# 1 Introduction

Review of the Introduction

1. Basic laws

   - Maxwell equations, integral form

$$\begin{aligned}
\oint_S \mathbf{D}\cdot\mathbf{n}\,da &= \int_V \rho\, d^3x \\
\oint_S \mathbf{B}\cdot\mathbf{n}\,da &= 0 \\
\oint_C \mathbf{H}\cdot d\mathbf{l} &= \int_S \left(\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}\right)\cdot\mathbf{n}\,d^2x \\
\oint_C \mathbf{E}\cdot d\mathbf{l} &= -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}\,d^2x
\end{aligned}$$

   - Maxwell equations, differential form

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho \\
\nabla\cdot\mathbf{B} &= 0 \\
\nabla\times\mathbf{H}-\frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J} \\
\nabla\times\mathbf{E}+\frac{\partial \mathbf{B}}{\partial t} &= 0
\end{aligned}$$

   - Continuity equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\mathbf{J}=0$$

   - Lorentz force law

$$\mathbf{F}=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

2. Mass of photon. The non-relativistic limit of the Klein-Gordon equation,

$$-\frac{\partial^2\varphi}{\partial t^2}+\nabla^2\varphi=\frac{m^2c^2}{\hbar^2}\varphi$$

   is the Schrödinger equation. The Klein-Gordon equation is a wave equation for a relativistic particle of mass $m$. When $m=0$, the resulting equation is

$$-\frac{\partial^2\varphi}{\partial t^2}+\nabla^2\varphi=0$$

   This equation is satisfied by the electric potential. Therefore, it is natural to assume that if the photon had mass, the electric potential would take the form of the Klein-Gordon equation. Consider the Klein-Gordon equation for a static, spherically symmetric potential. Then we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)=\frac{m^2c^2}{\hbar^2}\varphi$$

Substituting

$$\varphi(r) = \frac{A}{r}e^{-\mu r}$$

we find

$$\begin{aligned}
\frac{m^2c^2}{\hbar^2}\frac{A}{r}e^{-\mu r} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{A}{r}e^{-\mu r}\right)\right) \\
\frac{m^2c^2}{\hbar^2 r}e^{-\mu r} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{1}{r^2}e^{-\mu r} - \frac{\mu}{r}e^{-\mu r}\right)\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left((-1-\mu r)\,e^{-\mu r}\right) \\
&= \frac{1}{r^2}\left(-\mu\,(-1-\mu r)\,e^{-\mu r} - \mu e^{-\mu r}\right) \\
&= \frac{1}{r}\mu^2 e^{-\mu r}
\end{aligned}$$

so that we have a solution if $\mu = \frac{mc}{\hbar}$. Therefore, studies which place limits on the photon mass assume that the electric potential takes the form $\frac{A}{r}e^{-\mu r}$ and put limits on the value of $\mu$.

3. Linear superposition. This continues to hold in matter as long as the medium is linear

$$\begin{aligned}
D_i &= \sum_{j=1}^{3}\varepsilon_{ij}E_j \\
H_i &= \sum_{j=1}^{3}\mu'_{ij}B_j
\end{aligned}$$

4. Boundary conditions. We can use the integral form of Maxwell's equations to derive boundary conditions for the electric and magnetic fields. For the first pair of surface integrals,

$$\oint_S \mathbf{D}\cdot\mathbf{n}\,da, \oint_S \mathbf{B}\cdot\mathbf{n}\,da$$

we imagine a cylindrical volume perpendicular to the interface between two materials. The materials are characterized by the relationship between $(\mathbf{D},\mathbf{H})$ and $(\mathbf{E},\mathbf{B})$. With the cylinder piercing the boundary so that its curved sides are infinitesimally high, $\delta$, the finite part of the surface integrals is given by the flat end caps. We therefore find

$$\lim_{\delta\to 0}\oint_S \mathbf{D}\cdot\mathbf{n}\,da = (\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n}A$$

where $A$ is the area of each end cap. In the same limit, the volume integral of the charge density becomes

$$\begin{aligned}
\lim_{\delta\to 0}\int_V \rho\,d^3x &= \lim_{\delta\to 0}\rho A\delta \\
&= \sigma A
\end{aligned}$$

where $\sigma$ is the charge per unit area on the boundary surface. Therefore, we have

