1 Complex analysis

1.1 Cauchy Riemann equations

We would like to differentiate and integrate using complex numbers. However, the complex numbers lie in a plane, so there are two independent directions (and any linear combination of these) from which we may take limits. These different limits must agree.

Specifically, we define

$$\frac{df}{dz} = \lim_{\varepsilon \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}$$

where ε is an arbitrary complex number. If we let $\varepsilon = (a+bi)\delta$ and set

$$\begin{aligned} z &= x + iy \\ f(z) &= u(x,y) + iv(x,y) \end{aligned}$$

then we have

$$\begin{aligned} \frac{df}{dz} &= \lim_{\varepsilon \to 0} \frac{u(x+a\delta, y+b\delta) + iv(x+a\delta, y+b\delta) - u(x,y) - iv(x,y)}{(a+bi)\delta} \\ &= \frac{1}{a^2+b^2} \lim_{\varepsilon \to 0} (a-bi) \frac{u(x+a\delta, y+b\delta) + iv(x+a\delta, y+b\delta) - u(x,y) - iv(x,y)}{\delta} \\ &= \frac{1}{a^2+b^2} \lim_{\varepsilon \to 0} \frac{au(x+a\delta, y+b\delta) + bv(x+a\delta, y+b\delta) - au(x,y) - bv(x,y)}{\delta} \\ &\quad + \frac{i}{a^2+b^2} \lim_{\varepsilon \to 0} \frac{-bu(x+a\delta, y+b\delta) + av(x+a\delta, y+b\delta) + bu(x,y) - av(x,y)}{\delta} \\ &= \frac{1}{a^2+b^2} \left(a \lim_{\varepsilon \to 0} \frac{u(x+a\delta, y+b\delta) - u(x,y)}{\delta} + b \lim_{\varepsilon \to 0} \frac{v(x+a\delta, y+b\delta) - v(x,y)}{\delta} \right) \\ &\quad + \frac{i}{a^2+b^2} \left(a \lim_{\varepsilon \to 0} \frac{v(x+a\delta, y+b\delta) - v(x,y)}{\delta} - b \lim_{\varepsilon \to 0} \frac{u(x+a\delta, y+b\delta) - u(x,y)}{\delta} \right) \\ &= \frac{1}{a^2+b^2} \left(a^2 \frac{\partial u}{\partial x} + ab \frac{\partial u}{\partial y} + ab \frac{\partial v}{\partial x} + b^2 \frac{\partial v}{\partial y} \right) \\ &\quad + \frac{i}{a^2+b^2} \left(a^2 \frac{\partial v}{\partial x} + ab \frac{\partial v}{\partial y} - ab \frac{\partial u}{\partial x} - b^2 \frac{\partial u}{\partial y} \right) \end{aligned}$$

Now require

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned}$$

Then

$$\begin{aligned} \frac{df}{dz} &= \frac{1}{a^2+b^2} \left(a^2 \frac{\partial u}{\partial x} + b^2 \frac{\partial v}{\partial y} \right) + \frac{i}{a^2+b^2} \left(a^2 \frac{\partial v}{\partial x} - b^2 \frac{\partial u}{\partial y} \right) \\ &= \frac{1}{a^2+b^2} \left(a^2 \frac{\partial u}{\partial x} + b^2 \frac{\partial u}{\partial x} - ia^2 \frac{\partial u}{\partial y} - ib^2 \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \end{aligned}$$

which is independent of a, b and therefore independent of the direction in which we take the limit. The necessary and sufficient conditions for a well-defined derivative are therefore,

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned}$$

and these are called the Cauchy-Riemann conditions.

