Magnetic monopoles

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1 Modification of the Maxwell equations to include magnetic charges

We explore the physical effects of a magnetic monopole. Suppose we modify the Maxwell equations to

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{\rho_E}{\varepsilon_0}\\ \nabla\cdot\mathbf{B} &= \rho_M\\ \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{D}}{\partial t} &= \mu_0\mathbf{J}_E\\ \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= -\mathbf{J}_M\end{aligned}$$

by adding magnetic sources. The reason the magnetic current enters with a minus sign is so that both electric and magnetic charges are conserved. This follows from the continuity equations for each. Recall that for electric charge, we have

$$\begin{aligned}\frac{\partial\rho_E}{\partial t} &= \frac{\partial}{\partial t}\left(\varepsilon_0\nabla\cdot\mathbf{E}\right)\\ &= \nabla\cdot\left(\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}\right)\\ &= -c^2\mu_0\nabla\cdot\left(\varepsilon_0\mathbf{J}_E\right)\\ &= -\nabla\cdot\mathbf{J}_E\end{aligned}$$

giving the continuity equation $\frac{\partial\rho_E}{\partial t}+\nabla\cdot\mathbf{J}_E=0$. If we integrate this equation over any volume $\mathcal{V}$ and use the divergence theorem,

$$\begin{aligned}\int_{\mathcal{V}}\frac{\partial\rho_E}{\partial t}d^3x &= -\int_{\mathcal{V}}\nabla\cdot\mathbf{J}_E d^3x\\ \frac{d}{dt}\int_{\mathcal{V}}\rho_E d^3x &= -\oint_{\mathcal{S}}\hat{\mathbf{n}}\cdot\mathbf{J}_E d^2x\\ \frac{dQ_{\mathcal{V}}}{dt} &= -\oint_{\mathcal{S}}\hat{\mathbf{n}}\cdot\mathbf{J}_E d^2x\end{aligned}$$

so the total charge $Q_{\mathcal{V}}$ in $\mathcal{V}$ changes only by what flows out over the boundary.

We demand a similar relation for the total magnetic charge

$$\begin{aligned}\frac{\partial\rho_M}{\partial t} &= \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{B}\right)\\ &= \nabla\cdot\left(\frac{\partial}{\partial t}\mathbf{B}\right)\\ &= -\nabla\cdot\mathbf{J}_M\end{aligned}$$

and we see that the negative sign is required.

If all particles had the same ratio of magnetic charge to electric charge, a redefinition of the fields could eliminate one of the sets of sources and restore the Maxwell equations to the original form. The redefinition may be thought of as a rotation in the electric-magnetic plane, so that the sources all lie in the new electric direction,

$$
\begin{aligned}
\mathbf{E}' &= \mathbf{E}\cos\eta - \mathbf{B}\sin\eta \\
\mathbf{B}' &= \mathbf{E}\sin\eta + \mathbf{B}\cos\eta \\
\rho'_E &= \rho_E\cos\eta - \rho_B\sin\eta \\
\rho'_B &= \rho_E\sin\eta + \rho_B\cos\eta \\
\mathbf{J}'_E &= \mathbf{J}_E\cos\eta - \mathbf{J}_B\sin\eta \\
\mathbf{J}'_B &= \mathbf{J}_E\sin\eta + \mathbf{J}_B\cos\eta
\end{aligned}
$$

However this does not appear to be the case, so that we cannot rotate away the magnetic sources. We explore some of the consequences.

2 Linear momentum of nearby electric and magnetic charges

Suppose we have a magnetic charge and an electric charge at rest, a distance d apart along the x-axis. Then using the two versions of Coulomb's law, we have electric and magnetic fields,

$$
\begin{aligned}
\mathbf{E} &= \frac{e}{4\pi\varepsilon_0 r^3}\mathbf{r} \\
\mathbf{B} &= \frac{g}{4\pi r'^3}\mathbf{r}'
\end{aligned}
$$

where $r' = \sqrt{(x+d)^2 + y^2 + z^2}$ and $\mathbf{r}' = \mathbf{r} - d\mathbf{i}$. Taking d small enough that we may consider only terms of order $\frac{d}{r}$, so that

