1 Introduction

Review of the Introduction

1. Basic laws

   - Maxwell equations, integral form

$$\begin{aligned}
\oint_S \mathbf{D}\cdot\mathbf{n}\,da &= \int_V \rho\,d^3x \\
\oint_S \mathbf{B}\cdot\mathbf{n}\,da &= 0 \\
\oint_C \mathbf{H}\cdot d\mathbf{l} &= \int_S \left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{n}\,d^2x \\
\oint_C \mathbf{E}\cdot d\mathbf{l} &= -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}\,d^2x
\end{aligned}$$

   - Maxwell equations, differential form

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho \\
\nabla\cdot\mathbf{B} &= 0 \\
\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t} &= \mathbf{J} \\
\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0
\end{aligned}$$

   - Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$$

   - Lorentz force law

$$\mathbf{F}=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

2. Mass of photon. The non-relativistic limit of the Klein-Gordon equation,

$$-\frac{\partial^2\varphi}{\partial t^2}+\nabla^2\varphi=\frac{m^2c^2}{\hbar^2}\varphi$$

   is the Schrödinger equation. The Klein-Gordon equation is a wave equation for a relativistic particle of mass m. When $m=0$, the resulting equation is

$$-\frac{\partial^2\varphi}{\partial t^2}+\nabla^2\varphi=0$$

   This equation is satisfied by the electric potential. Therefore, it is natural to assume that if the photon had mass, the electric potential would take the form of the Klein-Gordon equation. Consider the Klein-Gordon equation for a static, spherically symmetric potential. Then we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)=\frac{m^2c^2}{\hbar^2}\varphi$$

Substituting

$$\varphi(r) = \frac{A}{r} e^{-\mu r}$$

we find

$$\begin{aligned}
\frac{m^2 c^2}{\hbar^2} \frac{A}{r} e^{-\mu r} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \left(\frac{A}{r} e^{-\mu r} \right) \right) \\
\frac{m^2 c^2}{\hbar^2 r} e^{-\mu r} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \left(-\frac{1}{r^2} e^{-\mu r} - \frac{\mu}{r} e^{-\mu r} \right) \right) \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left((-1 - \mu r) e^{-\mu r} \right) \\
&= \frac{1}{r^2} \left(-\mu (-1 - \mu r) e^{-\mu r} - \mu e^{-\mu r} \right) \\
&= \frac{1}{r} \mu^2 e^{-\mu r}
\end{aligned}$$

so that we have a solution if $\mu = \frac{mc}{\hbar}$. Therefore, studies which place limits on the photon mass assume that the electric potential takes the form $\frac{A}{r} e^{-\mu r}$ and put limits on the value of μ.

3. Linear superposition. This continues to hold in matter as long as the medium is linear

$$\begin{aligned}
D_i &= \sum_{j=1}^{3} \varepsilon_{ij} E_j \\
H_i &= \sum_{j=1}^{3} \mu'_{ij} B_j
\end{aligned}$$

4. Boundary conditions. We can use the integral form of Maxwell's equations to derive boundary conditions for the electric and magnetic fields. For the first pair of surface integrals,

$$\oint_S \mathbf{D} \cdot \mathbf{n}\, da, \oint_S \mathbf{B} \cdot \mathbf{n}\, da$$

we imagine a cylindrical volume perpendicular to the interface between two materials. The materials are characterized by the relationship between $(\mathbf{D}, \mathbf{H})$ and $(\mathbf{E}, \mathbf{B})$. With the cylinder piercing the boundary so that its curved sides are infinitesimally high, δ, the finite part of the surface integrals is given by the flat end caps. We therefore find

$$\lim_{\delta \to 0} \oint_S \mathbf{D} \cdot \mathbf{n}\, da = (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} A$$

where A is the area of each end cap. In the same limit, the volume integral of the charge density becomes

$$\begin{aligned}
\lim_{\delta \to 0} \int_V \rho\, d^3 x &= \lim_{\delta \to 0} \rho A \delta \\
&= \sigma A
\end{aligned}$$

where σ is the charge per unit area on the boundary surface. Therefore, we have

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma$$

so that the normal component of $\mathbf{D}$ has a discontinuity equal to the surface charge density, σ. The same result holds for $\mathbf{B}$ except there can be no magnetic charge density, so that the normal component of the magnetic induction $\mathbf{B}$ is continuous,