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma$$

so that the normal component of **D** has a discontinuity equal to the surface charge density, $\sigma$. The same result holds for **B** except there can be no magnetic charge density, so that the normal component of the magnetic induction **B** is continuous,

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0$$

For the remaining two equations, we take the curve $C$ to be a rectangle with two sides, **l**, parallel to the surface and two short sides (of length $\delta$) through the surface. Then the contour integrals become

$$\begin{aligned} \lim_{\delta \to 0} \oint_C \mathbf{H} \cdot d\mathbf{l} &= (\mathbf{H_2} - \mathbf{H_1}) \cdot \mathbf{l} \\ \lim_{\delta \to 0} \oint_C \mathbf{E} \cdot d\mathbf{l} &= (\mathbf{E_2} - \mathbf{E_1}) \cdot \mathbf{l} \end{aligned}$$

Because **l** is parallel to the surface, we get a relation for the tangential components of the electric and magnetic fields. For the right side of these equations, we have

$$\begin{aligned} \lim_{\delta \to 0} \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n} \, d^2x &= \lim_{\delta \to 0} \int_S \mathbf{J} \cdot \mathbf{n} \, d^2x + \lim_{\delta \to 0} \left[ \left( \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n} \right) l\delta \right] \\ &= \mathbf{K} \cdot \mathbf{l} + 0 \\ &= K \end{aligned}$$

where $\mathbf{K} \cdot \mathbf{l} = Kl$ is the surface current density and , and $\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n}$ is finite at the surface. For the magnetic induction term,

$$\begin{aligned} \lim_{\delta \to 0} \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, d^2x &= \lim_{\delta \to 0} \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) l\delta \\ &= 0 \end{aligned}$$

and we have

$$\begin{aligned} (\mathbf{H_2} - \mathbf{H_1}) \cdot \mathbf{l} &= Kl \\ (\mathbf{E_2} - \mathbf{E_1}) \cdot \mathbf{l} &= 0 \end{aligned}$$

Using the normal to the surface, we can write these as vector equations,

$$\begin{aligned} \mathbf{n} \times (\mathbf{H_2} - \mathbf{H_1}) &= \mathbf{K} \\ \mathbf{n} \times (\mathbf{E_2} - \mathbf{E_1}) &= 0 \end{aligned}$$

We can do this because the result is independent of the orientation of the vector **l**, as long as it is parallel to the surface.

5. Idealizations: Our volume elements will be taken as microscopically large and macroscopically small. This allows charge distributions to be taken as continuous functions of position instead of discrete charges at isolated locations.

6. Dirac delta function

# 2 Chapter 1

We need a purely mathematical result: Green's theorem

## 2.1 Green's Theorem

First, we establish a simple vector calculus identity,

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g) = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f\boldsymbol{\nabla}^2 g$$

This relation is not hard to prove. We can just expand the del operator in Cartesian coordinates,

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

Substituting in the left hand side,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) \\
&= \hat{\mathbf{i}}\frac{\partial}{\partial x} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) + \hat{\mathbf{j}}\frac{\partial}{\partial y} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} +\right.\right.
\end{aligned}$$

It is easiest to take the dot products first. Since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are independent of position (unlike, say, $\hat{\boldsymbol{r}}$,$\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$), we can bring them inside the derivative. Then we have

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g) &= \hat{\mathbf{i}}\frac{\partial}{\partial x} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) + \hat{\mathbf{j}}\frac{\partial}{\partial y} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z} \cdot \left(f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} +\right.\right. \\
&= \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right) \\
&= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) \\
&= \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f\boldsymbol{\nabla}^2 g
\end{aligned}$$

Since the dot product, gradient and Laplacian are vector operators which are independent of coordinate system,

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g) = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f\boldsymbol{\nabla}^2 g$$

must now hold in any coordinates, not just Cartesian.