$$
\begin{aligned}
r' &= \sqrt{(x-d)^2 + y^2 + z^2} \\
&= \sqrt{r^2 - 2xd + d^2} \\
&= r\sqrt{1 - \frac{2xd}{r^2} + \frac{d^2}{r^2}} \\
&\approx r\left(1 - \frac{xd}{r^2}\right) \\
&= r - \frac{xd}{r}
\end{aligned}
$$

the momentum density vector is

$$
\begin{aligned}
\mathbf{g} &= \frac{1}{c^2}\mathbf{E}\times\mathbf{H} \\
&= \frac{1}{c^2}\frac{e}{4\pi\varepsilon_0 r^3}\mathbf{r}\times\frac{g}{4\pi\mu_0 r'^3}(\mathbf{r} - d\mathbf{i}) \\
&= \frac{1}{c^2}\frac{eg}{16\pi^2\mu_0\varepsilon_0 r^3\left(r - \frac{xd}{r}\right)^3}\mathbf{r}\times(\mathbf{r} - d\mathbf{i}) \\
&= -\frac{egd}{16\pi^2 r^6}\left(1 + \frac{3xd}{r^2}\right)\mathbf{r}\times\mathbf{i} \\
&\approx -\frac{egd}{16\pi^2 r^6}\mathbf{r}\times\mathbf{i}
\end{aligned}
$$

Because $\mathbf{g}$ is already of order d, we have dropped the additional $\frac{3xd}{r^2}$ term.

Now integrate the momentum density over all space to find the total momentum,

$$\begin{aligned}\mathbf{P} &= \int \mathbf{g} d^3x \\ &= -\frac{egd}{16\pi^2}\int \frac{\mathbf{r}\times\mathbf{i}}{r^6} d^3x \\ &= \frac{egd}{16\pi^2}\int \frac{y\mathbf{k}-z\mathbf{j}}{r^6} d^3x\end{aligned}$$

We show that it vanishes. Each of the three integrals $(dx, dy$ and $dz)$, contains the form

$$\int_{-\infty}^{\infty} dx$$

If we change coordinates, replacing (x, y, z) with $(-x, -y, -z)$ each such term is unchanged,

$$\int_{-\infty}^{\infty} dx \rightarrow \int_{\infty}^{-\infty} (-dx) = \int_{-\infty}^{\infty} dx$$

Furthermore, the *length* of a vector, $r = \sqrt{x^2+y^2+z^2}$, is even under this transformation. On the other hand, the vector $y\mathbf{k} - z\mathbf{j}$ inverts to $-y\mathbf{k} + z\mathbf{j}$. We therefore have

$$\begin{aligned}\int \frac{y\mathbf{k}-z\mathbf{j}}{r^6} d^3x &= \int \frac{y'\mathbf{k}-z'\mathbf{j}}{r'^6} d^3x' \\ &= \int \frac{-y\mathbf{k}+z\mathbf{j}}{r^6} d^3x \\ &= -\int \frac{y\mathbf{k}-z\mathbf{j}}{r^6} d^3x\end{aligned}$$

so the integral equals minus itself and vanishes.

3 Angular momentum of nearby electric and magnetic charges

However, there does exist a net angular momentum, given by the integral of $\mathbf{r}\times\mathbf{g}$,

$$\begin{aligned}\mathbf{L} &= \int \mathbf{r}\times\mathbf{g} d^3x \\ &= \frac{1}{c^2}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})\, d^3x \\ &= \frac{1}{c^2}\int \mathbf{r}\times\left(\frac{e}{4\pi\varepsilon_0 r^3}\mathbf{r}\times\mathbf{H}\right) d^3x \\ &= \frac{e}{4\pi\varepsilon_0 c^2}\int \frac{1}{r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{H})\, d^3x \\ &= \frac{e}{4\pi\varepsilon_0 c^2}\int \frac{1}{r}\left(\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{H}\right) - \mathbf{H}\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)\right) d^3x \\ &= \frac{e}{4\pi}\int \frac{1}{r}\left(\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{B}\right) - \mathbf{B}\right) d^3x\end{aligned}$$

