y^{'} y = k_x^{''} x + k_y^{''} y$$

We choose the incoming wave to lie in the xz-plane, so that $k_y = 0$. Let the incoming wave make and angle of incidence, i, with the normal to the interface; let the refracted wave make an acute angle r and the reflected wave an acute angle r' with the normal. Varying x and y independently, it immediately follows that $k_y^{'} = 0$ and $k_y^{''} = 0$, so the three waves are coplanar. For the x components, we have

$$k \sin i = k' \sin r = k'' \sin r'$$

Since the incident and reflected waves are in the same medium, $k = k''$, and therefore the angle of reflection equals the angle of incidence,

$$r' = i$$

while, using the relationship for index of refraction, $n = \frac{c}{v} = c\sqrt{\mu\epsilon} = \frac{ck}{\omega}$, we have

$$\frac{n'}{n} = \frac{k'}{k}$$

and therefore, Snell's law for the angle of refraction,

$$n \sin i = n' \sin r$$

Now we turn to the boundary conditions. With $\mathbf{n}$ the unit normal to the interface (that is, a unit vector in the positive z direction), we have:

1. Continuity of the normal component of $\mathbf{D}$,

$$\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} = \epsilon'\boldsymbol{\mathcal{E}}'\cdot\mathbf{n}$$

2. Continuity of the tangential component of $\mathbf{E}$,

$$\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} = \boldsymbol{\mathcal{E}}'\times\mathbf{n}$$

3. Continuity of the normal component of $\mathbf{B}$. Using $\boldsymbol{\mathcal{B}} = \frac{1}{k}\sqrt{\mu\epsilon}\mathbf{k}\times\boldsymbol{\mathcal{E}} = \frac{1}{\omega}\mathbf{k}\times\boldsymbol{\mathcal{E}}$ and cancelling the overall ω,

$$\begin{aligned} \left(\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}''\right)\cdot\mathbf{n} &= \boldsymbol{\mathcal{B}}'\cdot\mathbf{n} \\ \left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \mathbf{k}'\times\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \end{aligned}$$

4. Continuity of the tangential component of $\mathbf{H}$,

$$\frac{1}{\mu}\left(\mathbf{k}\times\boldsymbol{\mathcal{E}}+\mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\times\mathbf{n}=\frac{1}{\mu'}\left(\mathbf{k}'\times\boldsymbol{\mathcal{E}}'\right)\times\mathbf{n}$$

Instead of applying these equations to a general incident polarization, we may treat the two independent linear polarization directions independently, with the general result being a linear combination of the two. We therefore need consider only two particular cases, (1) the electric field perpendicular to the plane incidence (i.e., the $+y$-direction), and (2) parallel to the plane of incidence.

Polarization perpendicular to the plane of incidence

Let the electric field vector, $\boldsymbol{\mathcal{E}}$, point in the positive y direction. Setting

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= \mathbf{j}\mathcal{E}\\ \boldsymbol{\mathcal{E}}' &= \mathbf{j}\mathcal{E}'\\ \boldsymbol{\mathcal{E}}'' &= \mathbf{j}\mathcal{E}''\end{aligned}$$

the first boundary condition vanishes identically, while the remaining boundary csonditions become (if you need more detail on this, look below at the second case),

$$\begin{aligned}\mathcal{E}+\mathcal{E}'' &= \mathcal{E}'\\ \left(k\mathcal{E}+k''\mathcal{E}''\right)\sin i &= k'\mathcal{E}'\sin r\\ \frac{1}{\mu}\left(k\mathcal{E}-k''\mathcal{E}''\right)\cos i &= \frac{1}{\mu'}k'\mathcal{E}'\cos r\end{aligned}$$

Using Snell's law in the form $k\sin i=k'\sin r$, the first two of these are identical, while using

$$\frac{k}{\mu}=\omega\sqrt{\frac{\epsilon}{\mu}}$$

to express the result in terms of properties of the medium, then cancelling an overall factor of ω, the third becomes

$$\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}-\mathcal{E}''\right)\cos i=\sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'\cos r$$