To prove necessity, we need:

$$\begin{aligned} 0 &= \frac{\partial}{\partial a}\left(\frac{1}{a^2+b^2}\left(a^2\frac{\partial u}{\partial x}+ab\frac{\partial u}{\partial y}+ab\frac{\partial v}{\partial x}+b^2\frac{\partial v}{\partial y}\right)\right) \\ &= -\frac{2a}{(a^2+b^2)^2}\left(a^2\frac{\partial u}{\partial x}+ab\frac{\partial u}{\partial y}+ab\frac{\partial v}{\partial x}+b^2\frac{\partial v}{\partial y}\right)+\frac{1}{a^2+b^2}\left(2a\frac{\partial u}{\partial x}+b\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right) \\ 0 &= -\frac{2a}{a^2+b^2}\left(a^2\frac{\partial u}{\partial x}+ab\frac{\partial u}{\partial y}+ab\frac{\partial v}{\partial x}+b^2\frac{\partial v}{\partial y}\right)+\left(2a\frac{\partial u}{\partial x}+b\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right) \\ 0 &= 2a\left(1-\frac{a^2}{a^2+b^2}\right)\frac{\partial u}{\partial x}+b\left(1-\frac{2a^2}{a^2+b^2}\right)\frac{\partial u}{\partial y}+b\left(1-\frac{2a^2}{a^2+b^2}\right)\frac{\partial v}{\partial x}-\frac{2ab^2}{a^2+b^2}\frac{\partial v}{\partial y} \end{aligned}$$

and similarly,

$$\begin{aligned} 0 &= \frac{\partial}{\partial b}\left(\frac{1}{a^2+b^2}\left(a^2\frac{\partial u}{\partial x}+ab\frac{\partial u}{\partial y}+ab\frac{\partial v}{\partial x}+b^2\frac{\partial v}{\partial y}\right)\right) \\ &= -\frac{2b}{(a^2+b^2)^2}\left(a^2\frac{\partial u}{\partial x}+ab\frac{\partial u}{\partial y}+ab\frac{\partial v}{\partial x}+b^2\frac{\partial v}{\partial y}\right)+\frac{1}{a^2+b^2}\left(a\frac{\partial u}{\partial y}+a\frac{\partial v}{\partial x}+2b\frac{\partial v}{\partial y}\right) \\ 0 &= -\frac{2b}{a^2+b^2}a^2\frac{\partial u}{\partial x}+a\left(1-\frac{2b^2}{a^2+b^2}\right)\frac{\partial u}{\partial y}+a\left(1-\frac{2b^2}{a^2+b^2}\right)\frac{\partial v}{\partial x}+2b\left(1-\frac{b^2}{a^2+b^2}\right)\frac{\partial v}{\partial y} \end{aligned}$$

Add b times the second to a times the first,

$$\begin{aligned} 0 &= 2a^2\left(1-\frac{a^2}{a^2+b^2}\right)\frac{\partial u}{\partial x}+ab\left(1-\frac{2a^2}{a^2+b^2}\right)\frac{\partial u}{\partial y}+ab\left(1-\frac{2a^2}{a^2+b^2}\right)\frac{\partial v}{\partial x}-\frac{2a^2b^2}{a^2+b^2}\frac{\partial v}{\partial y} \\ &\quad -\frac{2a^2b^2}{a^2+b^2}\frac{\partial u}{\partial x}+ab\left(1-\frac{2b^2}{a^2+b^2}\right)\frac{\partial u}{\partial y}+ab\left(1-\frac{2b^2}{a^2+b^2}\right)\frac{\partial v}{\partial x}+2b^2\left(1-\frac{b^2}{a^2+b^2}\right)\frac{\partial v}{\partial y} \\ &= ab\left(\frac{b^2-a^2}{a^2+b^2}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) \\ &= 0 \end{aligned}$$

so there is only one condition, which must be independent of both a and b. Thus, rearranging the first equation,

$$0 = \frac{2ab^2}{a^2+b^2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+b\left(\frac{b^2-a^2}{a^2+b^2}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

and since a and b are independent, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

Now, we have

$$\begin{aligned}\frac{df}{dz} &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\end{aligned}$$

Now let $F(x) = \frac{df}{dz}$ and consider the Cauchy-Riemann conditions for $F = U + iV$,

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x}\end{aligned}$$

Substituting for U and V,

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\ \frac{\partial}{\partial x}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\frac{\partial v}{\partial x}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x} \\ \frac{\partial}{\partial y}\frac{\partial u}{\partial x} &= -\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)\end{aligned}$$

we see that the Cauchy-Riemann conditions for F are identically satisfied by the equality of mixed partials,

$$\begin{aligned}\frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial y \partial x} &= \frac{\partial^2 v}{\partial x \partial y}\end{aligned}$$

Therefore, the second derivative of $f(z)$ exists whenever the first derivative exists, provided only that the separate real or imaginary parts, u, v are similarly differentiable. Since we may repeat this arguement ad infinitum, $f(z)$ satisfies the Cauchy-Riemann conditions at all orders if and only if u and v are C^∞ functions.