Using the identity

$$(\mathbf{a}\cdot\boldsymbol{\nabla})\,\hat{\mathbf{r}}f(r) = \frac{f(r)}{r}\left[\mathbf{a} - \hat{\mathbf{r}}\left(\mathbf{a}\cdot\hat{\mathbf{r}}\right)\right] + \hat{\mathbf{r}}\left(\mathbf{a}\cdot\hat{\mathbf{r}}\right)\frac{\partial f}{\partial r}$$

with $f = 1$ and $\mathbf{a} = \mathbf{B}$,

$$(\mathbf{B} \cdot \boldsymbol{\nabla})\, \hat{\mathbf{r}} = \frac{1}{r} \left(\mathbf{B} - \hat{\mathbf{r}} \left(\mathbf{B} \cdot \hat{\mathbf{r}} \right) \right)$$

the integral becomes

$$\begin{aligned}
\mathbf{L} &= -\frac{e}{4\pi} \int (\mathbf{B} \cdot \boldsymbol{\nabla})\, \hat{\mathbf{r}} d^3x \\
L_i &= -\frac{e}{4\pi} \sum_j \int (B_j \cdot \nabla_j)\, \hat{r}_i d^3x \\
&= -\frac{e}{4\pi} \sum_j \int \left(\nabla_j \left(B_j \hat{r}_i \right) - \hat{r}_i \nabla_j B_j \right) d^3x \\
&= -\frac{e}{4\pi} \sum_j \left(\oint_S \hat{r}_j \left(B_j \hat{r}_i \right) d^2x - \int \hat{r}_i \nabla_j B_j d^3x \right) \\
\mathbf{L} &= -\frac{e}{4\pi} \left(\oint_S \hat{\mathbf{r}} \left(\hat{\mathbf{r}} \cdot \mathbf{B} \right) d^2x - \int \hat{\mathbf{r}} \left(\boldsymbol{\nabla} \cdot \mathbf{B} \right) d^3x \right)
\end{aligned}$$

Since the first integral is taken at infinity,

$$\begin{aligned}
\oint_S \hat{\mathbf{r}} \left(\hat{\mathbf{r}} \cdot \mathbf{B} \right) d^2x &= \lim_{r \to \infty} \oint_S \left(\frac{g}{4\pi r^2 \left(1 + \frac{d^2}{r^2} - \frac{2d}{r} \cos\theta \right)^2} \left(\hat{\mathbf{r}} - \frac{d}{r} \mathbf{i} \right) \right) r^2 d\Omega \\
&= \frac{g}{4\pi} \oint_S \hat{\mathbf{r}} d\Omega \\
&= 0
\end{aligned}$$

since the average of $\hat{\mathbf{r}}$ over all directions vanishes. Therefore, replacing the divergence of the magnetic field by its point source,

$$\begin{aligned}
\mathbf{L} &= \frac{e}{4\pi} \int \hat{\mathbf{r}} \left(\boldsymbol{\nabla} \cdot \mathbf{B} \right) d^3x \\
&= \frac{eg}{4\pi} \int \hat{\mathbf{r}} \delta^3 \left(\mathbf{r} - d\mathbf{i} \right) d^3x \\
&= \frac{eg}{4\pi} \mathbf{i}
\end{aligned}$$

4 Quantization

Dirac points out that since angular momentum is quantized in integer multiples of $\frac{\hbar}{2}$, we have

$$\frac{eg}{2\pi} = n\hbar$$

The discreteness of this product means that both electric and magnetic charge must be quantized. The result above agrees with a more complete quantum mechanical treatment, first given by Dirac.

Dirac's argument begins by describing a monopole as an infinite solenoid. Picture one end at the origin with the solenoid running off to infinity along the negative z-axis. Then the dipole field of the solenoid becomes a point source at the origin. It is not too difficult to write the vector potential for this configuration, but it goes singular on the negative z-axis where we placed the idealized solenoid. To avoid this problem, Dirac required the wave function for an electron in the magnetic field to be independent of where the solenoid is. Changing the position of the solenoid changes the wave function by $\exp\left(ieg\Omega/4\pi\hbar\right)$, where Ω is the solid angle subtended relative to the electron by the loop enclosed by the two solenoids. The value of Ω changes by 4π any time the electron crosses this surface, and this must not affect the wave function. Therefore, we need

$$\frac{eg}{\hbar} = 2\pi n$$

Let's look at this in detail.

