

# Dispersion

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In materials, the dielectric “constant” and permeability are actually frequency dependent. This does not affect our results for single frequency modes, but when we have a superposition of frequencies it leads to dispersion. We begin with a simple model for this behavior. The variation of the permeability is often quite weak, and we may take  $\mu = \mu_0$ .

## Frequency dependence of the permittivity

Recall that our earlier treatment of the dielectric constant began with electric dipole moment produced by a static electric field  $\mathbf{E}$  in the presence of a linear restoring force,  $\mathbf{F} = -m\omega_0^2\mathbf{x}$ , to produce a molecular polarization

$$\begin{aligned}\mathbf{p}_{mol} &= e\mathbf{x} \\ &= \frac{e\mathbf{E}}{m\omega_0^2}\end{aligned}$$

Then, if there are  $N$  molecules per unit volume with  $Z$  electrons per molecule, the dipole moment per unit volume is

$$\mathbf{P} = NZ\mathbf{p}_{mol} \equiv \epsilon_0\chi_e\mathbf{E}$$

so that

$$\epsilon_0\chi_e = \frac{NZe^2}{m\omega_0^2}$$

Next, using

$$\begin{aligned}\mathbf{D} &= \epsilon_0\mathbf{E} + \mathbf{P} \\ \epsilon\mathbf{E} &= \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E}\end{aligned}$$

the dielectric constant is

$$\begin{aligned}\frac{\epsilon}{\epsilon_0} &= 1 + \chi_e \\ &= 1 + \frac{NZe^2}{m\omega_0^2\epsilon_0}\end{aligned}$$

This result changes when there is time dependence to the electric field, with the dielectric constant showing frequency dependence.

Suppose the material is sufficiently diffuse that the applied electric field is about equal to the electric field at each atom, and that the response of the atomic electrons may be modeled as harmonic. Again let  $\mathbf{x}$  represent the displacement of the charge from equilibrium, and include damping, so that now we have

$$m[\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}] = -e\mathbf{E}(\mathbf{x}, t)$$

We assume everything here – negligible magnetic effects, low-amplitude oscillations. The model is still enough to give important general features.

Let the electric field vary harmonically,

$$\mathbf{E} = \mathbf{E}(\mathbf{x}) e^{-i\omega t}$$

then the position of the electron will have the same time dependence, so

$$m [-\omega^2 - i\omega\gamma + \omega_0^2] \mathbf{x} = -e\mathbf{E}(\mathbf{x})$$

and the electric dipole moment is

$$\begin{aligned} \mathbf{p} &= -e\mathbf{x} \\ &= \frac{e^2\mathbf{E}}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \end{aligned}$$

Let there be  $N$  molecules per unit volume with  $Z$  electrons per molecule, with a fraction  $f_i$  of the electrons having binding frequency  $\omega_{i0}$  and damping  $\gamma_i$ . This is reasonable since the different electrons in each molecule are bound differently to the nucleus. The total of all the  $f_i$  should be the total number of electrons,  $\sum_i f_i = Z$ . The dipole moment for each molecule is then

$$\mathbf{p}_{mol} = \sum_i \frac{f_i e^2}{m(\omega_{i0}^2 - \omega^2 - i\omega\gamma_i)} \mathbf{E}$$

Then, since the total dipole moment per unit volume is  $\mathbf{P} = N\mathbf{p}_{mol} = \epsilon_0\chi_e\mathbf{E}$ , we have

$$\sum_i \frac{Nf_i e^2}{m(\omega_{i0}^2 - \omega^2 - i\omega\gamma_i)} \mathbf{E} = \epsilon_0\chi_e\mathbf{E}$$

Now, using  $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E}$ , the dielectric constant is

$$\begin{aligned} \frac{\epsilon}{\epsilon_0} &= 1 + \chi_e \\ &= 1 + \frac{N}{\epsilon_0} \sum \frac{f_i e^2}{m(\omega_{i0}^2 - \omega^2 - i\omega\gamma_i)} \\ &= 1 + \frac{Ne^2}{\epsilon_0 m} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \end{aligned}$$

This expression is accurate if  $f_i, \omega_{i0}$  and  $\gamma_i$  are found quantum mechanically.

