Plane electromagnetic waves

March 2, 2014

We start with the Maxwell equations in a uniform medium without sources:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D} &= 0\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t} &= 0\\
\boldsymbol{\nabla}\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0
\end{aligned}$$

and assume harmonic time dependence of the fields and linearity of the medium. We can then build more complicated waves by superposition. We set

$$\mathbf{E}(\mathbf{x},t)=\mathbf{E}(\mathbf{x},\omega)\,e^{-i\omega t}$$

with the same dependence for the other fields. Notice that we have made an analytic extension to complex fields. Because the Maxwell equations are linear, either the real or the imaginary parts will solve the equations. It is also useful to allow ε and μ be complex functions of frequency for the description of dissipation and dispersion. For now, think of ε and μ as real.

Then

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D} &= 0\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\times\mathbf{H}+i\omega\mathbf{D} &= 0\\
\boldsymbol{\nabla}\times\mathbf{E}-i\omega\mathbf{B} &= 0
\end{aligned}$$

Notice that since the divergenc of a curl vanishes, the two divergence equations follow automatically from the second pair of equations. Taking a second curl of the third equation and using the last,

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})+i\omega\epsilon\mu\boldsymbol{\nabla}\times\mathbf{E} &= 0\\
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B})-\nabla^2\mathbf{B}+i\omega\epsilon\mu\,(i\omega\mathbf{B}) &= 0\\
\nabla^2\mathbf{B}+\epsilon\mu\omega^2\mathbf{B} &= 0
\end{aligned}$$

with a similar calculation for the electric field yielding $\nabla^2\mathbf{E}+\epsilon\mu\omega^2\mathbf{E}=0$ If we assume a plane wave travelling in the $\hat{\mathbf{n}}$ direction, by setting $\mathbf{k}=k\hat{\mathbf{n}}$ and

$$\begin{aligned}
\mathbf{E} &= \boldsymbol{\mathcal{E}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\\
\mathbf{B} &= \boldsymbol{\mathcal{B}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}
\end{aligned}$$

Substituting,

$$\begin{aligned}
(i\mathbf{k})^2+\epsilon\mu\omega^2 &= 0\\
k^2 &= \epsilon\mu\omega^2\\
k &= \pm\sqrt{\epsilon\mu}\omega
\end{aligned}$$

A point of constant phase of the wave is now given by

$$\begin{aligned}\varphi_0 &= \mathbf{k}\cdot\mathbf{x}-\omega t\\ &= \left(\pm\sqrt{\epsilon\mu}\,\hat{\mathbf{n}}\cdot\mathbf{x}-t\right)\omega\end{aligned}$$

so, taking the differential, that point moves so that

$$\begin{aligned}0 &= \pm\sqrt{\epsilon\mu}\,\omega\hat{\mathbf{n}}\cdot d\mathbf{x}-\omega dt\\ \hat{\mathbf{n}}\cdot\frac{d\mathbf{x}}{dt} &= \pm\frac{1}{\sqrt{\epsilon\mu}}\end{aligned}$$

There is no motion orthogonal to $\hat{\mathbf{n}}$, and the speed of the point of constant phase (for example, the crest of a wave) is

$$v=\frac{1}{\sqrt{\epsilon\mu}}$$

This is called the phase velocity.

We must still solve the original equations. Substituting into each, we find

$$\begin{aligned}\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{E}} &= 0\\ \hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{B}} &= 0\\ ik\hat{\mathbf{n}}\times\boldsymbol{\mathcal{B}}+i\omega\epsilon\mu\boldsymbol{\mathcal{E}} &= 0\\ ik\hat{\mathbf{n}}\times\boldsymbol{\mathcal{E}}-i\omega\boldsymbol{\mathcal{B}} &= 0\end{aligned}$$

so that the electric and magnetic fields are orthogonal to the direction, $\hat{\mathbf{n}}$, of propagation. In addition, the final equation requires

$$\boldsymbol{\mathcal{B}}=\sqrt{\epsilon\mu}\,\hat{\mathbf{n}}\times\boldsymbol{\mathcal{E}}$$

Substituting this, the third equation becomes

$$\begin{aligned}\hat{\mathbf{n}}\times\left(\sqrt{\epsilon\mu}\,\hat{\mathbf{n}}\times\boldsymbol{\mathcal{E}}\right) &= -\sqrt{\epsilon\mu}\boldsymbol{\mathcal{E}}\\ \hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\boldsymbol{\mathcal{E}}\right) &= -\boldsymbol{\mathcal{E}}\\ \hat{\mathbf{n}}\left(\boldsymbol{\mathcal{E}}\cdot\hat{\mathbf{n}}\right)-\boldsymbol{\mathcal{E}} &= -\boldsymbol{\mathcal{E}}\end{aligned}$$

so it is identically satisfied.

We define the index of refraction, n, as the ratio of the speed of light in vacuum to the speed of light in a medium:

$$n\equiv\frac{c}{\sqrt{\mu\epsilon}}$$

For a plane electromagnetic wave, with real index of refraction, we therefore have the electric field, magnetic field, and direction of propagation all mutually perpendicular. The magnitude of the magnetic field is related to the magnitude of the electric field by

$$\mathcal{B}=\frac{c}{n}\mathcal{E}$$

To completely specify the wave we therefore need:

1. The frequency, ω.
2. The direction of propagation, $\hat{\mathbf{n}}$.
3. The magnitude and direction of $\boldsymbol{\mathcal{E}}$ in the plane perpendicular to $\hat{\mathbf{n}}$.

To specify the direction of the electric field, it is useful to introduce a set of orthonormal basis vectors, $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \mathbf{n}$, with $\boldsymbol{\varepsilon}_1$ or $\boldsymbol{\varepsilon}_2$ giving the direction of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\varepsilon}_2 = \mathbf{n} \times \boldsymbol{\varepsilon}_1$ or $-\boldsymbol{\varepsilon}_1$ giving the direction of $\boldsymbol{\mathcal{B}}$. The direction of the electric field is called the polarization. A general wave may be a superposition of different frequencies, different directions $\mathbf{n}$, and/or different polarizations.

We may also find the Poynting vector complexified. To get the right expression, consider the real part of our solution above,

$$\begin{aligned} \mathbf{E} &= \boldsymbol{\varepsilon}_1 \mathcal{E} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \\ \mathbf{B} &= \boldsymbol{\mathcal{B}} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \\ &= \boldsymbol{\varepsilon}_2 \frac{c}{n} \mathcal{E} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \end{aligned}$$

For these, the Poynting vector is

$$\begin{aligned} \mathbf{S} &= \mathbf{E} \times \mathbf{H} \\ &= \frac{c\mathcal{E}^2}{\mu n} \mathbf{n} \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t) \\ &= \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t) \end{aligned}$$

so the time average is

$$\begin{aligned} \langle \mathbf{S} \rangle &= \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \left\langle \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t) \right\rangle \\ &= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \end{aligned}$$

This differs from Jackson's result by a factor of $\frac{1}{2}$. However, Jackson states that the time averaged flux of energy is given by the real part of

$$\begin{aligned} \mathbf{S} &= \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \\ &= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \end{aligned}$$

In our example, this becomes

$$\mathbf{S} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n}$$

What happens is that the factor of $\frac{1}{2}$ gives the time average of a sine or cosine wave, while the complex conjugation cancels the phase factors altogether. However, eq. 7.13 cannot be correct because it has no time dependence, whereas a real wave will have oscillating flux,

$$\mathbf{S} = \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t)$$