Multipole Radiation

March 17, 2014

1 Zones

We will see that the problem of harmonic radiation divides into three approximate regions, depending on the relative magnitudes of the distance of the observation point, r, and the wavelength, λ. We assume throughout that the extent of the source is small compared to the both of these, $d \ll \lambda$ and $d \ll r$. We consider the cases:

$$\begin{aligned} d &\ll r \ll \lambda \quad (static\, zone) \\ d &\ll r \sim \lambda \quad (induction\, zone) \\ d &\ll \lambda \ll r \quad (radiation\, zone) \end{aligned}$$

or simply the near, intermediate, and far zones. The behavior of the fields is quite different, depending on the zone.

The near and far zones allow us to use different approximations. Since d is always much smaller than r or λ, we may always expand in powers of $\frac{d}{r}$ and/or $\frac{d}{\lambda}$. Starting from

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}(\mathbf{x}') e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

we expand

$$\begin{aligned} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} &= \frac{e^{ik\left(r-\hat{\mathbf{r}}\cdot\mathbf{x}'+\cdots\right)}}{r-\hat{\mathbf{r}}\cdot\mathbf{x}'+\cdots} \\ &= \frac{e^{ikr}}{r}\frac{e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}}{1-\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'} \\ &= \frac{e^{ikr}}{r}\left(1-ik\hat{\mathbf{r}}\cdot\mathbf{x}'+\cdots\right)\left(1+\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'+\cdots\right) \\ &= \frac{e^{ikr}}{r}\left(1+\left(\frac{1}{r}-ik\right)\hat{\mathbf{r}}\cdot\mathbf{x}'+\cdots\right) \end{aligned}$$

We can work from this expansion, or we can restrict to a particular zone to simplify the expansion.

There is another possible expansion, in terms of spherical harmonics. For the denominator,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}} Y_{lm}(\theta,\varphi) Y^*_{lm}(\theta',\varphi')$$

This is most useful in the near zone, where the factor of $e^{ik|\mathbf{x}-\mathbf{x}'|}$ is close to unity.

1.1 The near zone

1.1.1 Near zone

In the near zone, $r \ll \lambda$ so

$$\begin{aligned} k\left|\mathbf{x}-\mathbf{x}'\right| &= k\sqrt{r^2+x'^2-2\mathbf{x}\cdot\mathbf{x}'} \\ &< k\sqrt{r^2+d^2-2rd\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'} \\ &< kr\sqrt{1-\frac{2d}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'+\frac{d^2}{r^2}} \\ &\ll 1 \\ e^{ik\left|\mathbf{x}-\mathbf{x}'\right|} &\approx 1 \end{aligned}$$

With this approximation, the wavelength dependence drops out and the potential becomes

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|} \\ &= \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\int d^3x'\mathbf{J}(\mathbf{x}')\,r'^l Y^*_{lm}(\theta',\varphi') \end{aligned}$$

The only time dependence is the sinusoidal oscillation, $e^{-i\omega t}$, with the potential given in terms of the moments,

$$K_{lm} \equiv \int d^3x'\mathbf{J}(\mathbf{x}')\,r'^l Y^*_{lm}(\theta',\varphi')$$

of the current distribution. These are just like the multipole moments for the electrostatic potential, but with the charge density replaced by the current density. The lowest nonvanishing term in the series will dominate the field, since increasing l decreases the potential by powers of order $\frac{d}{r}$.

1.2 Intermediate zone

In the intermediate zone, with $r \sim \lambda$, an exact expansion of the Green function is useful. This is found by expanding

$$G(\mathbf{x},\mathbf{x}') = \frac{e^{ik\left|\mathbf{x}-\mathbf{x}'\right|}}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}$$

in spherical harmonics,

$$G(\mathbf{x},\mathbf{x}') = \sum_{l,m} g_{lm}(r,r')\,Y_{lm}(\theta,\varphi)\,Y^*_{lm}(\theta',\varphi')$$

and solving the rest of the Helmholtz equation for the radial function. The result is spherical Bessel functions, $j_l(x), n_l(x)$, and the related spherical Henkel functions, $h_l^{(1)}, h_l^{(2)}$, which are essentially Bessel function times $\frac{1}{\sqrt{r}}$. They are discussed in Jackson, Section 9.6. The vector potential then takes the form

$$\mathbf{A}(\mathbf{x}) = ik\mu_0\sum_{l,m} h_l^{(1)}(kr)\,Y_{lm}(\theta,\varphi)\int d^3x'\mathbf{J}(\mathbf{x}')\,j_l(kr')\,Y^*_{lm}(\theta',\varphi')$$

The spherical Bessel function may be expanded in powers of kr to recover the previous near zone approximations.

There are also applicable approximation methods. Although r and λ are of the same order of magnitude, both are much greater than d, so we may expand in a double power series. This gives the multipole expansion in subsequent sections.

1.3 Far zone

1.3.1 Truncation of the series in r

In the far zone, we still have both both $\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'\ll 1$ and $ik\hat{\mathbf{r}}\cdot\mathbf{x}'\ll 1$, but now $\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'\ll k\hat{\mathbf{r}}\cdot\mathbf{x}'$, so we expand only to zeroth order in $\frac{x'}{r}\sim\frac{d}{r}$ first. With

$$|\mathbf{x}-\mathbf{x}'|\approx r$$

the integrand becomes

$$\begin{aligned}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} &\approx \frac{e^{ik\left(r-\hat{\mathbf{r}}\cdot\mathbf{x}'\right)}}{r}\\ &= \frac{e^{ikr}}{r}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}\end{aligned}$$

The potential is then

$$\mathbf{A}\left(\mathbf{x}\right)=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}$$

Now we use $kx'\lesssim kd\ll 1$ and expand the exponential. We may carry this power series in kx' to order N as long as we can still neglect $\left(kx'\right)^N$ relative to $\frac{d}{r}$,

$$\frac{d}{r}\ll\left(kd\right)^N$$

The expansion is then

$$\begin{aligned}\mathbf{A}\left(\mathbf{x}\right) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}\\ &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}\frac{\left(-ik\right)^n}{n!}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)r'^n\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^n\end{aligned}$$

Again, only the lowest nonvanishing moment of the current distribution

$$\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)r'^n\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^n$$

dominates the radiation field. Notice that $\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}$ is just the Fourier transform of the current density.

2 Multipole expansion of time dependent electromagnetic fields

2.1 The fields in terms of the potentials

Consider a localized, oscillating source, located in otherwise empty space. Let the source fields be confined in a region $d\ll\lambda$ where λ is the wavelength of the radiation, and let the time dependence be harmonic, with frequency ω,

$$\begin{aligned}\mathbf{A}\left(\mathbf{x},t\right) &= \mathbf{A}\left(\mathbf{x}\right)e^{-i\omega t}\\ \phi\left(\mathbf{x},t\right) &= \phi\left(\mathbf{x}\right)e^{-i\omega t}\\ \mathbf{J}\left(\mathbf{x},t\right) &= \mathbf{J}\left(\mathbf{x}\right)e^{-i\omega t}\\ \rho\left(\mathbf{x},t\right) &= \rho\left(\mathbf{x}\right)e^{-i\omega t}\end{aligned}$$

We are interested in the field at distances $r\gg d$.