1.2 Analytic extension

Consider any real values function with a convergent Taylor series in x,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n x^n$$

Then we define the *analytic extension* of f to be the complex valued function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n z^n$$

This is always convergent since we may write z in polar notation,

$$\begin{aligned}f(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n e^{in\phi} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi + i \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \sin n\phi\end{aligned}$$

For each n, we have

$$\left|\frac{1}{n!}a_n r^n \cos n\phi\right| < \frac{1}{n!}a_n r^n$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \cos n\phi &< \sum_{n=0}^{\infty}\left|\frac{1}{n!}a_n r^n \cos n\phi\right| \\ &< \sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \\ &= f(r)\end{aligned}$$

and similarly for the imaginary part. Furthermore, $f(z)$ satisfies the Cauchy-Riemann conditions, since with $f(z) = u(r,\phi) + iv(r,\phi)$

$$\begin{aligned}u &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \cos n\phi \\ v &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \sin n\phi\end{aligned}$$

For this we need the Caucy-Riemann conditions in polar coordinates,

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\ &= \frac{x}{r}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\end{aligned}$$

We can find the partials from the coordinate transformation,

$$\begin{aligned}\tan\phi &= \frac{y}{x} \\ \frac{1}{\cos^2\phi}d\phi &= \frac{xdy - ydx}{x^2} \\ \frac{x^2}{\cos^2\phi}d\phi &= xdy - ydx \\ r^2 d\phi &= xdy - ydx \\ d\phi &= \frac{x}{r^2}dy - \frac{y}{r^2}dx \\ d\phi &= \frac{\cos\phi}{r}dy - \frac{\sin\phi}{r}dx\end{aligned}$$

so we see that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= -\frac{\sin\phi}{r} \\ \frac{\partial\phi}{\partial y} &= \frac{\cos\phi}{r}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial y} &= \sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial\phi}\end{aligned}$$

and the Cauchy-Riemann conditions become

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \cos\phi\frac{\partial u}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial u}{\partial \phi} &= \sin\phi\frac{\partial v}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial v}{\partial \phi}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \sin\phi\frac{\partial u}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial u}{\partial \phi} &= -\cos\phi\frac{\partial v}{\partial r} + \frac{1}{r}\sin\phi\frac{\partial v}{\partial \phi}\end{aligned}$$

Now substitute for u and v. In the first,

$$\begin{aligned}\cos\phi\frac{\partial u}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial u}{\partial \phi} &= \left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial \phi}\right)\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \cos n\phi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n\left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial \phi}\right) r^n\cos n\phi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n\left(nr^{n-1}\cos\phi\cos n\phi + nr^{n-1}\sin\phi\sin n\phi\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{(n-1)!}a_n r^{n-1}\cos(n-1)\phi \\ \sin\phi\frac{\partial v}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial v}{\partial \phi} &= \left(\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial \phi}\right)\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \sin n\phi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n\left(nr^{n-1}\sin\phi\sin n\phi + nr^{n-1}\cos\phi\cos n\phi\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{(n-1)!}a_n r^{n-1}\cos(n-1)\phi\end{aligned}$$

and these are indeed equal. For the second,

$$\begin{aligned}\cos\phi\frac{\partial v}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial v}{\partial \phi} &= \left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial \phi}\right)\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \sin n\phi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n nr^{n-1}\left(\cos\phi\sin n\phi - \sin\phi\cos n\phi\right) \\ \sin\phi\frac{\partial u}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial u}{\partial \phi} &= \left(\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial \phi}\right)\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n \cos n\phi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n nr^{n-1}\left(\sin\phi\cos n\phi - \cos\phi\sin n\phi\right)\end{aligned}$$

and these are negatives of one another as expected.

