Waves in plasma with a magnetic field

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Earth's ionosphere is a plasma, but our previous solution does not include the presence of a magnetic field. We consider a new model which shows some of the effects of a magnetic field on wave propagation.

Consider an electromagnetic wave passing through a medium with a strong, static, uniform magnetic induction $\mathbf{B}_0$ in the same direction as the wave propagation. We neglect damping (due to collisions of the particles of the medium) and we neglect the additional magnetic field produced by the movement of the charges. Then Newton's second law for the motion of an electron with charge $-e$ becomes

$$m\ddot{\mathbf{x}} = -e\dot{\mathbf{x}} \times \mathbf{B}_0 - e\mathbf{E}e^{-i\omega t}$$

Think of the waves as a superposition of the two possible circular polarizations,

$$\mathbf{E} = (\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)\, E$$

Since the magnetic field lies in the direction of propagation, it is orthogonal to both polarization vectors.

Since we ignore all magnetic fields except $\mathbf{B}_0$, the motion of the electron stays in the plane of the wave front. Therefore, we my set the position of the electron to be

$$\mathbf{x} = (x_1\boldsymbol{\varepsilon}_1 \pm i x_2\boldsymbol{\varepsilon}_2)\, e^{-i\omega t}$$

Take a moment to understand what this expression for the position means. Expanding

$$\begin{aligned}
\mathbf{x} &= (x_1\boldsymbol{\varepsilon}_1 \pm i x_2\boldsymbol{\varepsilon}_2)\, e^{-i\omega t} \\
&= (x_1\boldsymbol{\varepsilon}_1 \pm i x_2\boldsymbol{\varepsilon}_2)(\cos\omega t - i\sin\omega t) \\
&= (x_1\boldsymbol{\varepsilon}_1 \cos\omega t \pm x_2\boldsymbol{\varepsilon}_2 \sin\omega t) + i(-x_1\boldsymbol{\varepsilon}_1 \sin\omega t \pm x_2\boldsymbol{\varepsilon}_2 \cos\omega t)
\end{aligned}$$

we take the real part,

$$Re\,\mathbf{x} = x_1\boldsymbol{\varepsilon}_1 \cos\omega t \pm x_2\boldsymbol{\varepsilon}_2 \sin\omega t$$

Thus, if $x_1 = x_2$, the position vector rotates in a clockwise or counterclockwise circle. For different x_1, x_2, we get an ellipse.

The equation of motion becomes

$$\begin{aligned}
m\ddot{\mathbf{x}} &= -e\dot{\mathbf{x}} \times \mathbf{B}_0 - e\mathbf{E}e^{-i\omega t} \\
-\omega^2 m\,(x_1\boldsymbol{\varepsilon}_1 \pm i x_2\boldsymbol{\varepsilon}_2)\, e^{-i\omega t} &= +i\omega e\,(x_1\boldsymbol{\varepsilon}_1 \pm i x_2\boldsymbol{\varepsilon}_2)\, e^{-i\omega t} \times \mathbf{B}_0 - e\,(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)\, E e^{-i\omega t}
\end{aligned}$$

Expanding the cross product term

$$\begin{aligned}
i\omega e\,(x_1\boldsymbol{\varepsilon}_1 \pm i x_2\boldsymbol{\varepsilon}_2)\, e^{-i\omega t} \times \mathbf{B}_0 &= +i\omega e\,(x_1\boldsymbol{\varepsilon}_1 \times \mathbf{B}_0 \pm i x_2\boldsymbol{\varepsilon}_2 \times \mathbf{B}_0)\, e^{-i\omega t} \\
&= +i\omega e\,(-x_1\boldsymbol{\varepsilon}_2 \pm i x_2\boldsymbol{\varepsilon}_1)\, B_0 e^{-i\omega t}
\end{aligned}$$

we write the separate components

$$\begin{aligned}
-\omega^2 m\,(x_1\boldsymbol{\varepsilon}_1) &= +i\omega e\,(\pm i x_2\boldsymbol{\varepsilon}_1)\, B_0 - e\boldsymbol{\varepsilon}_1 E \\
-\omega^2 m\,(\pm i x_2\boldsymbol{\varepsilon}_2) &= +i\omega e\,(-x_1\boldsymbol{\varepsilon}_2)\, B_0 - e\,(\pm i\boldsymbol{\varepsilon}_2)\, E
\end{aligned}$$