In general, we have the fields in terms of the scalar and vector potentials,

$$\begin{aligned}\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}\end{aligned}$$

However, for most configurations the vector potential alone is sufficient, since the Maxwell equation, $\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0$, shows that

$$-i\omega\epsilon_0\mathbf{E} = \boldsymbol{\nabla}\times\mathbf{H}$$

Dividing by $-i\omega\epsilon_0 = -ikc\epsilon_0 = -ik\sqrt{\frac{\epsilon_0}{\mu_0}}$ and defining the impedence of free space, $Z = \sqrt{\frac{\mu_0}{\epsilon_0}}$, gives the electric field in the form

$$\mathbf{E} = \frac{iZ}{k}\boldsymbol{\nabla}\times\mathbf{H}$$

2.2 Electric monopole

There is one exception to this conclusion: the case of vanishing magnetic field. In this case, we may use the scalar potential satisfying

$$\Box\phi = -\frac{\rho}{\epsilon_0}$$

with solution

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\int dt'\frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)$$

with $k = \frac{\omega}{c}$. For $\frac{d}{r} \ll 1$ we have $|\mathbf{x}-\mathbf{x}'| \approx r$ and this becomes

$$\begin{aligned}\phi(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0 r}\int d^3x'\rho\left(\mathbf{x}',t-\frac{r}{c}\right) \\ &= \frac{q_{tot}\left(t-\frac{r}{c}\right)}{4\pi\epsilon_0 r}\end{aligned}$$

that is, just the Coulomb potential for the total charge at time $t = \frac{r}{c}$. But for an isolated system, the continuity equation, $\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$, may be integrated over any region outside d, giving

$$\begin{aligned}\int d^3x\left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}\right) &= 0 \\ \frac{d}{dt}\int d^3x\rho + \int d^3x\boldsymbol{\nabla}\cdot\mathbf{J} &= 0 \\ \frac{d}{dt}\int d^3x\rho &= -\oint d^2x\mathbf{n}\cdot\mathbf{J} \\ \frac{dq_{tot}}{dt} &= 0\end{aligned}$$

Therefore, the total charge cannot change, so $q_{tot}\left(t-\frac{r}{c}\right) = q_{tot} = constant$, and the potential is independent of time,

$$\phi(\mathbf{x},t) = \frac{q_{tot}}{4\pi\epsilon_0 r}$$

As a result, there is no electric monopole radiation.

Thus, when the conserved electric charge is confined to a bounded region, all radiation effects follow from the resulting *currents*, via the *vector* potential, with the fields following from

$$\begin{aligned}\mathbf{E} &= \frac{iZ}{k}\boldsymbol{\nabla}\times\mathbf{H} \\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}\end{aligned}$$

We first consider general results for $\mathbf{A}(\mathbf{x},t)$, then some cases which hold only in certain regions.

2.3 The vector potential

We now examine the vector potential. Consider a localized, oscillating source, located in otherwise empty space. We know that the solution for the vector potential (e.g. using the Green function for the outer boundary at infinity) is

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\int dt'\frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)$$

Then

$$\begin{aligned}\mathbf{A}(\mathbf{x})e^{-i\omega t} &= \frac{\mu_0}{4\pi}\int d^3x'\int dt'\frac{\mathbf{J}(\mathbf{x}')e^{-i\omega t'}}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\\ &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')e^{-i\omega\left(t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}}{|\mathbf{x}-\mathbf{x}'|}\end{aligned}$$

so that with $k=\frac{\omega}{c}$, we have

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

A general approach to finding the vector potential, similar to the static case, is to expand the denominator,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l,m}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}}Y_{lm}(\theta,\varphi)Y^*_{lm}(\theta',\varphi')$$

Then the vector potential is

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi)\int d^3x'\mathbf{J}(\mathbf{x}')r'^le^{ik|\mathbf{x}-\mathbf{x}'|}Y^*_{lm}(\theta',\varphi')$$

and since the integral is bounded

$$\int d^3x'\mathbf{J}(\mathbf{x}')r'^le^{ik|\mathbf{x}-\mathbf{x}'|}Y^*_{lm}(\theta',\varphi')<\frac{4}{3}\pi d^{l+3}\mathbf{J}_{max}$$

the terms fall off with increasing r as $\left(\frac{d}{r}\right)^{l+1}$. We may therefore approximate the vector potential by the lowest nonvanishing term.

An easier way to keep track of increasing powers of $\frac{d}{r}$ is to simply expand the denominator,

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r}\left(1+\frac{r'^2}{r^2}-\frac{2r'}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^{-1/2}\\ &= \frac{1}{r}\left(1-\frac{1}{2}\left(\frac{r'^2}{r^2}-\frac{2r'}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)+\frac{3}{8}\left(\frac{r'^2}{r^2}-\frac{2r'}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2+\mathcal{O}\left(\frac{d^3}{r^3}\right)\right)\\ &= \frac{1}{r}\left(1+\frac{r'}{r}(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')-\frac{1}{2}\frac{r'^2}{r^2}\left(1-3(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')^2\right)+\mathcal{O}\left(\frac{d^3}{r^3}\right)\right)\end{aligned}$$

and also expand the exponent,

$$\begin{aligned}|\mathbf{x}-\mathbf{x}'| &= \sqrt{r^2+r'^2-2\mathbf{x}\cdot\mathbf{x}'}\\ &= r\sqrt{1-\frac{2r'}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'+\frac{r'^2}{r^2}}\\ &= r\left(1+\frac{1}{2}\left(-\frac{2r'}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'+\frac{r'^2}{r^2}\right)-\frac{1}{2!}\frac{1}{4}\left(-\frac{2r'}{r}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'+\frac{r'^2}{r^2}\right)^2+\mathcal{O}\left(\frac{d^3}{r^3}\right)\right)\\ &= r\left(1-\frac{r'}{r}(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')+\frac{1}{2}\frac{r'^2}{r^2}\left(1-(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')^2\right)+\mathcal{O}\left(\frac{d^3}{r^3}\right)\right)\end{aligned}$$

so that

$$\begin{aligned}
e^{ik|\mathbf{x}-\mathbf{x}'|} &= e^{ikr-ikr\frac{r'}{r}\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)+\frac{1}{2}ikr\frac{r'^2}{r^2}\left(1-\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2\right)+\mathcal{O}\left(\frac{d^3}{r^3}\right)} \\
&= e^{ikr}e^{-ikr'\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)}e^{\frac{1}{2}ikr\frac{r'^2}{r^2}\left(1-\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2\right)} \\
&= e^{ikr}\left(1-ikr'\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)-\frac{1}{2}k^2r'^2\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2+\cdots\right)\left(1+\frac{1}{2}i\left(kr'\right)\left(\frac{r'}{r}\right)\left(1-\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2\right)+\cdots\right)
\end{aligned}$$

and since both kr' and $\frac{r'}{r}$ are small,

$$e^{ik|\mathbf{x}-\mathbf{x}'|}=e^{ikr}\left(1-ikr'\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)-\frac{1}{2}k^2r'^2\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2+\frac{1}{2}ikr'\left(\frac{r'}{r}\right)\left(1-\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2\right)+\cdots\right)$$