Therefore, analytic extension of any real Taylor series gives an analytic function $f(z)$. Conversely, any complex Taylor series gives an analytic function.

Exercise: Show that the composition of two analytic functions is analytic. That is, if $f(z)$ and $g(w)$ both satisfy the Cauchy-Riemann conditions, show that $g(f(z))$ also satisfies the Cauchy-Riemann conditions.

1.3 Contour Integrals

Now consider a function $f(z)$ with derivatives of all orders in some region of the complex plane, and consider the integral of $f(z)$ around a closed curve, C,

$$\oint_C f(z)\,dz$$

We may expand this as a pair of functions of two variables,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C (u(x,y) + iv(x,y))(dx + idy) \\ &= \oint_C (udx + iudy + ivdx - vdy) \\ &= \oint_C ((u+iv)\,dx + i(u+iv)\,dy) \\ &= \oint_C (udx - vdy) + i\oint_C (udy + vdx)\end{aligned}$$

Think of $\vec{\mathbf{u}} = (u, -v, 0)$ as a vector field in R^3 and $d\vec{\mathbf{x}} = (dx, dy, dz)$ as an infinitesimal displacement. Then we can use Stoke's theorem. The first integral becomes

$$\begin{aligned}\oint_C (udx - vdy) &= \oint_C \vec{\mathbf{u}}\cdot d\vec{\mathbf{x}} \\ &= \oint_C (\nabla\times\vec{\mathbf{u}})\cdot\hat{\mathbf{n}}d^2x\end{aligned}$$

where the normal is in the z-direction. The curl of $\vec{\mathbf{u}}$, however, is

$$\begin{aligned}\nabla\times\vec{\mathbf{u}} &= \hat{\mathbf{i}}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) \\ &= \hat{\mathbf{k}}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) \\ &= \hat{\mathbf{k}}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &= 0\end{aligned}$$

where the result vanishes by the Cauchy-Riemann conditions. For the second integral, let $\vec{\mathbf{w}} = (v, u, 0)$

$$\begin{aligned}\oint_C (udy + vdx) &= \oint_C \vec{\mathbf{w}}\cdot d\vec{\mathbf{x}} \\ &= \oint_C (\nabla\times\vec{\mathbf{w}})\cdot\hat{\mathbf{n}}d^2x\end{aligned}$$

and the curl is

$$\nabla\times\vec{\mathbf{w}} = \hat{\mathbf{i}}\left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial w_z}{\partial x} - \frac{\partial w_x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x}\right)$$

$$
\begin{aligned}
&= \hat{\mathbf{k}}\left(\frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x}\right) \\
&= \hat{\mathbf{k}}\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) \\
&= 0
\end{aligned}
$$

using the other Riemann-Cauchy condition.

Therefore, for any analytic function f, we have

$$\oint_C f(z)\, dz = 0$$

If two curves share a common segment, then we can add the curves together to get a larger curve. Starting with a given curve, we can therefore imagine adding a small second loop in such a way that the combined contour is slightly altered from the first. This is called a deformation of the contour, and it will not change the value of the integral as long as the small loop we add lies entirely within a region where f is analytic.

1.4 The Residue Theorem

We can use this result to simplify integrals where the function is not analytic in the entire complex plane. Suppose a function is analytic everywhere except a single point, z_0. Then in addition to a Taylor series for the function, there may be an expansion which includes poles at z_0,

$$\frac{1}{(z - z_0)^n}$$

Such terms are fine away from the point z_0, so they do not affect analticity elsewhere. Consider the class of functions which have a Laurent series, i.e., for some finite number N, the function may be expressed as

