Magnetostatics

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1 Magnetostatics

Magnetostatics is based on the time-independent forms of Ampère's law and the absence of magnetic monopoles,

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

From the vanishing divergence, we know that $\mathbf{B}$ may be written as the curl of a vector,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

The vector $\mathbf{A}$ is called the vector potential, but it is not unique. If we add the gradient of any function to $\mathbf{A}$,

$$\mathbf{A}' = \mathbf{A} + \nabla f$$

then we get the same magnetic field, since the curl of a gradient vanishes,

$$\begin{aligned} \mathbf{B}' &= \nabla \times \mathbf{A}' \\ &= \nabla \times (\mathbf{A} + \nabla f) \\ &= \nabla \times \mathbf{A} + \nabla \times \nabla f \\ &= \mathbf{B} \end{aligned}$$

The function f is called the gauge. This means that we can constrain $\nabla \times \mathbf{A}$ in a convenient way. For our present purpose, suppose we have

$$\mathbf{B} = \nabla \times \mathbf{A}$$

for some given magnetic field. Then the divergence of the potential is some particular function, $\nabla \cdot \mathbf{A} = g$, and we can find the divergence in any other gauge by writing

$$\begin{aligned} \nabla \cdot \mathbf{A}' &= \nabla \cdot (\mathbf{A} + \nabla f) \\ &= g + \nabla^2 f \end{aligned}$$

Since the choice of f is arbitrary, we choose it to solve

$$\nabla^2 f = -g$$

and since this is the Poisson equation for f we know it always has a solution. Using this f means that we have a gauge for which both

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}' \\ \nabla \cdot \mathbf{A}' &= 0 \end{aligned}$$

This is called the Coulomb gauge.

We continue to work in the Coulomb gauge, but to keep things simple drop the prime on the vector potential,

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A}\\ \nabla\cdot\mathbf{A} &= 0\end{aligned}$$

We now want to find what vector potential that satisfying these two conditions, also satisfies the magnetostatic equations,

$$\begin{aligned}\nabla\times\mathbf{B} &= \mu_0\mathbf{J}\\ \nabla\cdot\mathbf{B} &= 0\end{aligned}$$

Of course, the second is satisfied automatically since that is how we constructed $\mathbf{A}$ in the first place. For Ampère's law, we find

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times\mathbf{B}\\ &= \nabla\times(\nabla\times\mathbf{A})\\ &= -\nabla^2\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})\\ &= -\nabla^2\mathbf{A}\end{aligned}$$

and this is once again the Poisson equation (for each Cartesian component),

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

It is possible to introduce a Green function for the vector potential, so we may solve for arbitrary boundary conditions, but we will consider only the case of the field vanishing at infinity. Then the solution is immediate,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

The curl of this potential gives the magnetic field.

Is the gauge condition satisfied by this solution? Check:

$$\begin{aligned}\nabla\cdot\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\nabla\cdot\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\ &= \frac{\mu_0}{4\pi}\int\left(\frac{\nabla\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}+\mathbf{J}(\mathbf{x}')\cdot\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\\ &= \frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{x}')\cdot\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\ &= -\frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\ &= -\frac{\mu_0}{4\pi}\int\left(\nabla'\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\end{aligned}$$

The divergence of $\mathbf{J}(\mathbf{x}')$ vanishes by the continuity equation since there is no time dependence,

$$\begin{aligned}0 &= \frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}(\mathbf{x})\\ &= \nabla\cdot\mathbf{J}(\mathbf{x})\end{aligned}$$

so the second term vanishes. For the first, we use the divergence theorem,

$$\begin{aligned}\nabla \cdot \mathbf{A}(\mathbf{x}) &= -\frac{\mu_0}{4\pi}\int \nabla' \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= -\frac{\mu_0}{4\pi}\oint \frac{\mathbf{J}(\mathbf{x}')\cdot\mathbf{n}}{|\mathbf{x}-\mathbf{x}'|}d^2x'\end{aligned}$$

Since we have vanishing boundary condition at infinity, there can be no currents there and the integral vanishes.