Substituting this into both the exponential and the denominator,

$$\begin{aligned}
\mathbf{A}\left(\mathbf{x}\right) &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}\left(\mathbf{x}'\right)e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)+\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)\left(\hat{\mathbf{x}}\cdot\mathbf{x}'\right) \\
&\quad -\frac{\mu_0}{8\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)r'^2\left(\frac{1}{r^2}-\frac{ik}{r}+\left(k^2+\frac{3ik}{r}-\frac{3}{r^2}\right)\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)^2\right)+\mathcal{O}\left(\frac{d^3}{r^3}\right)
\end{aligned}$$

The first integral is of order $\frac{1}{r}$, the second is of order $max\left(\frac{1}{r}\frac{d}{r},\frac{1}{r}kd\right)$ and the third of order $max\left(\frac{1}{r}\frac{d^2}{r},\frac{1}{r}\frac{kd^2}{r},\frac{1}{r}k^2d^2\right)$, where both $\frac{d}{r}$ and kd are small. We show below that the first of these gives the electric dipole radiation, while the second gives both electric quadrupole and magnetic dipole radiation. The third and higher terms give higher electric and magnetic multipoles.

$$\begin{aligned}
\mathbf{A}\left(\mathbf{x}\right) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)+\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r^2}\left(1-ikr\right)\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right) \\
&\quad +\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r^3}\left(1-ikr-\frac{1}{2}k^2r^2\right)\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)^2+\cdots
\end{aligned}$$

3 Electric dipole radiation

The dominant term in the multipole expansion of $\mathbf{A}\left(\mathbf{x}\right)$ is

$$\mathbf{A}\left(\mathbf{x}\right)=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)$$

From the continuity equation,

$$\begin{aligned}
0 &= \frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{J} \\
&= -i\omega\rho\left(\mathbf{x}\right)+\boldsymbol{\nabla}\cdot\mathbf{J}
\end{aligned}$$

and a cute trick. Since the current vanishes at infinity, we may write a vanishing total divergence of $x_j\mathbf{J}\left(\mathbf{x}'\right)$:

$$\begin{aligned}
\int d^3x'\boldsymbol{\nabla}\cdot\left(x'_j\mathbf{J}\left(\mathbf{x}'\right)\right) &= \int d^2x'\hat{\mathbf{r}}\cdot\left(x'_j\mathbf{J}\left(\mathbf{x}'\right)\right) \\
&= 0
\end{aligned}$$

Then

$$\begin{aligned}
0 &= \int d^3x' \boldsymbol{\nabla}' \cdot \left(x'_j \mathbf{J}(\mathbf{x}')\right) \\
&= \sum_i \int d^3x' \nabla'_i \left(x'_j J_i(\mathbf{x}')\right) \\
&= \sum_i \int d^3x' \left(\nabla'_i x'_j J_i(\mathbf{x}') + x'_j \nabla'_i J_i(\mathbf{x}')\right) \\
&= \sum_i \int d^3x' \left(\delta_{ij} J_i(\mathbf{x}') + x'_j \nabla'_i J_i(\mathbf{x}')\right) \\
&= \int d^3x' \left(J_j(\mathbf{x}') + x'_j \boldsymbol{\nabla}' \cdot \mathbf{J}\right)
\end{aligned}$$

and we have

$$\begin{aligned}
\int d^3x' J_j(\mathbf{x}') &= -\int d^3x'\, x'_j \boldsymbol{\nabla}' \cdot \mathbf{J} \\
&= -i\omega \int d^3x'\, x'_j \rho(\mathbf{x}')
\end{aligned}$$

This integral is the electric dipole moment,

$$\mathbf{p} = \int d^3x'\, \mathbf{x}' \rho(\mathbf{x}')$$

and the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}$$

The magnetic field is the curl of this,

$$\begin{aligned}
\mathbf{H}(\mathbf{x}) &= \frac{1}{\mu_0}\boldsymbol{\nabla} \times \mathbf{A} \\
&= -\frac{i\omega}{4\pi}\boldsymbol{\nabla} \times \left(\frac{e^{ikr}}{r}\mathbf{p}\right) \\
&= -\frac{i\omega}{4\pi}\left(\boldsymbol{\nabla}\frac{e^{ikr}}{r} \times \mathbf{p}\right) \\
&= -\frac{i\omega}{4\pi}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)\hat{\mathbf{r}} \times \mathbf{p} \\
&= -\frac{i\omega}{4\pi}\left(\frac{ike^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right)\hat{\mathbf{r}} \times \mathbf{p} \\
&= \frac{\omega k}{4\pi}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \mathbf{p} \\
&= \frac{k^2 c}{4\pi}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \mathbf{p}
\end{aligned}$$

and is therefore transverse to the radial direction. For the electric field,

$$\begin{aligned}
\mathbf{E} &= \frac{iZ}{k}\boldsymbol{\nabla} \times \mathbf{H} \\
&= \frac{iZkc}{4\pi}\boldsymbol{\nabla} \times \left(\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \mathbf{p}\right)
\end{aligned}$$

Using

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\,(\nabla \cdot \mathbf{b}) - \mathbf{b}\,(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\,\mathbf{a} - (\mathbf{a} \cdot \nabla)\,\mathbf{b}$$

this becomes

$$\mathbf{E} \;=\; \frac{iZkc}{4\pi}\left[(\mathbf{p} \cdot \nabla)\left[\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right] - \mathbf{p}\left(\nabla \cdot \left[\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right]\right)\right]$$

We easily compute the first term in the brackets using the identity

$$(\mathbf{a} \cdot \nabla)\,(\hat{\mathbf{r}} f\,(r)) = \frac{f\,(r)}{r}\left[\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] + (\mathbf{a} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\frac{\partial f}{\partial r}$$

Thus, the first term is

$$\begin{aligned}
(\mathbf{p} \cdot \nabla)\left[\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right] &= \frac{f\,(r)}{r}\left[\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] + (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\frac{\partial f}{\partial r} \\
&= \frac{1}{r}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\left[\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] + (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\left[\frac{1}{ikr^2}\frac{e^{ikr}}{r} - \left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r^2} + \left(1 - \frac{1}{ikr}\right)\frac{ike^{ikr}}{r}\right. \\
&= \frac{e^{ikr}}{r}\left[\left(\frac{1}{r} - \frac{1}{ikr^2}\right)\left[\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] + (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\left(ik - \frac{2}{r} + \frac{2}{ikr^2}\right)\right]
\end{aligned}$$

For the second term, we use

$$\nabla \cdot (\hat{\mathbf{r}} f\,(r)) = \frac{2}{r} f + \frac{\partial f}{\partial r}$$

so that

$$\begin{aligned}
\nabla \cdot \left[\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right] &= \frac{2}{r}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r} + \frac{\partial}{\partial r}\left(\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\right) \\
&= \frac{2}{r}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r} + \frac{1}{ikr^2}\frac{e^{ikr}}{r} - \left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r^2} + \left(1 - \frac{1}{ikr}\right)\frac{ike^{ikr}}{r} \\
&= \frac{e^{ikr}}{r}\left(\frac{2}{r} - \frac{2}{ikr^2} + \frac{1}{ikr^2} - \frac{1}{r} + \frac{1}{ikr^2} + ik - \frac{1}{r}\right) \\
&= \frac{ike^{ikr}}{r}
\end{aligned}$$