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0$$

For the remaining two equations, we take the curve C to be a rectangle with two sides, $\mathbf{l}$, parallel to the surface and two short sides (of length δ) through the surface. Then the contour integrals become

$$\begin{aligned} \lim_{\delta \to 0} \oint_C \mathbf{H} \cdot d\mathbf{l} &= (\mathbf{H_2} - \mathbf{H_1}) \cdot \mathbf{l} \\ \lim_{\delta \to 0} \oint_C \mathbf{E} \cdot d\mathbf{l} &= (\mathbf{E_2} - \mathbf{E_1}) \cdot \mathbf{l} \end{aligned}$$

Because $\mathbf{l}$ is parallel to the surface, we get a relation for the tangential components of the electric and magnetic fields. For the right side of these equations, we have

$$\begin{aligned} \lim_{\delta \to 0} \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n} \, d^2x &= \lim_{\delta \to 0} \int_S \mathbf{J} \cdot \mathbf{n} \, d^2x + \lim_{\delta \to 0} \left[\left(\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n} \right) l\delta \right] \\ &= \mathbf{K} \cdot \mathbf{l} + 0 \\ &= K \end{aligned}$$

where $\mathbf{K} \cdot \mathbf{l} = Kl$ is the surface current density and , and $\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n}$ is finite at the surface. For the magnetic induction term,

$$\begin{aligned} \lim_{\delta \to 0} \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, d^2x &= \lim_{\delta \to 0} \left(\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) l\delta \\ &= 0 \end{aligned}$$

and we have

$$\begin{aligned} (\mathbf{H_2} - \mathbf{H_1}) \cdot \mathbf{l} &= Kl \\ (\mathbf{E_2} - \mathbf{E_1}) \cdot \mathbf{l} &= 0 \end{aligned}$$

Using the normal to the surface, we can write these as vector equations,

$$\begin{aligned} \mathbf{n} \times (\mathbf{H_2} - \mathbf{H_1}) &= \mathbf{K} \\ \mathbf{n} \times (\mathbf{E_2} - \mathbf{E_1}) &= 0 \end{aligned}$$

We can do this because the result is independent of the orientation of the vector $\mathbf{l}$, as long as it is parallel to the surface.

5. Idealizations: Our volume elements will be taken as microscopically large and macroscopically small. This allows charge distributions to be taken as continuous functions of position instead of discrete charges at isolated locations.

6. Dirac delta function

2 Chapter 1

Before starting our discussion of Green functions, we clarify some points about the Dirac delta function and the Laplacian of $\frac{1}{r}$.

2.1 The Laplacian of $\frac{1}{r}$

Consider the potential of a "point charge" q at $\mathbf{x}'$

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

For convenience, choose coordinates so that $\mathbf{x}'$ is at the origin. Then in spherical coordinates, the potential is proportional to $\frac{1}{r}$. As a function, $f = \frac{1}{r}$ is defined on the *open* interval $(0,\infty)$, but not at the origin. Its Laplacian is also defined on this interval, and is quickly seen to vanish everywhere,

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right) \\
&= \frac{1}{r^2}\frac{d}{dr}(-1) \\
&= 0
\end{aligned}$$

This leads to a difficulty when we consider the divergence theorem:

$$\int\limits_V \nabla^2\left(\frac{1}{r}\right)d^3x = \oint\limits_S \hat{\mathbf{n}}\cdot\nabla\left(\frac{1}{r}\right)d^2x$$

since the right hand side is well-defined but the left is not. Indeed, for a sphere of radius ε, the integral on the right becomes

$$\begin{aligned}
\oint\limits_S \hat{\mathbf{n}}\cdot\nabla\left(\frac{1}{r}\right)d^2x &= \int\limits_0^{\pi}\int\limits_0^{2\pi}\hat{\mathbf{r}}\cdot\left(-\hat{\mathbf{r}}\frac{1}{\varepsilon^2}\right)\varepsilon^2\sin\theta d\theta d\varphi \\
&= -4\pi
\end{aligned}$$

However, the integral on the left is undefined.