Electromagnetic gauge transformations and the Schrödinger equation

The Schrödinger equation, coupled to electromagnetism, is

$$\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and this equation clearly depends on the choice of gauge for the vector potential. That is, if we replace $\mathbf{A}$ by

$$\mathbf{A}' = \mathbf{A} + \nabla\phi$$

the Schrödinger equation changes. However, it turns out that if we simultaneously change

$$\begin{aligned}
\mathbf{A}' &= \mathbf{A} + \nabla\phi \\
\psi' &= \exp\left(-\frac{ie}{\hbar}\phi\right)\psi
\end{aligned}$$

then the solution is the same. Check this. Substituting into the wave equation in the primed system, we have:

$$\begin{aligned}
\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}'\right)^2\psi' + V\psi' &= i\hbar\frac{\partial\psi'}{\partial t} \\
\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A} - e\nabla\phi\right)^2\exp\left(-\frac{ie}{\hbar}\phi\right)\psi + V\exp\left(-\frac{ie}{\hbar}\phi\right)\psi &= i\hbar\frac{\partial}{\partial t}\left(\exp\left(-\frac{ie}{\hbar}\phi\right)\psi\right) \\
\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A} - e\nabla\phi\right)^2 e^{-\frac{ie}{\hbar}\phi}\psi + Ve^{-\frac{ie}{\hbar}\phi}\psi &= i\hbar e^{-\frac{ie}{\hbar}\phi}\frac{\partial\psi}{\partial t}
\end{aligned}$$

Now check the first factor,

$$\begin{aligned}
\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A} - e\nabla\phi\right)e^{-\frac{ie}{\hbar}\phi}\psi &= \left(i\hbar\boldsymbol{\nabla}\left(e^{-\frac{ie}{\hbar}\phi}\psi\right) - e\mathbf{A}e^{-\frac{ie}{\hbar}\phi}\psi - e\nabla\phi e^{-\frac{ie}{\hbar}\phi}\psi\right) \\
&= e\boldsymbol{\nabla}\phi\left(e^{-\frac{ie}{\hbar}\phi}\psi\right) + i\hbar e^{-\frac{ie}{\hbar}\phi}\boldsymbol{\nabla}\psi - e\mathbf{A}e^{-\frac{ie}{\hbar}\phi}\psi - e\nabla\phi\left(e^{-\frac{ie}{\hbar}\phi}\psi\right) \\
&= i\hbar e^{-\frac{ie}{\hbar}\phi}\boldsymbol{\nabla}\psi - e\mathbf{A}e^{-\frac{ie}{\hbar}\phi}\psi \\
&= e^{-\frac{ie}{\hbar}\phi}\left(i\hbar\boldsymbol{\nabla}\psi - e\mathbf{A}\psi\right) \\
&= e^{-\frac{ie}{\hbar}\phi}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)\psi
\end{aligned}$$

The effect of the primed operator on $e^{-\frac{ie}{\hbar}\phi}\psi$ is the same as the phase times the unprimed operator on ψ. Therefore, a second application of the operator $\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A} - e\nabla\phi\right)$ to $e^{-\frac{ie}{\hbar}\phi}\left(i\hbar\boldsymbol{\nabla}\psi - e\mathbf{A}\psi\right)$ will give

$$\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A} - e\nabla\phi\right)\cdot\left[e^{-\frac{ie}{\hbar}\phi}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)\psi\right] = e^{-\frac{ie}{\hbar}\phi}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)\cdot\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)\psi$$

Therefore, the full primed equation reduces to the unprimed equation,

$$\begin{aligned}
\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A} - e\nabla\phi\right)^2 e^{-\frac{ie}{\hbar}\phi}\psi + Ve^{-\frac{ie}{\hbar}\phi}\psi &= i\hbar e^{-\frac{ie}{\hbar}\phi}\frac{\partial\psi}{\partial t} \\
e^{-\frac{ie}{\hbar}\phi}\left[\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)^2\psi + V\psi\right] &= i\hbar e^{-\frac{ie}{\hbar}\phi}\frac{\partial\psi}{\partial t} \\
\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - e\mathbf{A}\right)^2\psi + V\psi &= i\hbar\frac{\partial\psi}{\partial t}
\end{aligned}$$

so we get the same solutions.