The first gives

$$\begin{aligned} -\omega^2 m x_1 &= \mp\omega e x_2 \boldsymbol{\varepsilon}_1 B_0 - eE \\ x_1 &= \pm\frac{eB_0}{m\omega} x_2 + \frac{eE}{m\omega^2} \end{aligned}$$

and the second

$$\begin{aligned} \pm i\omega^2 m x_2 &= +i\omega e x_1 B_0 \pm ieE \\ x_2 &= \pm\frac{eB_0}{\omega m} x_1 + \frac{eE}{m\omega^2} \end{aligned}$$

Now substitute the first into the second to find

$$\begin{aligned} x_2 &= \pm\frac{eB_0}{\omega m}\left(\pm\frac{eB_0}{m\omega} x_2 + \frac{eE}{m\omega^2}\right) + \frac{eE}{m\omega^2} \\ x_2 &= \left(\frac{eB_0}{\omega m}\right)^2 x_2 \pm \frac{e^2 B_0 E}{m^2\omega^3} + \frac{eE}{m\omega^2} \\ \left(1 - \left(\frac{eB_0}{\omega m}\right)^2\right) x_2 &= \frac{eE}{m\omega^2}\left(1 \pm \frac{eB_0}{m\omega}\right) \\ x_2 &= \frac{eE}{m\omega^2}\frac{1 \pm \frac{eB_0}{m\omega}}{1 - \left(\frac{eB_0}{\omega m}\right)^2} \end{aligned}$$

Define the precession frequency,

$$\omega_B \equiv \frac{eB_0}{m}$$

Then, factoring the denominator on the right, we have

$$\begin{aligned} x_2 &= \frac{eE}{m\omega^2}\frac{1 \pm \frac{\omega_B}{\omega}}{\left(1 - \frac{\omega_B}{\omega}\right)\left(1 + \frac{\omega_B}{\omega}\right)} \\ &= \frac{eE}{m\omega^2}\frac{1}{1 \mp \frac{\omega_B}{\omega}} \\ &= \frac{eE}{m\omega}\frac{1}{\omega \mp \omega_B} \end{aligned}$$

The result for x_1 is therefore,

$$\begin{aligned} x_1 &= \pm\frac{eB_0}{m\omega} x_2 + \frac{eE}{m\omega^2} \\ &= \pm\frac{\omega_B}{\omega}\frac{eE}{m\omega}\frac{1}{\omega \mp \omega_B} + \frac{eE}{m\omega^2} \\ &= \frac{eE}{m\omega^2}\left(1 \pm \frac{\omega_B}{\omega \mp \omega_B}\right) \\ &= \frac{eE}{m\omega^2}\left(\frac{\omega \mp \omega_B \pm \omega_B}{\omega \mp \omega_B}\right) \\ &= \frac{eE}{m\omega}\left(\frac{1}{\omega \mp \omega_B}\right) \end{aligned}$$

so reconstructing the full position vector,

$$\begin{aligned} \mathbf{x} &= \left(x_1 \boldsymbol{\varepsilon}_1 \pm i x_2 \boldsymbol{\varepsilon}_2\right) e^{-i\omega t} \\ &= \left(\frac{eE}{m\omega}\left(\frac{1}{\omega \mp \omega_B}\right)\boldsymbol{\varepsilon}_1 \pm i\frac{eE}{m\omega}\frac{1}{\omega \mp \omega_B}\boldsymbol{\varepsilon}_2\right) e^{-i\omega t} \\ &= \frac{e}{m\omega}\left(\frac{1}{\omega \mp \omega_B}\right)\left(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2\right) E e^{-i\omega t} \end{aligned}$$

and finally,

$$\mathbf{x} = \frac{e}{m\omega}\left(\frac{1}{\omega \mp \omega_B}\right)\mathbf{E}$$

Physically, what happens is that each circular polarization drives the electron position in a corresponding circle, with the amplitude of the circle diverging if the wave is at the precession frequency of the electron.