Now, to prove Green's first identity, we begin with the divergence theorem:

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{A}\,d^3x = \oint_S \mathbf{A}\cdot\mathbf{n}\,d^2x$$

This holds for any vector field $\mathbf{A}$. Let the vector field $\mathbf{A}$ have the particular form

$$\mathbf{A} = f\boldsymbol{\nabla}g$$

where $f(\mathbf{x}), g(\mathbf{x})$ are any two functions of position. Substituting,

$$\begin{aligned}
\int_V \boldsymbol{\nabla}\cdot\mathbf{A}\,d^3x &= \oint_S \mathbf{A}\cdot\mathbf{n}\,d^2x \\
\int_V \boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}g)\,d^3x &= \oint_S (f\boldsymbol{\nabla}g)\cdot\mathbf{n}\,d^2x \\
\int_V \left(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\boldsymbol{\nabla}^2 g\right)d^3x &= \oint_S (f\boldsymbol{\nabla}g)\cdot\mathbf{n}\,d^2x
\end{aligned}$$

where we used the relation derived above. This is Green's first identity.

Suppose we pick some $f$ and $g$. Then we know that

$$\int_V \left(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\boldsymbol{\nabla}^2 g\right)d^3x = \oint_S (f\boldsymbol{\nabla}g)\cdot\mathbf{n}\,d^2x$$

However, since $f$ and $g$ are arbitrary, we may also write

$$\int_V \left(\boldsymbol{\nabla}g\cdot\boldsymbol{\nabla}f + g\boldsymbol{\nabla}^2 f\right)d^3x = \oint_S (g\boldsymbol{\nabla}f)\cdot\mathbf{n}\,d^2x$$

for the same $f$ and $g$. Taking the difference of these two expressions (and using the symmetry of the dot product $\boldsymbol{\nabla}g\cdot\boldsymbol{\nabla}f = \boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g$), we have

$$\int_V \left(f\boldsymbol{\nabla}^2 g - g\boldsymbol{\nabla}^2 f\right)d^3x = \oint_S (f\boldsymbol{\nabla}g - g\boldsymbol{\nabla}f)\cdot\mathbf{n}\,d^2x$$

Since $\mathbf{n}$ is the outward unit normal to the surface $S$ bounding the volume $V$

$$\mathbf{n}\cdot\boldsymbol{\nabla}g = \frac{\partial g}{\partial n}$$

is just the derivative of $g$ normal to $S$. We can therefore simplify the notation a little,

$$\int_V \left(f\boldsymbol{\nabla}^2 g - g\boldsymbol{\nabla}^2 f\right)d^3x = \oint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial g}{\partial n}\right)d^2x$$

This is Green's theorem.

## 2.2 Example of the use of Green's theorem

As an example of the use of Green's theorem, we solve problem 10 from Jackson

**.J1.10 Mean value theorem**

Prove that

$$\Phi\left(\mathcal{P}\right) = \oint_{Sphere} \Phi\left(x\right) d^2x$$

in charge-free space, where the integral is over any sphere centered on the point $\mathcal{P}$.

Start wtih Green's theorem

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right) d^3x = \oint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d^2x$$

Choose $f = \Phi$ and $g = \frac{1}{r}$. Then

$$\int_V \left(\Phi\nabla^2\left(\frac{1}{r}\right) - \frac{1}{r}\nabla^2\Phi\right) d^3x = \oint_S \left(\Phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial n}\right) d^2x$$

Since we are in charge-free space we know that

$$\nabla^2\Phi = 0$$

and we also have

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3\left(r\right)$$

Substituting,

$$-4\pi\int_V \delta^3\left(r\right)\Phi d^3x = \oint_S \left(\Phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial n}\right) d^2x$$

Now, with the surface $S$ a sphere centered on $r = 0$, we have

$$\frac{\partial}{\partial n}\frac{1}{r} = \hat{\mathbf{r}}\cdot\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}\right)\frac{1}{r} = -\frac{1}{r^2}$$

Finally, the last term may be evaluated using Gauss's law:

$$\begin{aligned}
\oint_S \frac{1}{r}\frac{\partial\Phi}{\partial n} d^2x &= -\oint_S \frac{1}{r}\mathbf{E}\cdot\hat{\mathbf{r}} d^2x \\
&= -\frac{1}{R}\oint_S \mathbf{E}\cdot\hat{\mathbf{r}} d^2x \\
&= -\frac{Q_{tot}}{\varepsilon_0 R} \\
&= 0
\end{aligned}$$

Putting this all together, and carrying out the integral over the delta function,

$$\begin{aligned}
-4\pi\int_V \delta^3\left(r\right)\Phi d^3x &= \oint_S \left(\Phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial n}\right) d^2x \\
-4\pi\Phi\left(0\right) &= \oint_S \left(-\frac{1}{r^2}\right)\Phi d^2x \\
\Phi\left(0\right) &= \frac{1}{4\pi r^2}\oint_S \Phi d^2x
\end{aligned}$$

The right hand side is the average of the potential over the sphere, and the left side is the potential at the center.