Eliminating $\mathcal{E}'$ using the first equation, we solve for $\frac{\mathcal{E}''}{\mathcal{E}}$,

$$\begin{aligned}\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}-\mathcal{E}''\right)\cos i &= \sqrt{\frac{\epsilon'}{\mu'}}\left(\mathcal{E}+\mathcal{E}''\right)\cos r\\ \sqrt{\frac{\epsilon}{\mu}}\mathcal{E}\cos i-\sqrt{\frac{\epsilon}{\mu}}\mathcal{E}''\cos i &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}\sqrt{1-\sin^2 r}+\sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}''\sqrt{1-\sin^2 r}\\ \left(\sqrt{\frac{\epsilon}{\mu}}\cos i-\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}\right)\mathcal{E} &= \left(\sqrt{\frac{\epsilon}{\mu}}\cos i+\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}\right)\mathcal{E}''\end{aligned}$$

so that

$$\frac{\mathcal{E}''}{\mathcal{E}}=\frac{\sqrt{\frac{\epsilon}{\mu}}\cos i-\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}}{\sqrt{\frac{\epsilon}{\mu}}\cos i+\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}}$$

Next we want to express this in terms of the index of refraction,

$$n = \frac{c}{v} = c\sqrt{\mu\epsilon}$$

so that multiplying through the numerator and denominator by c, we replace

$$c\sqrt{\frac{\epsilon}{\mu}} = \frac{n}{\mu}$$

and similarly $c\sqrt{\frac{\epsilon'}{\mu'}} = \frac{n'}{\mu'}$,

$$\begin{aligned}
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{c\sqrt{\frac{\epsilon}{\mu}}\cos i - c\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}}{c\sqrt{\frac{\epsilon}{\mu}}\cos i + c\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}} \\
&= \frac{\frac{1}{\mu}n\cos i - \frac{1}{\mu'}n'\sqrt{1-\sin^2 r}}{\frac{1}{\mu}n\cos i + \frac{1}{\mu'}n'\sqrt{1-\sin^2 r}} \\
&= \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

Now multiply through by μ and use Snell's law to write this in terms of the angle of incidence only as

$$\frac{\mathcal{E}''}{\mathcal{E}} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Finally, substituting this result into $\mathcal{E} + \mathcal{E}'' = \mathcal{E}'$, we have

$$\begin{aligned}
\mathcal{E}' &= \mathcal{E} + \mathcal{E}'' \\
\frac{\mathcal{E}'}{\mathcal{E}} &= 1 + \frac{\mathcal{E}''}{\mathcal{E}} \\
&= 1 + \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\
&= \frac{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i} + n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\
&= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

so we have the final result for parallel reflection/refraction:

$$\begin{aligned}
\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

Polarization in the plane of incidence

Now let $\boldsymbol{\mathcal{E}}$ lie in the xz-plane. Then, setting

$$\begin{aligned}
\boldsymbol{\mathcal{E}} &= -\mathbf{i}\mathcal{E}\cos i + \mathbf{n}\mathcal{E}\sin i \\
\boldsymbol{\mathcal{E}}' &= -\mathbf{i}\mathcal{E}'\cos r + \mathbf{n}\mathcal{E}'\sin r \\
\boldsymbol{\mathcal{E}}'' &= \mathbf{i}\mathcal{E}''\cos i + \mathbf{n}\mathcal{E}''\sin i
\end{aligned}$$

we substitute into the boundary conditions,

$$\begin{aligned}
\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \epsilon'\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} &= \boldsymbol{\mathcal{E}}'\times\mathbf{n} \\
\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \mathbf{k}'\times\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\frac{1}{\mu}\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} &= \frac{1}{\mu'}\left(\mathbf{k}'\times\boldsymbol{\mathcal{E}}'\right)\times\mathbf{n}
\end{aligned}$$