Combining these results, and using $Zc = \sqrt{\frac{\mu_0}{\epsilon_0}}\sqrt{\frac{1}{\mu_0 \epsilon_0}} = \frac{1}{\epsilon_0}$,

$$\begin{aligned}
\mathbf{E} &= \frac{iZkc}{4\pi}\left[\frac{e^{ikr}}{r}\left[\left(\frac{1}{r} - \frac{1}{ikr^2}\right)\left[\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] + (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\left(ik - \frac{2}{r} + \frac{2}{ikr^2}\right)\right] - \mathbf{p}\left(\frac{ike^{ikr}}{r}\right)\right] \\
&= \frac{iZkc}{4\pi}\frac{e^{ikr}}{r}\left[\left(\frac{1}{r} - \frac{1}{ikr^2}\right)\left[\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] - \left(\frac{2}{r} - \frac{2}{ikr^2}\right)(\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - ik\,(\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}})\right] \\
&= \frac{iZkc}{4\pi}\frac{e^{ikr}}{r}\left[\left(-ik + \frac{1}{r}\left(1 - \frac{1}{ikr}\right)\right)(\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}) - \frac{2}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] \\
&= \frac{iZkc}{4\pi}\frac{e^{ikr}}{r}\left[-ik\,(\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}) + \frac{1}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}) - \frac{2}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[k^2\,(\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}) + \frac{ik}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{p} - 3\,(\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}})\right]
\end{aligned}$$

Finally, noting that

$$\hat{\mathbf{r}} \times (\mathbf{p} \times \hat{\mathbf{r}}) = \mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}$$

we write the elecrtric field as

$$\mathbf{E}(\mathbf{x},t)=\frac{1}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\left[\left(k^2+\frac{ik}{r}-\frac{1}{r^2}\right)\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})-\left(\frac{2ik}{r}-\frac{2}{r^2}\right)(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\right]$$

where the first term is transverse and the second is not.

The electric and magnetic fields for an oscillating dipole field are therefore,

$$\begin{aligned}\mathbf{H}(\mathbf{x},t) &= \frac{k^2c}{4\pi}\left(1-\frac{1}{ikr}\right)\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}\\ \mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\left[\frac{1}{r^2}\left(k^2r^2+ikr-1\right)\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})-\frac{2}{r^2}(ikr-1)(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\right]\end{aligned}$$

In the radiation zone, $kr\gg 1$, these simplify to

$$\begin{aligned}\mathbf{H}(\mathbf{x},t) &= \frac{k^2c}{4\pi}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}\\ \mathbf{E}(\mathbf{x},t) &= \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})\\ &= Z_0\mathbf{H}\times\hat{\mathbf{r}}\end{aligned}$$

while in the near zone, $kr\ll 1$,

$$\begin{aligned}\mathbf{H}(\mathbf{x},t) &= \frac{ikc}{4\pi}\frac{1}{r^2}e^{-i\omega t}\hat{\mathbf{r}}\times\mathbf{p}\\ &= \frac{ikr}{Z}\frac{1}{4\pi\epsilon_0 r^3}e^{-i\omega t}\hat{\mathbf{r}}\times\mathbf{p}\\ \mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0 r^3}e^{-i\omega t}\left(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}\right)\end{aligned}$$

Notice that in the near zone, the electric field is just $e^{-i\omega t}$ times a static dipole field, while

$$\begin{aligned}H &= \frac{kr}{Z}\frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\\ E &= \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left|3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\hat{\mathbf{p}}\right|\end{aligned}$$

so that $H\ll E$, while the spatial oscillation is negligible. In the far zone, by contrast, there is a transverse wave travelling radially outward with the electric and magnetic fields comparable.

4 Electric quadrupole and magnetic dipole radiation

To go to higher multipoles, we generalize our expression for multipole integrals of the current.

4.1 The trick, in general

Does the trick for expressing the moments of the current work for higher multipoles? Not quite! We show here that two distinct distributions are required: the charge density and the magnetic moment density.

First, we know how to find the zeroth moment of the current in terms of the first moment of the charge density,

$$\int d^3x'\,J_j(\mathbf{x}') = -i\omega\int d^3x'\,x'_j\rho(\mathbf{x}')$$

Now, suppose we know the first $n-1$ moments of a current distribution in terms of moments of the charge density, and want to find the n^{th},

$$\mathbf{j}_{k_1\ldots k_n} \equiv \int d^3x\, x_{k_1}\ldots x_{k_n}\mathbf{J}(\mathbf{x})$$

Then consider the (vanishing) integral of a total divergence,

$$\begin{aligned}
0 &= \int d^3x \boldsymbol{\nabla}\cdot\left(x_{k_1}x_{k_2}\ldots x_{k_{n+1}}\mathbf{J}(\mathbf{x})\right) \\
&= \sum_i \int d^3x \nabla_i\left(x_{k_1}x_{k_2}\ldots x_{k_{n+1}}J_i\right) \\
&= \sum_i \int d^3x\left[\left(\delta_{ik_1}x_{k_2}\ldots x_{k_{n+1}} + x_{k_1}\delta_{ik_2}\ldots x_{k_{n+1}} + \ldots + x_{k_1}x_{k_2}\ldots\delta_{ik_{n+1}}\right)J_i + x_{k_1}x_{k_2}\ldots x_{k_{n+1}}\nabla_i J_i\right] \\
&= \int d^3x\left[\left(J_{k_1}x_{k_2}\ldots x_{k_{n+1}} + x_{k_1}J_{k_2}\ldots x_{k_{n+1}} + \ldots + x_{k_1}x_{k_2}\ldots J_{k_{n+1}}\right) + i\omega x_{k_1}x_{k_2}\ldots x_{k_{n+1}}\rho\right]
\end{aligned}$$

Each of the first $n+1$ integrals,

$$\int d^3x\, J_{k_1}x_{k_2}\ldots x_{k_{n+1}}$$

is a component of $\mathbf{j}_{k_1\ldots k_n}$, since there are n factors of the coordinates, $x_{k_2}\ldots x_{k_{n+1}}$. The problem is that we have expressed the $(n+1)^{st}$ moment of the charge density in terms of the *symmetrized* moments of the current. We cannot solve for $\mathbf{j}_{k_1\ldots k_n}$ unless we also know the antisymmetric parts. Since the products of the coordinates are necessarily symmetric (i.e., $x_{k_2}\ldots x_{k_{n+1}}$ is the same regardless of the order of the k_i indices), the only antisymmetric piece is

$$\int d^3x\, \left(J_i x_k - J_k x_i\right)x_{k_2}\ldots x_{k_n}$$

This does not vanish, but it can be expressed in terms of the magnetic moment density. Recall

$$\begin{aligned}
\boldsymbol{\mathcal{M}} &= \frac{1}{2}\mathbf{x}\times\mathbf{J} \\
\mathcal{M}_i &= \frac{1}{2}\sum_{j,k}\varepsilon_{ijk}x_jJ_k \\
\sum_i \mathcal{M}_i\varepsilon_{imn} &= \frac{1}{2}\sum_{i,j,k}\varepsilon_{imn}\varepsilon_{ijk}x_jJ_k \\
&= \frac{1}{2}\sum_{j,k}\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right)x_jJ_k \\
&= \frac{1}{2}\left(x_mJ_n - x_nJ_m\right) \\
x_mJ_n &= x_nJ_m + 2\sum_i \mathcal{M}_i\varepsilon_{imn}
\end{aligned}$$