Dielectric constant

The dielectric constant is found in the same way as for our previous model. We repeat the argument with this new solution.

The electric dipole moment is

$$\begin{aligned}\mathbf{p}_{mol} &= -e\mathbf{x} \\ &= -\frac{e^2}{m\omega}\left(\frac{1}{\omega \mp \omega_B}\right)\mathbf{E}\end{aligned}$$

Then, since the total dipole moment per unit volume is $\mathbf{P} = N\mathbf{p}_{mol} = \epsilon_0\chi_e\mathbf{E}$, the dielectric constant is

$$\begin{aligned}\frac{\epsilon}{\epsilon_0} &= 1 + \chi_e \\ &= 1 - \frac{Ne^2}{m\omega\epsilon_0}\left(\frac{1}{\omega \mp \omega_B}\right) \\ &= 1 - \frac{\omega_p^2}{\omega\left(\omega \mp \omega_B\right)}\end{aligned}$$

where we again find the plasma frequency, $\omega_p^2 = \frac{NZe^2}{m\epsilon_0}$.

Wave vector

Now find the wave vector corresponding to this dielectric constant. With $\mu = \mu_0$, we have

$$\begin{aligned}k &= \sqrt{\mu\epsilon}\omega \\ &= \sqrt{\mu_0\epsilon_0\frac{\epsilon}{\epsilon_0}}\omega \\ &= \frac{\omega}{c}\sqrt{1 - \frac{\omega_P^2}{\omega\left(\omega \mp \omega_B\right)}}\end{aligned}$$

Not only does this become imaginary for some frequencies, but it happens differently for the different circular polarizations. This means that there are frequency ranges where e^{ikx} becomes a damping factor, and one circular polarization can propagate but the other cannot.

Concretely, consider the top sign (positive helicity, or left-handed circular polarization) and suppose, with $\omega > \omega_B$,

$$\begin{aligned}1 - \frac{\omega_P^2}{\omega\left(\omega - \omega_B\right)} &< 0 \\ \frac{\omega\left(\omega - \omega_B\right) - \omega_P^2}{\omega\left(\omega - \omega_B\right)} &< 0 \\ \omega^2 - \omega\omega_B - \omega_P^2 &< 0\end{aligned}$$

Transitions occur when $\omega^2 + \omega\omega_B - \omega_P^2 = 0$. This condition occurs when the frequency is

$$\begin{aligned} \omega_+ &= \frac{\omega_B + \sqrt{\omega_B^2 + 4\omega_P^2}}{2} \\ &= \frac{\omega_B}{2}\left(1 + \sqrt{1 + \frac{4\omega_P^2}{\omega_B^2}}\right) \end{aligned}$$

where we choose the positive value of the frequency. Below this frequency, the positive helicity (left handed) polarization has an imaginary wave vector and therefore decays exponentially and does not propagate.

For negative helicity (right-handed) waves, the corresponding calculation gives

$$\omega^2 + \omega\omega_B - \omega_P^2 \quad < \quad 0$$

and therefore

$$\omega_- < \frac{\omega_B}{2}\left(\sqrt{1 + \frac{4\omega_P^2}{\omega_B^2}} - 1\right)$$

as the condition for non-propagation. Clearly, $\omega_+ > \omega_-$ so that when

$$\omega_- < \omega < \omega_+$$

negative helicity waves will propagate, but not positive helicity.

An electromagnetic pulse sent up into the ionosphere will reflect if one of these conditions is met for the appropriate polarization. Since the plasma frequency,

$$\omega_p = \sqrt{\frac{NZe^2}{m\varepsilon_0}}$$

varies as the square root of the number density of electrons, we can measure the electron density by timing the round trip travel time of the pulse.