For the first,

$$\begin{aligned}
\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \epsilon'\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\epsilon\left(-\mathbf{i}\mathcal{E}\cos i + \mathbf{n}\mathcal{E}\sin i + \mathbf{i}\mathcal{E}''\cos i + \mathbf{n}\mathcal{E}''\sin i\right)\cdot\mathbf{n} &= \epsilon'\left(-\mathbf{i}\mathcal{E}'\cos r + \mathbf{n}\mathcal{E}'\sin r\right)\cdot\mathbf{n} \\
\epsilon\left(\mathcal{E}\sin i + \mathcal{E}''\sin i\right) &= \epsilon'\mathcal{E}'\sin r
\end{aligned}$$

the second becomes

$$\begin{aligned}
\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} &= \boldsymbol{\mathcal{E}}'\times\mathbf{n} \\
\left(-\mathbf{i}\mathcal{E}\cos i + \mathbf{n}\mathcal{E}\sin i + \mathbf{i}\mathcal{E}''\cos i + \mathbf{n}\mathcal{E}''\sin i\right)\times\mathbf{n} &= \left(-\mathbf{i}\mathcal{E}'\cos r + \mathbf{n}\mathcal{E}'\sin r\right)\times\mathbf{n} \\
\mathbf{j}\left(\mathcal{E}\cos i - \mathbf{i}\mathcal{E}''\cos i\right) &= \mathbf{j}\mathcal{E}'\cos r \\
\mathcal{E}\cos i - \mathcal{E}''\cos i &= \mathcal{E}'\cos r
\end{aligned}$$

the third is now

$$\begin{aligned}
\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \mathbf{k}'\times\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\left(-\mathbf{j}k\mathcal{E} - \mathbf{j}k''\mathcal{E}''\right)\cdot\mathbf{n} &= -\mathbf{j}k'\mathcal{E}'\cdot\mathbf{n} \\
0 &= 0
\end{aligned}$$

and, finally, the fourth gives

$$\begin{aligned}
\frac{1}{\mu}\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} &= \frac{1}{\mu'}\left(\mathbf{k}'\times\boldsymbol{\mathcal{E}}'\right)\times\mathbf{n} \\
\frac{1}{\mu}\left(-\mathbf{j}k\mathcal{E} - \mathbf{j}k''\mathcal{E}''\right)\times\mathbf{n} &= \frac{1}{\mu'}\left(-\mathbf{j}k'\mathcal{E}'\right)\times\mathbf{n} \\
-\frac{1}{\mu}\omega\sqrt{\mu\epsilon}\left(\mathcal{E} + \mathcal{E}''\right)\mathbf{i} &= -\frac{1}{\mu'}\omega\sqrt{\mu'\epsilon'}\mathcal{E}'\mathbf{i} \\
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E} + \mathcal{E}''\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

Use Snell's law on the first of these four,

$$\begin{aligned}
\epsilon\left(\mathcal{E}\sin i + \mathcal{E}''\sin i\right) &= \epsilon'\mathcal{E}'\sin r \\
\frac{\epsilon}{k}\left(\mathcal{E} + \mathcal{E}''\right) &= \frac{\epsilon'}{k'}\mathcal{E}'
\end{aligned}$$

$$\begin{aligned}
\frac{\epsilon}{\omega\sqrt{\mu\epsilon}}\left(\mathcal{E}+\mathcal{E}''\right) &= \frac{\epsilon'}{\omega\sqrt{\mu'\epsilon'}}\mathcal{E}' \\
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}''\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

showing that it reproduces the fourth. Therefore, we have just two equations,

$$\begin{aligned}
\mathcal{E}\cos i-\mathcal{E}''\cos i &= \mathcal{E}'\cos r \\
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}''\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