Consider the second term in our expression for the moments,

$$\int d^3x\left(x_{k_1}J_{k_2}\ldots x_{k_{n+1}}\right)$$

We can now turn it into the same form as the first term,

$$\begin{aligned}
\int d^3x\left(x_{k_1}J_{k_2}\ldots x_{k_{n+1}}\right) &= \int d^3x\left(\left(x_{k_2}J_{k_1} + 2\sum_i \mathcal{M}_i\varepsilon_{ik_1k_2}\right)\ldots x_{k_{n+1}}\right) \\
&= \int d^3x J_{k_1}x_{k_2}\ldots x_{k_{n+1}} + 2\sum_i\int d^3x \mathcal{M}_i\varepsilon_{ik_1k_2}x_{k_3}\ldots x_{k_{n+1}}
\end{aligned}$$

The first term on the right is now the same as the first term in our expression. Repeating this for each of the $n+1$ terms involving J_k gives

$$0 = \int d^3x\,(n+1)\,J_{k_1}x_{k_2}\ldots x_{k_{n+1}} + 2\sum_i \int d^3x \mathcal{M}_i \varepsilon_{ik_1k_2}x_{k_3}\ldots x_{k_{n+1}} + \ldots + 2\sum_i \int d^3x \mathcal{M}_i \varepsilon_{ik_1k_{n+1}}x_{k_2}x_{k_3}\ldots x_{k_{n+1}} + i\omega$$

and therefore,

$$\begin{aligned}
\mathbf{j}_{k_2\ldots k_{n+1}} &= -\frac{2}{n+1}\sum_i \int d^3x \left[\mathcal{M}_i\varepsilon_{ik_1k_2}x_{k_3}\ldots x_{k_{n+1}} + \ldots + \mathcal{M}_i\varepsilon_{ik_1k_{n+1}}x_{k_2}x_{k_3}\ldots x_{k_{n+1}}\right] - \frac{i\omega}{n+1}\mathbf{p}_{k_1\ldots k_{n+1}} \\
\mathbf{j}_{k_2\ldots k_{n+1}} &= -\frac{2n!}{n+1}\sum_i \varepsilon_{ik_1(k_2}m_{ik_3\ldots k_{n+1})} - \frac{i\omega}{n+1}\mathbf{p}_{k_1\ldots k_{n+1}}
\end{aligned}$$

where

$$\mathbf{m}_{k_1\ldots k_n} = \int d^3x \boldsymbol{\mathcal{M}} x_{k_1}\ldots x_{k_n}$$

are the higher magnetic moments. The parentheses notation means symmetrization. For example,

$$T_{(ijk)} \equiv \frac{1}{3!}\left(T_{ijk} + T_{jki} + T_{kij} + T_{jik} + T_{kji} + T_{ikj}\right)$$

This shows that there are two types of moments that we will encounter: electric multipole moments built from the charge density, and magnetic multipole moments built from the magnetic moment density. We see this explicitly in the calculation of the electric quadrupole term of our general expansion.

4.2 Electric quadrupole and magnetic dipole radiation

If the electric dipole term is absent, the dominant term in our expansion of the vector potential is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int d^3x'\mathbf{J}(\mathbf{x}')(\hat{\mathbf{x}}\cdot\mathbf{x}')$$

Now we need the next higher moment of the current,

$$\int d^3x'\mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}')$$

We have seen that we can express the n^{th} moment of the charge distribution in terms of symmetrized $(n-1)^{st}$ moments of the current. In order to get the symmetrized moments of $\mathbf{J}$ we may use a vector identity. Starting with the *magnetic* moment density

$$\boldsymbol{\mathcal{M}} = \frac{1}{2}(\mathbf{x}\times\mathbf{J})$$

we take a second curl

$$\begin{aligned}
\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}} &= \frac{1}{2}\hat{\mathbf{r}}\times(\mathbf{x}'\times\mathbf{J}) \\
&= \frac{1}{2}\left(\mathbf{x}'(\hat{\mathbf{r}}\cdot\mathbf{J}) - \mathbf{J}(\hat{\mathbf{r}}\cdot\mathbf{x}')\right)
\end{aligned}$$

so that

$$\mathbf{J}(\hat{\mathbf{r}}\cdot\mathbf{x}') = \mathbf{x}'(\hat{\mathbf{r}}\cdot\mathbf{J}) - 2\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}}$$

In components, this is

$$\sum_k \hat{r}_k(J_ix'_k) = \sum_k \hat{r}_k x'_i J_k - 2\sum_{j,k}\varepsilon_{ijk}\hat{r}_j\mathcal{M}_k$$

Therefore, the i^{th} component of the current moment is

$$\begin{aligned}
\left[\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\right]_i &= \int d^3x'\left(\sum_k \hat{r}_k J_i x'_k\right) \\
&= \int d^3x'\left(\frac{1}{2}\sum_k \hat{r}_k J_i x'_k + \frac{1}{2}\sum_k \hat{r}_k J_i x'_k\right) \\
&= \int d^3x'\left(\frac{1}{2}\sum_k \hat{r}_k J_i x'_k + \frac{1}{2}\left(\sum_k \hat{r}_k x'_i J_k - 2\sum_{j,k}\varepsilon_{ijk}\hat{r}_j \mathcal{M}_k\right)\right) \\
&= \int d^3x'\left(\sum_k \hat{r}_k \frac{1}{2}\left(J_i x'_k + x'_i J_k\right) - \sum_{j,k}\varepsilon_{ijk}\hat{r}_j \mathcal{M}_k\right) \\
&= -\frac{i\omega}{2}\sum_k \hat{r}_k \int d^3x'\left(x'_i x'_k \rho\right) + \int d^3x' \sum_{j,k}\varepsilon_{ijk}\hat{r}_k \mathcal{M}_j \\
&= \sum_k \hat{r}_k \int d^3x'\left(\sum_j \varepsilon_{ijk}\mathcal{M}_j - \frac{i\omega}{2}\int d^3x'\left(x'_i x'_k \rho\right)\right)
\end{aligned}$$

Notice that

$$\begin{aligned}
f_i &= \sum_k \hat{r}_k \int d^3x'\left(r'^2 \rho \delta_{ik}\right) \\
&= \hat{r}_i \int d^3x'\, r'^2 \rho
\end{aligned}$$

has vanishing curl,

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{f} &= \left(\boldsymbol{\nabla}\times\hat{\mathbf{r}}\right)\int d^3x'\, r'^2\rho \\
&= 0
\end{aligned}$$

This means that this term may be added to the vector potential without affecting the fields (Jackson never mentions this). This allows us to define the quadrupole moment of the charge distribution as the traceless matrix

$$\begin{aligned}
Q_{ik} &= \int d^3x'\left(3x'_i x'_k - \delta_{ik} r'^2\right)\rho\left(\mathbf{x}'\right) \\
\sum_k Q_{kk} &= 0
\end{aligned}$$

without changing the fields. Then, with the magnetic dipole moment equal to

$$\mathbf{m} = \int d^3x \boldsymbol{\mathcal{M}}$$

and setting

$$\left[\mathbf{Q}\left(\hat{\mathbf{r}}\right)\right]_i \equiv \sum_k \hat{r}_k Q_{ik}$$

the vector potential becomes

$$\mathbf{A}\left(\mathbf{x}\right) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)$$

$$
\begin{aligned}
&= \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left(\hat{\mathbf{r}}\times\mathbf{m}+\frac{i\omega}{6}\mathbf{Q}\left(\hat{\mathbf{r}}\right)\right)\\
&= \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left(\frac{i\omega}{6}\mathbf{Q}\left(\hat{\mathbf{r}}\right)+\hat{\mathbf{r}}\times\mathbf{m}\right)
\end{aligned}
$$