## 2.3 Uniqueness of solutions to the Poisson equation

We want to establish the uniqueness of solutions to our electrostatic problems,

$$\nabla^2 \Phi = \frac{\rho}{\varepsilon_0}$$

for a fixed source distribution, $\rho(\mathbf{x})$. However, we know that for any $\Phi(\mathbf{x})$ satisfying this equation, we may add a solution, $\Phi'(\mathbf{x})$, of the homogeneous (Laplace) equation,

$$\nabla^2 \Phi'(\mathbf{x}) = 0$$

Recall what happens with Newton's 2nd law in mechanics. We have a second order differential equation,

$$\mathbf{F}(\mathbf{x}, t) = m \frac{d^2 \mathbf{x}(t)}{dt^2}$$

which also does not have unique solutions. It is only when we add *initial conditions* for the position and velocity, $\mathbf{x}(0), \mathbf{v}(0)$ that the solution becomes unique and the motion is completely determined.

For electrostatic field theory, the situation is similar but does not involve time evolution. Instead, we specify *boundary conditions* for the solution. Thus, we solve the Poisson equation in a volume, $V$, then restrict the class of solutions to those satisfying certain conditions on the boundary $S$ of $V$. We now prove that this gives a unique solution. We give a proof by two solutions satisfying the same boundary conditions must be the same.

Suppose we have two solutions, $\Phi_1, \Phi_2$, satisfying the Poisson equation on $V$, and satisfying some specific condition on the boundary $S$. The boundary condition may be either the value of the potential on $S$ or the value of the normal derivative, $\frac{\partial \Phi}{\partial n}$, on $S$. In either case, the difference of the two potentials,

$$U = \Phi_2 - \Phi_1$$

must satisfy the Poisson equation as well, but will have vanishing boundary values, i.e., either

$$U(\mathbf{x})\big|_{x \in S} = 0$$

or

$$\left.\frac{\partial U(\mathbf{x})}{\partial n}\right|_{x \in S} = 0$$

Now use Green's first identity with $f = g = U$:

$$\begin{aligned}
\int_V \left( \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \boldsymbol{\nabla}^2 g \right) d^3x &= \oint_S f \frac{\partial g}{\partial n} n \, d^2x \\
\int_V \left( \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U + U \boldsymbol{\nabla}^2 U \right) d^3x &= \oint_S U \frac{\partial U}{\partial n} \, d^2x
\end{aligned}$$

The boundary condition for $U$ makes the right side vanish, while on the left side we have

$$\begin{aligned}
\boldsymbol{\nabla}^2 U &= \boldsymbol{\nabla}^2 \left(\Phi_2 - \Phi_1\right) \\
&= \boldsymbol{\nabla}^2 \Phi_2 - \boldsymbol{\nabla}^2 \Phi_1 \\
&= \frac{1}{\varepsilon_0}\rho - \frac{1}{\varepsilon_0}\rho \\
&= 0
\end{aligned}$$

The identity reduces to

$$\int_V \left(\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\right) d^3x = \int_V \left|\boldsymbol{\nabla} U\right|^2 d^3x = 0$$

This implies $\boldsymbol{\nabla} U = 0$, so that $U = c$. Only the value of this constant remains unspecified, and the field, $E = -\boldsymbol{\nabla}\Phi$ is unique.

Since the result holds for either $U = 0$ (Dirichlet boundary conditions) or $\frac{\partial U}{\partial n} = 0$ (Neumann boundary conditions) on the boundary, it would overspecify the solution to try to impose both conditions. It is possible, however, to specify $U = 0$ on one part of $S$, and $\frac{\partial U}{\partial n} = 0$ on the remaining part, as long as the right side vanishes.

## 2.4 Green Functions and the formal solution