Substituting the second into the first,

$$\begin{aligned}
\mathcal{E}\cos i-\mathcal{E}''\cos i &= \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}''\right)\cos r \\
\mathcal{E}\cos i-\mathcal{E}''\cos i &= \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}\cos r+\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}''\cos r \\
\mathcal{E}''\cos i+\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}''\cos r &= \mathcal{E}\cos i-\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}\cos r
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{\cos i-\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\cos r}{\cos i+\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\cos r} \\
&= \frac{\cos i-\frac{n\mu'}{n'\mu}\cos r}{\cos i+\frac{n\mu'}{n'\mu}\cos r} \\
&= \frac{n'^2\cos i-\frac{\mu'}{\mu}nn'\sqrt{1-\sin^2 r}}{n'^2\cos i+\frac{\mu'}{\mu}nn'\sqrt{1-\sin^2 r}} \\
&= \frac{n'^2\cos i-\frac{\mu'}{\mu}n\sqrt{n'^2-n'^2\sin^2 r}}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n'^2\sin^2 r}} \\
&= \frac{n'^2\cos i-\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}
\end{aligned}$$

Notice how the last few steps allow us to express $\cos r$ in terms of the incident angle, i.

Finally, substitute rewrite the second boundary condition as

$$\begin{aligned}
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}''\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}' \\
\sqrt{\frac{\epsilon}{\mu}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\frac{\mathcal{E}'}{\mathcal{E}} \\
\frac{\mathcal{E}'}{\mathcal{E}} &= \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) \\
&= \frac{\mu'}{\mu}\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right)
\end{aligned}$$

$$= \frac{n\mu'}{n'\mu}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right)$$

and substitute the first solution,

$$\begin{aligned}
\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{n\mu'}{n'\mu}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(1+\frac{n'^2\cos i-\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(\frac{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}+n'^2\cos i-\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(\frac{2n'^2\cos i}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\right) \\
&= \frac{\mu'}{\mu}\left(\frac{2nn'\cos i}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\right) \\
&= \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i+n\sqrt{n'^2-n^2\sin^2 i}}
\end{aligned}$$

Therefore, the electric field amplitudes for incidence with polarization in the plane of the scattering are given by

$$\begin{aligned}
\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i+n\sqrt{n'^2-n^2\sin^2 i}} \\
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{n'^2\cos i-\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}
\end{aligned}$$

Special cases

There are two special cases. For simplicity, we set $\mu = \mu'$.

Total internal reflection

For polarization perpendicular to the plane of incidence, the direction of the refracted wave is given by Snell's law,

$$n\sin i = n'\sin r$$

and r becomes $\frac{\pi}{2}$ if $n > n'$ and

$$\sin i = \frac{n'}{n}$$

This means that the refracted wave moves only in the x-direction, and not into the second medium at all. We have total internal reflection, and the wave stays in the first medium.

Total absorption

For polarization parallel to the plane of incidence, it is possible for the reflected wave to vanish. With $\mu = \mu'$, the conditon for this is

$$0 = \frac{\mathcal{E}''}{\mathcal{E}}$$

$$
\begin{aligned}
&= \frac{n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \\
n'^2\cos i &= n\sqrt{n'^2 - n^2\sin^2 i} \\
n'^4\cos^2 i &= n^2 n'^2 - n^4\sin^2 i \\
n'^4 - n'^4\sin^2 i &= n^2 n'^2 - n^4\sin^2 i \\
n'^4 - n^2 n'^2 &= \left(n'^4 - n^4\right)\sin^2 i \\
\left(n'^2 - n^2\right) n'^2 &= \left(n'^2 - n^2\right)\left(n'^2 + n^2\right)\sin^2 i \\
\frac{n'}{\sqrt{n'^2 + n^2}} &= \sin i \\
\cos i &= \sqrt{1 - \sin^2 i} \\
&= \sqrt{1 - \frac{n'^2}{n'^2 + n^2}} \\
&= \frac{n}{\sqrt{n'^2 + n^2}}
\end{aligned}
$$

and therefore,

$$\tan i = \frac{n'}{n}$$

This angle, $i_B = \tan^{-1}\frac{n'}{n}$, is called Brewster's angle. When light with a mixture of polarizations strikes a surface at or near Brewster's angle, only the polarization perpendicular to the plane of incidence is reflected.