The rigorous way to handle this is to *extend* the *function* $f = \frac{1}{r}$ to a *distribution.* A distribution is defined as the limit of a sequence of functions, giving an object which is only meaningful when integrated. Thus, if we define a distribution f to be the limit

$$f(x) \equiv \lim_{a\to 0} f_a(x)$$

where $f_a(x)$ is a collection of functions depending on a parameter a. The integral of the distribution is defined as the limit of the well-behaved integrals of the series of functions

$$\int f(x)\,dx \equiv \lim_{a\to 0}\int f_a(x)\,dx$$

and this may be perfectly finite even if $f(x)$ is not.

With this in mind, let $f_a(r) = \frac{1}{\sqrt{r^2+a^2}}$. This is defined for the closed interval $r \in [0,\infty]$, and so is its Laplacian

$$\begin{aligned}
\nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right) &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\frac{1}{\sqrt{r^2+a^2}}\right) \\
&= \frac{1}{r^2}\frac{d}{dr}\left(-\frac{1}{2}r^2\frac{2r}{\left(r^2+a^2\right)^{3/2}}\right) \\
&= -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^3}{\left(r^2+a^2\right)^{3/2}}\right) \\
&= -\frac{1}{r^2}\left(\frac{3r^2}{\left(r^2+a^2\right)^{3/2}} - \frac{3}{2}\frac{2r^4}{\left(r^2+a^2\right)^{5/2}}\right) \\
&= -\frac{3}{\left(r^2+a^2\right)^{3/2}} + \frac{3r^2}{\left(r^2+a^2\right)^{5/2}} \\
&= \frac{3r^2}{\left(r^2+a^2\right)^{5/2}} - \frac{3\left(r^2+a^2\right)}{\left(r^2+a^2\right)^{5/2}} \\
&= -\frac{3a^2}{\left(r^2+a^2\right)^{5/2}}
\end{aligned}$$

We may therefore define a distribution to extend $f(r) = \nabla^2\left(\frac{1}{r}\right)$ by

$$\begin{aligned}
f(r) &= \lim_{a\to 0} f_a(x) \\
&= \lim_{a\to 0}\left(-\frac{3a^2}{\left(r^2+a^2\right)^{5/2}}\right)
\end{aligned}$$

The integral is now well-defined:

$$\int_V \nabla^2\left(\frac{1}{r}\right) d^3x \equiv \lim_{a\to 0}\int_V \nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right) d^3x$$

$$
\begin{aligned}
&= -12\pi \lim_{a\to 0} a^2 \int_0^{\varepsilon} \frac{r^2 dr}{(r^2+a^2)^{5/2}} \\
&= -12\pi \lim_{a\to 0} a^2 \int_0^{\varepsilon} \frac{r^2 dr}{(r^2+a^2)^{5/2}} \\
&= -4\pi \lim_{a\to 0} \int_0^{\varepsilon} \frac{d}{dr} \frac{r^3}{(r^2+a^2)^{3/2}} dr \\
&= -\lim_{a\to 0} \frac{4\pi\varepsilon^3}{(\varepsilon^2+a^2)^{3/2}} \\
&= -4\pi
\end{aligned}
$$

for any finite ε. As a pleasant bonus, the divergence theorem is now satisfied as long as we understand $\nabla^2\left(\frac{1}{r}\right)$ to be a distribution.

2.2 Green's Theorem

To develop a general method for solving the Poisson equation, we need a purely mathematical result: Green's theorem.

First, we establish a simple vector calculus identity,

$$
\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g) = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \boldsymbol{\nabla}^2 g
$$

This relation is not hard to prove. We can just expand the del operator in Cartesian coordinates,

$$
\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}
$$

Substituting in the left hand side,

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) \\
&= \hat{\mathbf{i}}\frac{\partial}{\partial x} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) + \hat{\mathbf{j}}\frac{\partial}{\partial y} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) \\
&\quad + \hat{\mathbf{k}}\frac{\partial}{\partial z} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right)
\end{aligned}
$$

It is easiest to take the dot products first. Since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are independent of position (unlike, say, $\hat{\boldsymbol{r}}$,$\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$), we can bring them inside the derivative. Then we have

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g) &= \hat{\mathbf{i}}\frac{\partial}{\partial x} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) + \hat{\mathbf{j}}\frac{\partial}{\partial y} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) \\
&\quad + \hat{\mathbf{k}}\frac{\partial}{\partial z} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right) \\
&= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right) \\
&= \boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g+f\boldsymbol{\nabla}^2 g
\end{aligned}$$

Since the dot product, gradient and Laplacian are vector operators which are independent of coordinate system,

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} g)=\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g+f\boldsymbol{\nabla}^2 g$$

must now hold in any coordinates, not just Cartesian.