In conclusion, when we change the gauge of the vector potential, we also need to change the phase of the wave function, according to

$$\begin{aligned}\mathbf{A}' &= \mathbf{A} + \nabla\phi \\ \psi' &= \exp\left(-\frac{ie}{\hbar}\phi\right)\psi\end{aligned}$$

Potential of a solenoid

From Chapter 5, we know that a multipole expansion for the vector potential from isolated currents is dominated by the dipole term,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$$

Suppose we think of a solenoid as many infinitesimal magnetic dipole moments, $d\mathbf{m}$, combined end to end. With the vector potential from one at $\mathbf{x}'$ given by

$$\begin{aligned}d\mathbf{A} &= \frac{\mu_0}{4\pi}\frac{d\mathbf{m}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\frac{\mu_0}{4\pi}d\mathbf{m}\times\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\end{aligned}$$

Now consider Dirac's semi-infinite solenoid, idealized to infinitesimal length. In analogy to the strength of a dipole moment in terms of point charges, $\mathbf{p} = q\mathbf{d}$, let the magnetic moment be expressed as $d\mathbf{m} = \frac{1}{\mu_0}g d\mathbf{l}'$. Then the total vector potential for a solenoid lying along a curve C is given by the integral

$$\mathbf{A}(\mathbf{x}) = -\frac{g}{4\pi}\int\limits_C d\mathbf{l}'\times\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

Let the solenoid lie along the negative z-axis. Then

$$\begin{aligned}\mathbf{A} &= -\frac{g}{4\pi}\int\limits_C dz'\mathbf{k}\times\boldsymbol{\nabla}\frac{1}{|\mathbf{x}+z'\mathbf{k}|} \\ &= -\frac{g}{4\pi}\mathbf{k}\times\boldsymbol{\nabla}\int\limits_{-\infty}^{0}\frac{dz'}{|\mathbf{x}+z'\mathbf{k}|} \\ &= -\frac{g}{4\pi}\mathbf{k}\times\boldsymbol{\nabla}\int\limits_{-\infty}^{0}\frac{dz'}{\sqrt{x^2+y^2+(z+z')^2}} \\ &= -\frac{g}{4\pi}\mathbf{k}\times\boldsymbol{\nabla}\int\limits_{-\infty}^{z}d\zeta\frac{1}{\sqrt{\rho^2+\zeta^2}} \\ &= -\frac{g}{4\pi}\mathbf{k}\times\boldsymbol{\nabla}\left[\ln\left(\sqrt{\rho^2+\zeta^2}+\zeta\right)\right]_{-\infty}^{z} \\ &= -\frac{g}{4\pi}\mathbf{k}\times\boldsymbol{\nabla}\left[\ln\left(\sqrt{\rho^2+z^2}+z\right)-\lim_{z\to-\infty}\ln\left(\sqrt{\rho^2+z^2}+z\right)\right]\end{aligned}$$

where $\zeta = z + z'$.