For comparison, here is the divergence trick applied to the present case. Consider

$$
\begin{aligned}
0 &= \int d^3x\boldsymbol{\nabla}\cdot\left(x_jx_k\mathbf{J}\left(\mathbf{x}\right)\right)\\
&= \sum_i\int d^3x\nabla_i\left(x_jx_kJ_i\left(\mathbf{x}\right)\right)\\
&= \sum_i\int d^3x\left[\left(\delta_{ij}x_k+\delta_{ik}x_j\right)J_i+x_jx_k\nabla_iJ_i\right]\\
&= \int d^3x\left[J_jx_k+J_kx_j+i\omega x_jx_k\rho\right]\\
&= \int d^3x\left[2J_jx_k+\left(J_kx_j-J_jx_k\right)+i\omega x_jx_k\rho\right]\\
&= 2\int d^3x\;J_jx_k+\int d^3x\left(J_kx_j-J_jx_k\right)+i\omega\int d^3x\;\rho\;x_jx_k
\end{aligned}
$$

so that finally,

$$
\int d^3x\;J_jx_k=-\frac{1}{2}\int d^3x\left(J_kx_j-J_jx_k\right)-\frac{i\omega}{2}\int d^3x\;\rho\;x_jx_k
$$

Now using the definition of the magnetic moment density,

$$
\boldsymbol{\mathcal{M}}=\frac{1}{2}\left(\mathbf{x}\times\mathbf{J}\right)
$$

and noting that

$$
\begin{aligned}
\left[\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}}\right]_i &= \frac{1}{2}\sum_{jk}\varepsilon_{ijk}\hat{r}_j\left(\mathbf{x}'\times\mathbf{J}\right)_k\\
&= \frac{1}{2}\sum_{jkmn}\varepsilon_{ijk}\varepsilon_{kmn}\hat{r}_jx'_mJ_n\\
&= \frac{1}{2}\sum_{jmn}\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\hat{r}_jx'_mJ_n\\
&= \frac{1}{2}\sum_j\hat{r}_j\left(x'_iJ_j-x'_jJ_i\right)
\end{aligned}
$$

Returning to the expression for the vector potential

$$
\begin{aligned}
\mathbf{A}\left(\mathbf{x}\right) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int d^3x'\mathbf{J}\left(\mathbf{x}'\right)\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\\
A_i &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\sum_k\hat{r}_k\int d^3x'J_ix'_k\\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\sum_k\hat{r}_k\left(-\frac{1}{2}\int d^3x'\left(J_kx'_i-J_ix'_k\right)-\frac{i\omega}{2}\int d^3x'\;\rho\;x'_ix'_k\right)\\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\left(-\int d^3x'\left(\left[\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}}\right]_i\right)-\frac{i\omega}{2}\sum_k\hat{r}_k\int d^3x'\;\rho\;x'_ix'_k\right)
\end{aligned}
$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left(\frac{i\omega}{6}\mathbf{Q}(\hat{\mathbf{r}})+\hat{\mathbf{r}}\times\mathbf{m}\right)$$

so the result is just as above.

4.2.1 Magnetic dipole

The fields are now found in the usual way. The magnetic dipole potential is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{m}$$

Notice that the magnetic field for the electric dipole had exactly this form,

$$\frac{i\mu_0}{kc}\mathbf{H}_e(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{ike^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}$$

once we replace $\mathbf{p}\longrightarrow\mathbf{m}$. Since the electric field in that case was given by $\mathbf{E}_e(\mathbf{x},t) = \frac{iZ_0}{k}\boldsymbol{\nabla}\times\mathbf{H}_e$, we can write the curl of $\mathbf{A}$ immediately. We have

$$\begin{aligned}
\mathbf{H}_m(\mathbf{x},t) &= \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A}_m(\mathbf{x}) \\
&= \frac{1}{\mu_0}\boldsymbol{\nabla}\times\left(\left.\frac{i\mu_0}{kc}\mathbf{H}_e(\mathbf{x},t)\right|_{\mathbf{p}\longrightarrow\mathbf{m}}\right) \\
&= \frac{i}{kc}\frac{k}{iZ_0}\left.\mathbf{E}_e(\mathbf{x},t)\right|_{\mathbf{p}\longrightarrow\mathbf{m}} \\
&= \frac{1}{cZ}\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr-i\omega t}}{r}\left[k^2\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})+\frac{ik}{r}\left(1-\frac{1}{ikr}\right)(\mathbf{m}-3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right] \\
&= \frac{1}{4\pi}\frac{e^{ikr-i\omega t}}{r}\left[k^2\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})+\frac{ik}{r}\left(1-\frac{1}{ikr}\right)(\mathbf{m}-3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right]
\end{aligned}$$

We can resort to this sort of magic again, because we know that this form of $\mathbf{E}_e(\mathbf{x},t)$ was achieved by taking the curl of $\mathbf{H}_e$, and the Maxwell equations for harmonic sources tell us that

$$\begin{aligned}
i\omega\mathbf{B}_m &= \boldsymbol{\nabla}\times\mathbf{E}_m \\
i\omega\boldsymbol{\nabla}\times\left(\left.\frac{i\mu_0}{kc}\mathbf{H}_e(\mathbf{x},t)\right|_{\mathbf{p}\longrightarrow\mathbf{m}}\right) &= \boldsymbol{\nabla}\times\mathbf{E}_m \\
-\mu_0\left.\mathbf{H}_e(\mathbf{x},t)\right|_{\mathbf{p}\longrightarrow\mathbf{m}} &= \mathbf{E}_m
\end{aligned}$$

Notice that dropping the curl on both sides is not quite allowed, since the right and left sides could differ by a gradient, but the answer here is correct. The electric field for magnetic dipole radiation is correctly given by

$$\begin{aligned}
\mathbf{E}_m &= -\mu_0\left.\mathbf{H}_e(\mathbf{x},t)\right|_{\mathbf{p}\longrightarrow\mathbf{m}} \\
&= -\mu_0\left.\left(\frac{k^2c}{4\pi}\left(1-\frac{1}{ikr}\right)\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}\right)\right|_{\mathbf{p}\longrightarrow\mathbf{m}} \\
&= -\frac{Z_0}{4\pi}k^2\left(1-\frac{1}{ikr}\right)\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{m}
\end{aligned}$$

This gives the magnetic dipole fields as

$$\begin{aligned}
\mathbf{H} &= \frac{1}{4\pi}\frac{e^{ikr-i\omega t}}{r}\left[k^2\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})+\frac{ik}{r}\left(1-\frac{1}{ikr}\right)(\mathbf{m}-3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right] \\
\mathbf{E} &= -\frac{Z_0}{4\pi}k^2\left(1-\frac{1}{ikr}\right)\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{m}
\end{aligned}$$

in complete analogy to the electric dipole field, but with magnetic and electric parts interchanged.