Now, to prove Green's first identity, we begin with the divergence theorem:

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{A}\,d^3x=\oint_S \mathbf{A}\cdot\mathbf{n}\,d^2x$$

This holds for any vector field $\mathbf{A}$. Let the vector field $\mathbf{A}$ have the particular form

$$\mathbf{A}=f\boldsymbol{\nabla} g$$

where $f(\mathbf{x}), g(\mathbf{x})$ are any two functions of position. Substituting,

$$\begin{aligned}
\int_V \boldsymbol{\nabla}\cdot\mathbf{A}\,d^3x &= \oint_S \mathbf{A}\cdot\mathbf{n}\,d^2x \\
\int_V \boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} g)\,d^3x &= \oint_S (f\boldsymbol{\nabla} g)\cdot\mathbf{n}\,d^2x \\
\int_V \left(\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g+f\boldsymbol{\nabla}^2 g\right)d^3x &= \oint_S (f\boldsymbol{\nabla} g)\cdot\mathbf{n}\,d^2x
\end{aligned}$$

where we used the relation derived above. This is Green's first identity.

Suppose we pick some f and g. Then we know that

$$\int_V \left(\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g+f\boldsymbol{\nabla}^2 g\right)d^3x=\oint_S (f\boldsymbol{\nabla} g)\cdot\mathbf{n}\,d^2x$$

However, since f and g are arbitrary, we may also write

$$\int_V \left(\boldsymbol{\nabla} g\cdot\boldsymbol{\nabla} f+g\boldsymbol{\nabla}^2 f\right)d^3x=\oint_S (g\boldsymbol{\nabla} f)\cdot\mathbf{n}\,d^2x$$

for the same f and g. Taking the difference of these two expressions (and using the symmetry of the dot product $\boldsymbol{\nabla} g\cdot\boldsymbol{\nabla} f=\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g$), we have

$$\int_V \left(f\boldsymbol{\nabla}^2 g-g\boldsymbol{\nabla}^2 f\right)d^3x=\oint_S (f\boldsymbol{\nabla} g-g\boldsymbol{\nabla} f)\cdot\mathbf{n}\,d^2x$$

Since $\mathbf{n}$ is the outward unit normal to the surface S bounding the volume V

$$\mathbf{n}\cdot\boldsymbol{\nabla} g = \frac{\partial g}{\partial n}$$

is just the derivative of g normal to S. We can therefore simplify the notation a little,

$$\int_V \left(f\boldsymbol{\nabla}^2 g - g\boldsymbol{\nabla}^2 f\right) d^3x = \oint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d^2x$$

This is Green's theorem.

2.3 Example of the use of Green's theorem

As an example of the use of Green's theorem, we solve problem 10 from Jackson

Jackson 1.10: Mean value theorem

Prove that

$$\Phi\left(\mathcal{P}\right) = \oint_{Sphere} \Phi\left(x\right) d^2x$$

in charge-free space, where the integral is over any sphere centered on the point $\mathcal{P}$.

Start wtih Green's theorem

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right) d^3x = \oint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d^2x$$

Choose $f = \Phi$ and $g = \frac{1}{r}$. Then

$$\int_V \left(\Phi\nabla^2\left(\frac{1}{r}\right) - \frac{1}{r}\nabla^2\Phi\right) d^3x = \oint_S \left(\Phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial n}\right) d^2x$$

Since we are in charge-free space we know that

$$\nabla^2\Phi = 0$$

and we also have

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3\left(r\right)$$

Substituting,

$$-4\pi\int_V \delta^3\left(r\right)\Phi d^3x = \oint_S \left(\Phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial n}\right) d^2x$$

Now, with the surface S a sphere centered on $r = 0$, we have

$$\frac{\partial}{\partial n}\frac{1}{r} = \hat{\mathbf{r}}\cdot\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}\right)\frac{1}{r} = -\frac{1}{r^2}$$