## Anomalous dispersion and resonant absorption

The frequency dependence of the dielectric constant has certain regular properties. Separating real and imaginary parts,

$$\begin{aligned} \epsilon &= \epsilon_0 + \frac{Ne^2}{m} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &= \epsilon_0 + \frac{Ne^2}{m} \sum \frac{f_i(\omega_{i0}^2 - \omega^2 + i\omega\gamma_i)}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2\gamma_i^2} \end{aligned}$$

we have

$$\begin{aligned} Re \epsilon &= \epsilon_0 + \frac{Ne^2}{m} \sum \frac{f_i(\omega_{i0}^2 - \omega^2)}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2\gamma_i^2} \\ Im \epsilon &= \frac{Ne^2\omega}{m} \sum \frac{f_i\gamma_i}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2\gamma_i^2} \end{aligned}$$

and we note that the damping constant  $\gamma_i$  is usually small. At low frequencies,

$$\omega < \omega_{i0}$$

for all  $i$ , each term in the real part of  $\epsilon$  is positive and  $\epsilon > \epsilon_0$ . As the frequency increases, more and more of the terms become negative, until at high frequency,

$$Re \epsilon = \epsilon_0 - \frac{Ne^2}{m} \sum \frac{f_i |\omega^2 - \omega_{i0}^2|}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

and  $\epsilon < \epsilon_0$ .

The real and imaginary parts of  $\epsilon$  have peaks whenever  $\omega$  is near one of the resonant frequencies,  $\omega_{i0}$ , of the molecule. At these frequencies, the corresponding term of the denominator becomes

$$(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2 \rightarrow \omega^2 \gamma_i^2$$

which is very small (but larger the higher the driving frequency). This produces a resonance peak in the imaginary part of  $\epsilon$ , but since the term is also changing sign in the real part, the peak is double – positive for  $\omega < \omega_{i0}$  and negative once  $\omega > \omega_{i0}$ . At these peaks, the material is absorbing energy from the field – the damping  $\gamma_i$  plays an important role. The effect is easily seen from the wave vector  $\mathbf{k}$ .

Suppose we have a plane wave travelling in the  $z$  direction,  $\mathbf{E} = \mathcal{E} e^{i(kz - \omega t)}$ , which passes near a resonant atom. Then

$$\begin{aligned} k &= \sqrt{\mu \epsilon \omega} \\ &= \sqrt{\mu_0 \epsilon_0 \frac{Re \epsilon + i Im \epsilon}{\epsilon_0}} \omega \\ &= \frac{\omega}{c} \sqrt{\frac{Re \epsilon + i Im \epsilon}{\epsilon_0}} \end{aligned}$$

Then, setting

$$k \equiv \beta + \frac{1}{2} i \alpha$$

we have

$$\begin{aligned} \beta + \frac{1}{2} i \alpha &= \frac{\omega}{c} \sqrt{\frac{Re \epsilon + i Im \epsilon}{\epsilon_0}} \\ \beta^2 + i \alpha \beta - \frac{1}{4} \alpha^2 &= \frac{\omega^2}{c^2} \left( Re \frac{\epsilon}{\epsilon_0} + i Im \frac{\epsilon}{\epsilon_0} \right) \end{aligned}$$

so that

$$\begin{aligned} \beta^2 - \frac{1}{4} \alpha^2 &= \frac{\omega^2}{c^2} Re \frac{\epsilon}{\epsilon_0} \\ \alpha \beta &= \frac{\omega^2}{c^2} Im \frac{\epsilon}{\epsilon_0} \end{aligned}$$

and the electric field is given by

$$\begin{aligned} \mathbf{E} &= \mathcal{E} e^{i(kz - \omega t)} \\ &= \mathcal{E} e^{-\frac{1}{2} \alpha z} e^{i(\beta z - \omega t)} \end{aligned}$$

The intensity of the wave, which varies as  $\mathbf{E}^2$  then falls off as

$$e^{-\alpha z}$$

and  $\alpha$  is called the attenuation constant. We see that the attenuation is larger near each resonant frequency,  $\omega_{i0}$ , where we expect the radiation to be exciting the electrons of the molecule. Relating this back to the resonance behavior, with  $\alpha \ll \beta$ ,

$$\begin{aligned}\beta^2 &\approx \beta^2 - \frac{1}{4}\alpha^2 = \frac{\omega^2}{c^2} \operatorname{Re} \frac{\epsilon}{\epsilon_0} \\ \alpha &= \frac{\omega^2}{c^2 \beta} \operatorname{Im} \frac{\epsilon}{\epsilon_0} \\ &\approx \frac{\omega}{c \sqrt{\operatorname{Re} \frac{\epsilon}{\epsilon_0}}} \operatorname{Im} \frac{\epsilon}{\epsilon_0}\end{aligned}$$

Suppose, for simplicity, we have only a single resonance. Then with  $\omega \approx \omega_0$

$$\begin{aligned}\operatorname{Re} \frac{\epsilon}{\epsilon_0} &= 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \\ &\approx 1 \\ \operatorname{Im} \frac{\epsilon}{\epsilon_0} &= \frac{Ne^2 \omega}{m\epsilon_0} \frac{\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \\ &\approx \frac{Ne^2}{m\epsilon_0 \omega \gamma}\end{aligned}$$

so the attenuation constant is approximately

$$\alpha = \frac{Ne^2}{m\epsilon_0 \omega \gamma}$$

## Conductivity and low frequency behavior

In a conductor there are free electrons. Since these have no restoring force, they may be thought of as having a resonance frequency of zero. This makes the response of conductors at low frequency very different from that of insulators.