In the radiation zone, these become

$$\begin{aligned}\mathbf{H} &= \frac{1}{4\pi}k^2\frac{e^{ikr-i\omega t}}{r}\left[\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})\right] \\ \mathbf{E} &= -\frac{Z_0}{4\pi}k^2\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{m}\end{aligned}$$

so they are once again transverse and comparable in magnitude. In the near zone,

$$\begin{aligned}\mathbf{H} &= \frac{e^{-i\omega t}}{4\pi r^3}\left(3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}\right) \\ \mathbf{E} &= -\frac{iZ_0kr}{4\pi}\frac{e^{-i\omega t}}{r^3}\hat{\mathbf{r}}\times\mathbf{m}\end{aligned}$$

so the electric field is much weaker than the magnetic, which takes the form of a dipole.

4.2.2 Electric quadrupole

For the quadrupole fields, we begin with the quadrupole piece of the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{i\omega\mu_0}{24\pi}\frac{ike^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\mathbf{Q}(\hat{\mathbf{r}})$$

where writing

$$\left[\mathbf{Q}(\hat{\mathbf{r}})\right]_i \equiv \sum_k \hat{r}_k Q_{ik}$$

allows us to write the potential in vector form. The magnetic field is then

$$\mathbf{H}(\mathbf{x}) = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x})$$

Keeping only terms of order $\frac{1}{r}$, this gives

$$\begin{aligned}\mathbf{H}(\mathbf{x}) &= \frac{1}{\mu_0}\boldsymbol{\nabla}\times\left[\frac{i\omega\mu_0}{24\pi}\frac{ike^{ikr}}{r}\mathbf{Q}(\hat{\mathbf{r}})\right] \\ &= \frac{1}{\mu_0}\left[\frac{-i\omega k^2\mu_0}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right] \\ &= \frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\end{aligned}$$

since the only derivative term that does not increase the power of $\frac{1}{r}$ is $\left[\frac{i\omega\mu_0}{24\pi}\frac{ik}{r}\left(\boldsymbol{\nabla}e^{ikr}\right)\times\mathbf{Q}(\hat{\mathbf{r}})\right]$. The electric field is then

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= Z_0\mathbf{H}\times\hat{\mathbf{r}} \\ &= Z_0\left(\frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\times\hat{\mathbf{r}} \\ &= -\frac{ik^3}{24\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\end{aligned}$$

The fields in the radiation zone are therefore

$$\begin{aligned}\mathbf{H}(\mathbf{x},t) &= -\frac{ik^3}{24\pi\epsilon_0 Z}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}}) \\ \mathbf{E}(\mathbf{x},t) &= -\frac{ik^3}{24\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\end{aligned}$$

Near field?
With

$$\mathbf{A}\left(\mathbf{x}\right)=\frac{i\omega\mu_0}{24\pi}\frac{ike^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\mathbf{Q}\left(\hat{\mathbf{r}}\right)$$

the magnetic field is then

$$\begin{aligned}
\mathbf{H}\left(\mathbf{x}\right) &= \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A}\left(\mathbf{x}\right)\\
&= \frac{1}{\mu_0}\boldsymbol{\nabla}\times\left(\frac{i\omega\mu_0}{24\pi}\frac{ike^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\mathbf{Q}\left(\hat{\mathbf{r}}\right)\right)\\
H_i\left(\mathbf{x}\right) &= -\frac{k^2c}{24\pi}\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)Q_k\left(\hat{\mathbf{r}}\right)\right)\\
&= -\frac{k^2c}{24\pi}\sum_{j,k,m}Q_{km}\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\hat{r}_m\right)\\
&= -\frac{k^2c}{24\pi}\sum_{j,k,m}Q_{km}\varepsilon_{ijk}\left(-\frac{1}{r^2}\hat{r}_je^{ikr}\left(1-\frac{1}{ikr}\right)\hat{r}_m+\frac{ik\hat{r}_je^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\hat{r}_m+\frac{e^{ikr}}{r}\frac{1}{ikr^2}\hat{r}_j\hat{r}_m+\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left(\frac{1}{r}\delta\right.\right.\\
&= -\frac{k^2c}{24\pi}\frac{e^{ikr}}{r^2}\sum_{j,k,m}Q_{km}\varepsilon_{ijk}\left(\left(ikr-2+\frac{2}{ikr}\right)\hat{r}_j\hat{r}_m+\left(1-\frac{1}{ikr}\right)\left(\delta_{jm}-\hat{r}_j\hat{r}_m\right)\right)\\
&= -\frac{k^2c}{24\pi}\frac{e^{ikr}}{r^2}\left(\left(ikr-2+\frac{2}{ikr}\right)\left[\hat{\mathbf{r}}\times\mathbf{Q}\right]_i+\left(1-\frac{1}{ikr}\right)\left(\sum_{j,k}Q_{kj}\varepsilon_{ijk}-\left[\hat{\mathbf{r}}\times\mathbf{Q}\right]_i\right)\right)\\
&= -\frac{k^2c}{24\pi}\frac{e^{ikr}}{r^2}\left(ikr-3+\frac{3}{ikr}\right)\left[\hat{\mathbf{r}}\times\mathbf{Q}\right]_i
\end{aligned}$$

that is,

$$\mathbf{H}\left(\mathbf{x}\right) = -\frac{k^2}{24\pi\epsilon_0 Z}\frac{e^{ikr}}{r^2}\left(ikr-3+\frac{3}{ikr}\right)\hat{\mathbf{r}}\times\mathbf{Q}$$

with the electric field given by

$$\begin{aligned}
\mathbf{E} &= \frac{iZ}{k}\boldsymbol{\nabla}\times\mathbf{H}\\
&= -\frac{ik}{24\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r^2}\left(ikr-3+\frac{3}{ikr}\right)\right)\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\mathbf{Q}\right)\\
&\quad -\frac{ik}{24\pi\epsilon_0}\left(\frac{e^{ikr}}{r^2}\left(ikr-3+\frac{3}{ikr}\right)\boldsymbol{\nabla}\times\left(\hat{\mathbf{r}}\times\mathbf{Q}\right)\right)\\
&= -\frac{ik}{24\pi\epsilon_0}\frac{e^{ikr}}{r^3}\left(-k^2r^2-4ikr+9-\frac{9}{ikr}\right)\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\mathbf{Q}\right)\\
&\quad +\frac{ik}{24\pi\epsilon_0}\left(\frac{e^{ikr}}{r^3}\left(ikr-3+\frac{3}{ikr}\right)\left(\left(\mathbf{Q}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}+2\mathbf{Q}\right)\right)\\
&= -\frac{ik}{24\pi\epsilon_0}\frac{e^{ikr}}{r^3}\left(-k^2r^2-4ikr+9-\frac{9}{ikr}\right)\left(\left(\mathbf{Q}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{Q}\right)\\
&\quad +\frac{ik}{24\pi\epsilon_0}\left(\frac{e^{ikr}}{r^3}\left(ikr-3+\frac{3}{ikr}\right)\left(\left(\mathbf{Q}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}+2\mathbf{Q}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{ik}{24\pi\epsilon_0}\frac{e^{ikr}}{r^3}\left(-k^2r^2 - 5ikr + 12 - \frac{12}{ikr}\right)(\mathbf{Q}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} \\
&\quad + \frac{ik}{24\pi\epsilon_0}\frac{e^{ikr}}{r^3}\left(-k^2r^2 - 2ikr + 3 - \frac{3}{ikr}\right)\mathbf{Q}
\end{aligned}
$$

where we use

$$
\begin{aligned}
[\boldsymbol{\nabla}\times(\hat{\mathbf{r}}\times\mathbf{Q})]_i &= \sum_{jklm}\varepsilon_{ijk}\nabla_j\left(\varepsilon_{klm}\frac{r_l}{r}Q_{mn}\frac{r_n}{r}\right) \\
&= \sum_{jklm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,Q_{mn}\left(\frac{r_l\delta_{jn}}{r^2} + \frac{\delta_{lj}r_n}{r^2} - \frac{r_l r_n r_j}{r^4}\right) \\
&= Q_{nn}\frac{r_i}{r^2} + \frac{1}{r}Q_i - Q_j\hat{r}_j\frac{\hat{r}_i}{r} - 3Q_i\frac{1}{r} \\
&= -\frac{1}{r}\left((\mathbf{Q}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + 2\mathbf{Q}\right)
\end{aligned}
$$