Unfortunately, the infinite limit diverges, giving an infinite constant. However, if we take the derivative first, we have

$$\begin{aligned}
\mathbf{A}_- &= -\frac{g}{4\pi}\mathbf{k}\times\boldsymbol{\nabla}\int_z^{-\infty}\frac{d\zeta'}{\sqrt{\rho^2+\zeta'^2}} \\
&= \frac{g}{4\pi}\left(\mathbf{i}\frac{\partial}{\partial y}-\mathbf{j}\frac{\partial}{\partial x}\right)\int_z^{-\infty}\frac{d\zeta'}{\sqrt{\rho^2+\zeta'^2}} \\
&= \frac{g}{4\pi}\int_z^{-\infty}\left(-\mathbf{i}\frac{yd\zeta'}{\left(\rho^2+\zeta'^2\right)^{3/2}}+\mathbf{j}\frac{xd\zeta'}{\left(\rho^2+\zeta'^2\right)^{3/2}}\right) \\
&= \left(-\mathbf{i}y+\mathbf{j}x\right)\frac{g}{4\pi}\int_z^{-\infty}\frac{d\zeta'}{\left(\rho^2+\zeta'^2\right)^{3/2}} \\
&= \left(-\mathbf{i}y+\mathbf{j}x\right)\frac{g}{4\pi}\int_z^{-\infty}\frac{\rho d\alpha}{\cos^2\alpha\left(\rho^2+\rho^2\tan^2\alpha\right)^{3/2}} \\
&= \left(-\mathbf{i}\frac{y}{\rho}+\mathbf{j}\frac{x}{\rho}\right)\frac{g}{4\pi\rho}\int_z^{-\infty}\cos\alpha d\alpha \\
&= \hat{\boldsymbol{\varphi}}\frac{g}{4\pi\rho}\left[\sin\alpha\right]_z^{-\infty} \\
&= \hat{\boldsymbol{\varphi}}\frac{g}{4\pi\rho}\left[\frac{\tan\alpha'}{\sqrt{1+\tan^2\alpha'}}\right]_z^{-\infty} \\
&= \hat{\boldsymbol{\varphi}}\frac{g}{4\pi\rho}\left[\frac{\zeta'}{\sqrt{\rho^2+\zeta'^2}}\right]_z^{-\infty} \\
&= \hat{\boldsymbol{\varphi}}\frac{g}{4\pi\rho}\left(1-\frac{z}{\sqrt{\rho^2+z^2}}\right) \\
&= \frac{g\left(1-\cos\theta\right)}{4\pi r\sin\theta}\hat{\boldsymbol{\varphi}}
\end{aligned}$$

which is well-behaved except on the negative z-axis.

The magnetic field is also well-behaved everywhere except along the solenoid. The magnetic field is

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} \\
&= \frac{g}{4\pi}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left(\mathbf{k}\int_{-\infty}^{0}dz'\frac{1}{|\mathbf{x}+z'\mathbf{k}|}\right)\right) \\
&= \frac{g}{4\pi}\left(\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\int_{-\infty}^{0}dz'\frac{\mathbf{k}}{|\mathbf{x}+z'\mathbf{k}|}\right)-\nabla^2\left(\int_{-\infty}^{0}dz'\frac{\mathbf{k}}{|\mathbf{x}+z'\mathbf{k}|}\right)\right) \\
&= \frac{g}{4\pi}\boldsymbol{\nabla}\left(\int_{-\infty}^{0}dz'\frac{\partial}{\partial z}\frac{1}{|\mathbf{x}+z'\mathbf{k}|}\right)-\frac{g}{4\pi}\mathbf{k}\int_{-\infty}^{0}dz'\nabla^2\frac{1}{|\mathbf{x}+z'\mathbf{k}|}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{g}{4\pi}\boldsymbol{\nabla}\left(\int_{-\infty}^{0} dz'\frac{\partial}{\partial z'}\frac{1}{|\mathbf{x}+z'\mathbf{k}|}\right)+g\mathbf{k}\int_{-\infty}^{0} dz'\delta^{3}\left(\mathbf{x}+z'\mathbf{k}\right) \\
&= \frac{g}{4\pi}\boldsymbol{\nabla}\left(-\frac{1}{|\mathbf{x}|}+\frac{1}{|\mathbf{x}-\infty\mathbf{k}|}\right)+g\mathbf{k}\delta\left(x\right)\delta\left(y\right)\Theta\left(-z\right) \\
&= \frac{g\mathbf{r}}{4\pi r^{3}}+g\mathbf{k}\delta\left(x\right)\delta\left(y\right)\Theta\left(-z\right)
\end{aligned}
$$

This is the magnetic field of a magnetic monopole at the origin, plus an infinite contribution on the negative z-axis due to the idealized solenoid.