Finally, the last term may be evaluated using Gauss's law:

$$\begin{aligned}
\oint_S \frac{1}{r}\frac{\partial \Phi}{\partial n} d^2x &= -\oint_S \frac{1}{r}\mathbf{E}\cdot\hat{\mathbf{r}} d^2x \\
&= -\frac{1}{R}\oint_S \mathbf{E}\cdot\hat{\mathbf{r}} d^2x \\
&= -\frac{Q_{tot}}{\epsilon_0 R} \\
&= 0
\end{aligned}$$

Putting this all together, and carrying out the integral over the delta function,

$$\begin{aligned}
-4\pi \int_V \delta^3(r)\,\Phi d^3x &= \oint_S \left(\Phi \frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial \Phi}{\partial n}\right) d^2x \\
-4\pi\Phi(0) &= \oint_S \left(-\frac{1}{r^2}\right)\Phi d^2x \\
\Phi(0) &= \frac{1}{4\pi r^2}\oint_S \Phi d^2x
\end{aligned}$$

The right hand side is the average of the potential over the sphere, and the left side is the potential at the center.

2.4 Uniqueness of solutions to the Poisson equation

We want to establish the uniqueness of solutions to our electrostatic problems,

$$\nabla^2\Phi = \frac{\rho}{\epsilon_0}$$

for a fixed source distribution, $\rho(\mathbf{x})$. However, we know that for any $\Phi(\mathbf{x})$ satisfying this equation, we may add a solution, $\Phi'(\mathbf{x})$, of the homogeneous (Laplace) equation,

$$\nabla^2\Phi'(\mathbf{x}) = 0$$

Recall what happens with Newton's 2nd law in mechanics. We have a second order differential equation,

$$\mathbf{F}(\mathbf{x}, t) = m\frac{d^2\mathbf{x}(t)}{dt^2}$$

which also does not have unique solutions. It is only when we add *initial conditions* for the position and velocity, $\mathbf{x}(0), \mathbf{v}(0)$ that the solution becomes unique and the motion is completely determined.

For electrostatic field theory, the situation is similar but does not involve time evolution. Instead, we specify *boundary conditions* for the solution. Thus, we solve the Poisson equation in a volume, V, then restrict the class of solutions to those satisfying certain conditions on the boundary S of V. We now prove that this gives a unique solution. We give a proof by two solutions satisfying the same boundary conditions must be the same.

Suppose we have two solutions, Φ_1, Φ_2, satisfying the Poisson equation on V, and satisfying some specific condition on the boundary S. The boundary condition may be either the value of the potential on S or the value of the normal derivative, $\frac{\partial \Phi}{\partial n}$, on S. In either case, the difference of the two potentials,

$$U = \Phi_2 - \Phi_1$$

must satisfy the Poisson equation as well, but will have vanishing boundary values, i.e., either

$$U(\mathbf{x})|_{x \in S} = 0$$

or

$$\left. \frac{\partial U(\mathbf{x})}{\partial n} \right|_{x \in S} = 0$$

Now use Green's first identity with $f = g = U$:

$$\begin{aligned} \int_V \left(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \boldsymbol{\nabla}^2 g \right) d^3x &= \oint_S f \frac{\partial g}{\partial n} n \, d^2x \\ \int_V \left(\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U + U \boldsymbol{\nabla}^2 U \right) d^3x &= \oint_S U \frac{\partial U}{\partial n} \, d^2x \end{aligned}$$

The boundary condition for U makes the right side vanish, while on the left side we have

$$\begin{aligned} \boldsymbol{\nabla}^2 U &= \boldsymbol{\nabla}^2 (\Phi_2 - \Phi_1) \\ &= \boldsymbol{\nabla}^2 \Phi_2 - \boldsymbol{\nabla}^2 \Phi_1 \\ &= \frac{1}{\epsilon_0} \rho - \frac{1}{\epsilon_0} \rho \\ &= 0 \end{aligned}$$

The identity reduces to

$$\int_V (\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U) \, d^3x = \int_V |\boldsymbol{\nabla} U|^2 \, d^3x = 0$$

This implies $\boldsymbol{\nabla} U = 0$, so that $U = c$. Only the value of this constant remains unspecified, and the field, $E = -\boldsymbol{\nabla} \Phi$ is unique.

Since the result holds for either $U = 0$ (Dirichlet boundary conditions) or $\frac{\partial U}{\partial n} = 0$ (Neumann boundary conditions) on the boundary, it would overspecify the solution to try to impose both conditions. It is possible, however, to specify $U = 0$ on one part of S, and $\frac{\partial U}{\partial n} = 0$ on the remaining part, as long as the right side vanishes.