For insulators, the lowest resonant frequency is  $\omega_{10} > 0$  and the equations above give a good approximation to the actual response.

For conductors, there is a very strong response at zero frequency. Consider the fraction of free electrons,  $f_0$ , separately, writing

$$\begin{aligned}\epsilon &= \epsilon_0 + \frac{Ne^2}{m} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &= \epsilon_0 + \frac{Ne^2}{m} \sum_{i \neq 0} \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} + \frac{Ne^2}{m} \frac{f_0}{-\omega^2 - i\omega\gamma_0} \\ &= \epsilon_b + i \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\end{aligned}$$

To see what is happening in terms of conduction, consider the Maxwell equation involving current,

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$$

We compare two different ways of handling this expression. First, let  $\epsilon_b$  be real (so that if there are any other resonances they are far enough away in frequency to be negligible), and assume the medium satisfies Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E}$$

Then, with the harmonic applied field and a real dielectric constant,  $\epsilon_b$ , we would have

$$\begin{aligned}\nabla \times \mathbf{H} - \frac{\partial(\epsilon_b \mathbf{E})}{\partial t} &= \sigma \mathbf{E} \\ \nabla \times \mathbf{H} + i\omega \epsilon_b \mathbf{E} &= \sigma \mathbf{E} \\ \nabla \times \mathbf{H} &= (\sigma - i\omega \epsilon_b) \mathbf{E} \\ &= -i\omega \left( \epsilon_b + \frac{i\sigma}{\omega} \right) \mathbf{E}\end{aligned}$$

Now compare this to the result without current but with the conduction electrons included in the dielectric constant. Then we have instead

$$\begin{aligned}\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= 0 \\ \nabla \times \mathbf{H} &= -i\omega \epsilon \mathbf{E} \\ &= -i\omega \left( \epsilon_b + i \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \right) \mathbf{E}\end{aligned}$$

Comparing the two expressions, we see that the effect of the zero-frequency resonance is the same as a conductivity  $\sigma$  given by

$$\begin{aligned}\frac{i\sigma}{\omega} &= i \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \\ \sigma &= \frac{Ne^2 f_0}{m(\gamma_0 - i\omega)}\end{aligned}$$

a result produced in 1900 by Drude. It requires substantial correction because the free electrons actually form a Fermi gas.

## High frequency

We expand our expression for the dielectric constant for  $\omega \gg \omega_{i0}$  for all  $i$ . The denominator is approximately  $\omega_{i0}^2 - \omega^2 - i\omega\gamma_i \approx \omega^2$  and we have

$$\begin{aligned}\epsilon &= \epsilon_0 + \frac{Ne^2}{m} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &\approx \epsilon_0 - \frac{Ne^2}{m\omega^2} \sum f_i\end{aligned}$$

Replacing the sum over fractions by the atomic number,  $\sum f_i = Z$ ,

$$\begin{aligned}\frac{\epsilon}{\epsilon_0} &= 1 - \frac{NZe^2}{\epsilon_0 m \omega^2} \\ &= 1 - \frac{\omega_p^2}{\omega^2}\end{aligned}$$

where

$$\omega_p = \sqrt{\frac{NZe^2}{\epsilon_0 m}}$$

is called the plasma frequency. The plasma frequency depends only on  $NZ$ , the total number of electrons in the system. Combining with the wave number  $k$ , we have

$$k = \sqrt{\mu \epsilon \omega}$$

$$\begin{aligned}k^2 &= \frac{1}{c^2} \omega^2 \frac{\epsilon}{\epsilon_0} \\ &= \frac{\omega^2}{c^2} \left( 1 - \frac{\omega_p^2}{\omega^2} \right)\end{aligned}$$

so that

$$\omega^2 = k^2 c^2 + \omega_p^2$$

This provides a dispersion relation,  $\omega(k)$ , for plasmas. We will explore more about dispersion relations soon.

### **Example: Water**

Jackson has collected all available data (to the date of first publication, anyway) on the index of refraction and attenuation coefficient of water as functions of frequency. The graphs on page 315 are fascinating! Take some time to digest what is going on. The window in the visible range is particularly interesting.