Dirac's quantization condition

To solve the problem of the infinite field in the solenoid, Dirac required the wave function to vanish at the position of the solenoid. This prescription must be independent of the position of the solenoid, so we also must require the result to be the same regardless of the solenoid position.

An equivalent approach is to use two different vector potentials. We found the potential for a solenoid down the negative z-axis to be

$$
\mathbf{A}_{-} = \frac{g\left(1-\cos\theta\right)}{4\pi r\sin\theta}\hat{\boldsymbol{\varphi}}
$$

If we change the limits of that calculation so that the solenoid lies along the positive z-axis, then the potential is

$$
\mathbf{A}_{+} = -\frac{g\left(1+\cos\theta\right)}{4\pi r\sin\theta}\hat{\boldsymbol{\varphi}}
$$

The trick is to use $\mathbf{A}_{-}$ in a region that excludes the negative axis, and $\mathbf{A}_{+}$ in an overlapping region that excludes the positive axis. For consistency, the two must agree in the region where they overlap, but in fact they differ by

$$
\begin{aligned}
\mathbf{A}_{+}-\mathbf{A}_{-} &= \frac{g}{4\pi r\sin\theta}\left(-\left(1+\cos\theta\right)-\left(1-\cos\theta\right)\right)\hat{\boldsymbol{\varphi}} \\
&= -\frac{g}{2\pi r\sin\theta}\hat{\boldsymbol{\varphi}} \\
&= -\frac{g}{2\pi}\hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\varphi\right) \\
&= -\frac{g}{2\pi}\boldsymbol{\nabla}\varphi
\end{aligned}
$$

Though the difference doesn't vanish, it is pure gauge and doesn't affect the magnetic field. We can define a consistent interaction of particles with this field if the gauge change makes no difference to the wave function.

Above, we showed that when we change the gauge of the vector potential, we must also change the phase of the wave function, according to

$$
\begin{aligned}
\mathbf{A}' &= \mathbf{A}+\nabla f \\
\psi' &= \exp\left(-\frac{ie}{\hbar}f\right)\psi
\end{aligned}
$$

Therefore, when we change from using $\mathbf{A}_{+}$ to $\mathbf{A}_{-}$ we must also change the wave function by

$$
\psi' = \exp\left(\frac{ieg}{2\pi\hbar}\varphi\right)\psi
$$

In general, a phase doesn't change probabilities, but we must require the wave function to be single valued. Thus, when φ changes by 2π, the full phase must change by $2\pi n$,

$$\begin{aligned} \frac{eg}{2\pi\hbar}2\pi &= 2\pi n \\ eg &= 2\pi n\hbar \end{aligned}$$

in agreement with our previous result.

Based on this, it is often claimed that, if a magnetic monopole were ever to be discovered, it would explain the quantization of charge. The relation also gives the strength of the basic magnetic charge. Expressing the electric and magnetic charges in terms of dimensionless ratios (i.e., the fine structure constant),

$$\begin{aligned} \alpha_e &= \frac{e^2}{4\pi\epsilon_0\hbar c} \\ \alpha_g &= \frac{g^2}{4\pi\mu_0\hbar c} \end{aligned}$$

and setting $\alpha_e \approx \frac{1}{137}$, we estimate the strength of the magnetic monopole to be

$$\begin{aligned} \alpha_g &= \frac{g^2}{4\pi\mu_0\hbar c} \\ &= \frac{(2\pi\hbar)^2}{4\pi\mu_0\hbar c e^2} \\ &= \frac{4\pi\hbar}{4\mu_0 c e^2} \\ &= \frac{1}{4\mu_0 c^2\epsilon_0\alpha_e} \\ &= \frac{1}{4\alpha_e} \end{aligned}$$

This means that the coupling of a magnetic monopole is about 4700 times stronger than the coupling of a unit charge,

$$\frac{\alpha_g}{\alpha_e} = \frac{1}{4\alpha_e^2} = 4692$$

This should make such particles easy to detect, but they have not been seen.