Notice that since the Laplace equation is just the Poisson equation with $\rho = 0$, solutions to the Laplace equation with given boundary conditions are also unique.

2.5 Green Functions and the formal solution

We now use the results of the preceeding sections to write a complete formal solution to the Poisson equation, satisfying given boundary conditions.

Suppose we can find a function, $G(\mathbf{x}, \mathbf{x}')$ satisfying

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta^3(\mathbf{x} - \mathbf{x}')$$

and satisfying the given boundary conditions, where the Laplacian is with respect to the unprimed coordinates. We may think of $G(\mathbf{x}, \mathbf{x}')$ as the solution of the Poisson equation for a unit point charge at $\mathbf{x}'$. Then a full solution may be found by taking a superposition of such point charges. Thus, setting

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x'$$

we see that

$$\begin{aligned}
\nabla^2 \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \nabla^2 \int G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x' \\
&= \frac{1}{4\pi\epsilon_0} \int \left(\nabla^2 G(\mathbf{x}, \mathbf{x}')\right) \rho(\mathbf{x}') d^3x' \\
&= -\frac{1}{\epsilon_0} \int \delta^3(\mathbf{x} - \mathbf{x}') \rho(\mathbf{x}') d^3x' \\
&= -\frac{1}{\epsilon_0} \rho(\mathbf{x})
\end{aligned}$$

and we have a solution. The function $G(\mathbf{x}, \mathbf{x}')$ is called a Green function.

2.5.1 Example: Isolated point charge

The simplest example of a Green function is for a point charge, with the potential vanishing at infinity. We have already shown that

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta^3(\mathbf{x} - \mathbf{x}')$$

we immediately have

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x})$$

where F satisfies the Laplace equation, $\nabla^2 F = 0$. By uniqueness, the function F must be determined by the boundary conditions. In the present case, we ask for $G(\mathbf{x}, \mathbf{x}')$ to vanish at $\mathbf{x} \to \infty$. Since the first term in $G(\mathbf{x}, \mathbf{x}')$ already satisfies this, we require the same condition for F:

$$\begin{aligned}
\nabla^2 F &= 0 \\
F(\infty) &= 0
\end{aligned}$$

The argument of the preceeding section shows that $F(\mathbf{x}) = 0$ is the unique solution to this, so the Green function for an isolated point charge is $G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$.

If we have a localized distribution of charge, $\rho(\mathbf{x}')$, in empty space, the potential vanishes at infinity and we can use this Green function to find the potential everywhere by integrating

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int G(\mathbf{x}, \mathbf{x}')\,\rho(\mathbf{x}')\,d^3x' \\ &= \frac{1}{\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\end{aligned}$$

2.5.2 The existence of general solutions

The freedom to choose $F(\mathbf{x})$ allows us to solve with more general boundary conditions. Suppose our problem requires solutions with values $\Phi(S)$ on an arbitrary closed boundary surface S. Then using Green's theorem

$$\int_V \left(f\boldsymbol{\nabla}^2 g - g\boldsymbol{\nabla}^2 f\right) d^3x = \oint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d^2x$$

with

$$\begin{aligned}f &= \Phi(\mathbf{x}') \\ g &= G(\mathbf{x}, \mathbf{x}')\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\nabla}^2\Phi(\mathbf{x}) &= -\frac{\rho(\mathbf{x})}{\epsilon_0} \\ \boldsymbol{\nabla}^2 G(\mathbf{x}, \mathbf{x}') &= -4\pi\delta^3(\mathbf{x}-\mathbf{x}')\end{aligned}$$

we find

$$\begin{aligned}\int_V \left(\Phi(\mathbf{x}')\,\boldsymbol{\nabla}^2 G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\,\boldsymbol{\nabla}^2\Phi(\mathbf{x}')\right) d^3x' &= \oint_S \left(\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \Phi(\mathbf{x}')}{\partial n'}\right) d^2x' \\ \int_V \left(-4\pi\Phi(\mathbf{x}')\,\delta^3(\mathbf{x}-\mathbf{x}') + G(\mathbf{x}, \mathbf{x}')\frac{\rho(\mathbf{x}')}{\epsilon_0}\right) d^3x' &= \oint_S \left(\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \Phi(\mathbf{x}')}{\partial n'}\right) d^2x' \\ -4\pi\Phi(\mathbf{x}) + \frac{1}{\epsilon_0}\int_V G(\mathbf{x}, \mathbf{x}')\,\rho(\mathbf{x}')\,d^3x' &= \oint_S \left(\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \Phi(\mathbf{x}')}{\partial n'}\right) d^2x'\end{aligned}$$