In the radiation zone this is dominated by the first term,

$$
\begin{aligned}
\mathbf{H}(\mathbf{x}) &= -\frac{ik^3}{24\pi\epsilon_0 Z}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q} \\
\mathbf{E}(\mathbf{x}) &= -\frac{ik^3}{24\pi\epsilon_0}\frac{e^{ikr}}{r}\left(\mathbf{Q} - (\mathbf{Q}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\right) \\
&= \frac{ik^3}{24\pi\epsilon_0}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)
\end{aligned}
$$

and in the near zone by

$$
\begin{aligned}
\mathbf{H}(\mathbf{x}) &= \frac{ikr}{8\pi\epsilon_0 Z}\frac{e^{ikr}}{r^4}\hat{\mathbf{r}}\times\mathbf{Q} \\
\mathbf{E}(\mathbf{x}) &= -\frac{1}{8\pi\epsilon_0}\frac{e^{ikr}}{r^4}\left(\mathbf{Q} - 4(\mathbf{Q}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\right)
\end{aligned}
$$

5 Radiated power

The energy per unit area carried by an electromagnetic wave is given by the Poynting vector,

$$
\mathbf{S} = \mathbf{E}\times\mathbf{H}
$$

For a plane wave, we have

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x} - \omega t) \\
\mathbf{H} &= \frac{1}{\mu}\sqrt{\mu\varepsilon}\frac{1}{k}\mathbf{k}\times\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbf{S} &= \mathbf{E}\times\mathbf{H} \\
&= \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)\times\left(\frac{1}{\mu}\sqrt{\mu\varepsilon}\frac{1}{k}\mathbf{k}\times\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)\right) \\
&= \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}\mathbf{E}_0\times(\mathbf{k}\times\mathbf{E}_0)\cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t) \\
&= \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}E_0^2\mathbf{k}\cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t)
\end{aligned}
$$

with real part

$$\mathbf{S} = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}E_0^2\mathbf{k}\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t)$$

with time average

$$\mathbf{S} = \left(\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}E_0^2\right)\hat{\mathbf{k}}$$

For a complex representation of the wave,

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\\ \mathbf{H} &= \frac{1}{\mu}\sqrt{\mu\varepsilon}\frac{1}{k}\mathbf{k}\times\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\end{aligned}$$

we may write the same quantity as

$$\begin{aligned}\frac{1}{2}Re\left(\mathbf{E}\times\mathbf{H}^*\right) &= \frac{1}{2}Re\left(\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\times\frac{1}{\mu}\sqrt{\mu\varepsilon}\hat{\mathbf{k}}\times\mathbf{E}_0^* e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\\ &= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left|E_0\right|^2\hat{\mathbf{k}}\end{aligned}$$

so the time-averaged energy flow per unit area per unit time is

$$\mathbf{S} = \frac{1}{2}Re\left(\mathbf{E}\times\mathbf{H}^*\right)$$

Now consider the average power carried off by electric dipole, electric quadrupole, and magnetic dipole radiation.

5.1 Electric dipole

The radiation zone fields were found above to be

$$\begin{aligned}\mathbf{H}(\mathbf{x},t) &= \frac{k^2 c}{4\pi}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}\\ \mathbf{E}(\mathbf{x},t) &= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})\end{aligned}$$

so that

$$\begin{aligned}\frac{dP}{dA} &= \hat{\mathbf{r}}\cdot\mathbf{S}\\ &= \frac{1}{2}Re\left(\hat{\mathbf{r}}\cdot(\mathbf{E}\times\mathbf{H}^*)\right)\\ &= \frac{1}{2}Re\left(\frac{k^2}{4\pi\varepsilon_0}\frac{1}{r}\hat{\mathbf{r}}\cdot\left([\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})]\times\frac{k^2 c}{4\pi}\frac{1}{r}[\hat{\mathbf{r}}\times\mathbf{p}]\right)\right)\\ &= \frac{ck^4}{32\pi^2\varepsilon_0}\frac{1}{r^2}\left|\mathbf{p}\right|^2\sin^2\theta\\ &= \frac{c^2k^4\sqrt{\mu_0\varepsilon_0}}{32\pi^2\varepsilon_0}\frac{1}{r^2}\left|\mathbf{p}\right|^2\sin^2\theta\\ &= \frac{c^2 Z_0}{32\pi^2}\frac{1}{r^2}k^4\left|\mathbf{p}\right|^2\sin^2\theta\end{aligned}$$

This is the power per unit area. Since the area element at large distances is $dA = r^2 d\Omega$, where Ω is the solid angle, we may write the differential power radiated per unit solid angle using

$$\frac{dP}{dA} = \frac{1}{r^2}\frac{dP}{d\Omega}$$

so that

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left|\mathbf{p}\right|^2 \sin^2\theta$$

5.2 Magnetic dipole

The radiation zone fields for magnetic dipole radiation are

$$\begin{aligned}
\mathbf{H}(\mathbf{x},t) &= \frac{k^2}{4\pi}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}}) \\
\mathbf{E}(\mathbf{x},t) &= -\frac{Z_0 k^2}{4\pi}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{m}
\end{aligned}$$

so the result is the same as for the electric dipole with the substitution $\mathbf{p} \longrightarrow \mathbf{m}/c$,

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2} k^4 \left|\mathbf{m}\right|^2 \sin^2\theta$$

5.3 Electric quadrupole moment

For electric quadrupole radiation the fields are given by

$$\begin{aligned}
\mathbf{H}(\mathbf{x},t) &= \frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}}) \\
\mathbf{E}(\mathbf{x},t) &= -\frac{ik^3}{24\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}
\end{aligned}$$

giving an average power per unit solid angle of

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{r^2}{2}\left|Re\left(\mathbf{E}\times\mathbf{H}^*\right)\right| \\
&= \frac{r^2}{2}\left|Re\left[\left(-\frac{ik^3}{24\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right)\times\left(\frac{ick^3}{24\pi}\frac{e^{-ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\right]\right| \\
&= \frac{1}{2}\frac{k^3}{24\pi\varepsilon_0}\frac{ck^3}{24\pi}\left|\left(\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right)\times\left(\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\right| \\
&= \frac{ck^6}{1152\pi^2\varepsilon_0}\left|\left(\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right)\times\left(\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\right| \\
&= \frac{Z_0 c^2}{1152\pi^2}k^6\left|\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right|^2
\end{aligned}$$

Notice that the power radiated by the quadrupole moment depends on k^6, whereas the power radiated by the dipole moments both go as k^4. This pattern continues for higher moments.