Solving for the potential we have

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V G(\mathbf{x}, \mathbf{x}')\,\rho(\mathbf{x}')\,d^3x' - \oint_S \left(\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \Phi(\mathbf{x}')}{\partial n'}\right) d^2x'$$

For the case of Dirichlet boundary conditions, we require

$$G(\mathbf{x}, \mathbf{x}')\big|_{\mathbf{x}'\in S} = 0$$

This uniquely specifies $G(\mathbf{x},\mathbf{x}')$ within S, and gives the solution

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V G(\mathbf{x},\mathbf{x}')\,\rho(\mathbf{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\,\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\,d^2x'$$

Noting that we know both $G(\mathbf{x},\mathbf{x}')$ and $\Phi(S)$, this gives the potential everywhere inside S.

To see that we have satisfied the boundary conditions, let $\mathbf{x}$ lie on S,

$$\begin{aligned}
\Phi(\mathbf{x})\big|_{\mathbf{x}\in S} &= \frac{1}{4\pi\epsilon_0}\int_V G\left(\mathbf{x}|_S,\mathbf{x}'\right)\rho(\mathbf{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{\mathbf{x}\in S} d^2x' \\
&= -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{\mathbf{x}\in S} d^2x'
\end{aligned}$$

What is this derivative? Integrate across the boundary, where both $\mathbf{x}$ and $\mathbf{x}'$ lie on S,

$$\begin{aligned}
\int \boldsymbol{\nabla}^2 G(\mathbf{x},\mathbf{x}')\,dn' &= -4\pi\int \delta^3(\mathbf{x}-\mathbf{x}')\,dn' \\
\int \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} G(\mathbf{x},\mathbf{x}')\,dn' &= -4\pi\delta^2(\mathbf{x}-\mathbf{x}')
\end{aligned}$$

Now, since $G(\mathbf{x},\mathbf{x}')$ is constant on S, $\nabla G(\mathbf{x},\mathbf{x}')$ is in the normal direction so the first integral is

$$\begin{aligned}
\int \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} G(\mathbf{x},\mathbf{x}')\,dn' &= \int \frac{\partial^2 G}{\partial n'^2}(\mathbf{x},\mathbf{x}')\,dn' \\
&= \frac{\partial G}{\partial n'}(\mathbf{x},\mathbf{x}')
\end{aligned}$$

and we have

$$\frac{\partial G}{\partial n'}(\mathbf{x},\mathbf{x}') = -4\pi\delta^2(\mathbf{x}-\mathbf{x}')$$

Substituting into our expression for the potential,

$$\begin{aligned}
\Phi(\mathbf{x})\big|_{\mathbf{x}\in S} &= -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{\mathbf{x}\in S} d^2x' \\
&= \oint_S \Phi(\mathbf{x}')\,\delta^2(\mathbf{x}-\mathbf{x}')\,d^2x' \\
&= \Phi(\mathbf{x})\big|_{\mathbf{x}\in S}
\end{aligned}$$

and the boundary condition is satisfied.

2.6 Green function with spherical boundary conditions

To solve the Poisson equation with values of Φ specified on a sphere of radius a, we need a Green function satisfying

$$\begin{aligned}
\boldsymbol{\nabla}^2 G(\mathbf{x},\mathbf{x}') &= -4\pi\delta^3(\mathbf{x}-\mathbf{x}') \\
G(\mathbf{x},\mathbf{x}')\big|_{r=a} &= 0
\end{aligned}$$

and we would like the result to be expressed in terms of spherical coordinates. The second boundary may be taken either at the origin or infinity. For concreteness we look at the exterior case. The interior case is similar.

To begin, we note the solution for a single point charge q outside a grounded sphere at position $\mathbf{x}'$ is given by the image method as

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a/r'}{\left|\mathbf{x}-\left(\frac{a}{r'}\right)^2\mathbf{x}'\right|}\right)$$

This satisfies

$$\boldsymbol{\nabla}^2\Phi(\mathbf{x}) = -\frac{1}{\epsilon_0}q\delta^3(\mathbf{x}-\mathbf{x}')$$

so that rescaling,

$$\boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a/r'}{\left|\mathbf{x}-\left(\frac{a}{r'}\right)^2\mathbf{x}'\right|}\right) = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

and we have the Green function.

Now, so express this in spherical coordinates,

$$\begin{aligned}\mathbf{x} &= (r,\theta,\varphi)\\ \mathbf{x}' &= (r',\theta',\varphi')\end{aligned}$$

we use the expansion in spherical harmonics,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

For the image charge we have $\frac{a^2}{r'^2}r' < r$, so

$$\begin{aligned}\frac{a/r'}{\left|\mathbf{x}-\left(\frac{a}{r'}\right)^2\mathbf{x}'\right|} &= \frac{a}{r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{\left(\frac{a^2}{r'}\right)^l}{r^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{a^{2l+1}}{r^{l+1}r'^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\end{aligned}$$

Substituting, we have the Green function for boundary conditions on a sphere and at infinity:

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r'^{l+1}}\right)Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

Now, so solve

$$\begin{aligned}\boldsymbol{\nabla}^2\Phi(\mathbf{x}) &= -\frac{1}{\epsilon_0}\rho(\mathbf{x}) \\ \Phi_S(\mathbf{x}) &= f(\theta,\varphi)\end{aligned}$$

we use our general solution,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\, d^3x' - \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} f(\theta',\varphi')\left[-\frac{\partial G(r,\theta,\varphi;a,\theta',\varphi')}{\partial r'}\right] a^2\sin\theta' d\theta' d\varphi'$$

The minus sign in the derivative of the Green fucntion is because we need the outward normal from the (exterior) region of interest. Because on the boundary sphere, $r' = a < r$, this derivative has the form

$$\begin{aligned}\frac{\partial G(r,\theta,\varphi;a,\theta',\varphi')}{\partial r'} &= \left[\frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r'^{l+1}}\right)Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]_{r'=a} \\ &= \left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{\partial}{\partial r'}\left(\frac{r'^l}{r^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r'^{l+1}}\right)Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]_{r'=a} \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{la^{l-1}}{r^{l+1}} + \frac{(l+1)a^{2l+1}}{r^{l+1}a^{l+2}}\right)Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \\ &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{l-1}}{r^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\end{aligned}$$

2.7 Example

Let the charge density vanish and the potential at $r = a$ be given by

$$\Phi_S(\mathbf{x}) = A\cos\theta$$

Then the potential exterior to $r = a$ is given by

$$\begin{aligned}\Phi(\mathbf{x}) &= -\frac{A}{4\pi}\int_0^\pi\int_0^{2\pi}\cos\theta'\left[-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{l-1}}{r^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]a^2\sin\theta' d\theta' d\varphi' \\ &= A\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{l-1}}{r^{l+1}}Y_{lm}(\theta,\varphi)\int_0^\pi\int_0^{2\pi}Y_{lm}^*(\theta',\varphi')\cos\theta'\, a^2\sin\theta' d\theta' d\varphi'\end{aligned}$$

To evaluate the integrals, we rewrite $\cos\theta'$ in terms of spherical harmonics,

$$\cos\theta' = \sqrt{\frac{4\pi}{3}}Y_{10}(\theta',\varphi')$$

Then

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi} Y_{lm}^*\left(\theta',\varphi'\right)\cos\theta'\, a^2\sin\theta' d\theta' d\varphi' &= \sqrt{\frac{4\pi}{3}}\int_0^{\pi}\int_0^{2\pi} Y_{lm}^*\left(\theta',\varphi'\right) Y_{10}\left(\theta',\varphi'\right)\, a^2\sin\theta' d\theta' d\varphi' \\ &= \sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{m0}\end{aligned}$$

and we have

$$\begin{aligned}\Phi\left(\mathbf{x}\right) &= A\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{l-1}}{r^{l+1}}Y_{lm}\left(\theta,\varphi\right)\sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{m0}a^2 \\ &= \frac{Aa^2}{r^2}\cos\theta\end{aligned}$$

which clearly satisfies the required boundary conditions at $r = a$ and infinity.