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n$$

This has poles of orders $1, 2, \ldots, N$. Since the mapping $w = g(z) = z - z_0$ is analytic, we might as well write this as

$$f(w) = \sum_{n=-N}^{\infty} a_n w^n$$

where the poles are now at $w = 0$. Now consider a contour integral of the form

$$\oint_C f(w)\, dw$$

for any closed curve C. Since f is analytic everywhere except the origin, the integral vanishes if C does not enclose the origin. If C does include the origin, we may deform C until it is a circle of radius R about the origin, and the deformation does not affect the value of the integral. Then on the circle $dw = d\left(Re^{i\phi}\right) = iRe^{i\phi} d\phi$ so we have

$$
\begin{aligned}
\oint_C f(w)\, dw &= \int_0^{2\pi} \sum_{n=-N}^{\infty} a_n R^n e^{in\phi} \left(iRe^{i\phi} d\phi\right) \\
&= i \sum_{n=-N}^{\infty} a_n R^{n+1} \int_0^{2\pi} e^{i(n+1)\phi} d\phi
\end{aligned}
$$

$$
\begin{aligned}
&= i \sum_{n \neq -1}^{\infty} a_n R^{n+1} \left. \frac{e^{i(n+1)\phi}}{i(n+1)} \right|_0^{2\pi} + i a_{-1} \int_0^{2\pi} d\phi \\
&= i \sum_{n=-N}^{\infty} a_n R^n \frac{1}{i(n+1)} \left(e^{2\pi i (n+1)\phi} - 1 \right) + a_0 \int_0^{2\pi} d\phi \\
&= 2\pi i a_{-1}
\end{aligned}
$$

We see that the integral depends only on the coefficient of the simple pole (i.e., the pole of order 1). This coefficient is called the residue of f at z_0, and we write

$$
\begin{aligned}
Res\left(f(z)\right) &= Res\left(\sum_{n=-N}^{\infty} a_n (z - z_0)^n \right) \\
&= a_{-1}
\end{aligned}
$$

The residue theorem now states that the integral of a complex function about a pole equals $2\pi i$ times the residue of the function at the pole. If there are multiple poles, the result is the sume of the residues at all poles included within the contour C. Thus, the residue theorem becomes

$$
\oint_C f(w)\, dw = 2\pi i \sum Res(f)
$$

where the sum is over all poles included within C.

1.5 Example: Completeness relation for Fourier integrals

Suppose we can expand a function $f(\mathbf{x})$ as

$$
f(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}} d^3k
$$

We would like to show that this transformation is invertible, and this requires the completeness relation for Fourier transformations. To see this, consider inverting the transformation. Multiply both sides by $e^{i\mathbf{k}'\cdot\mathbf{x}}$ and integrate over all $\mathbf{x}$,

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mathbf{x})\, e^{i\mathbf{k}'\cdot\mathbf{x}} d^3x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mathbf{k}'\cdot\mathbf{x}} d^3x \int_{-\infty}^{\infty} g(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}} d^3k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\mathbf{k})\, e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}} d^3k\, d^3x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\mathbf{k})\, d^3k \int_{-\infty}^{\infty} e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}\, d^3x
\end{aligned}
$$

We desire the result of this integration to be the transform, $g(\mathbf{k})$, and this will be true if and only if

$$
\delta^3\left(\mathbf{k} - \mathbf{k}'\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}\, d^3x
$$

or equivalently

$$\delta^3(\mathbf{k}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3x$$

Using Cartesian coordinates, this breaks into three identical integrals of the form

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ikx}\, dx$$

which we may use contour integration to evaluate.

Our goal is to show that this integral is a Dirac delta function, which means that for any test function $g(k)$ (i.e., $g(k)$ is bounded, as differentiable as we like, and vanishes outside a compact region),

$$g(0) = \int\limits_{-\infty}^{\infty} g(k)\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ikx}\, dx\right] dk$$

Replace the infinite limit on the inner integral by R. We will let $R \to \infty$ at the end of the calculation. Then, carrying out the integral of the exponential,

$$\lim_{R\to\infty}\int\limits_{-\infty}^{\infty} g(k)\left[\frac{1}{2\pi}\int\limits_{-R}^{R} e^{-ikx}\, dx\right] dk = \lim_{R\to\infty}\int\limits_{-\infty}^{\infty} g(k)\left[-\frac{1}{2\pi ik}\left(e^{-ikR} - e^{ikR}\right)\right] dk$$

We can carry this out using contour integration.

In order to use contour integration, we need to enclose the simple pole at $k = 0$ with a curve. There are two problems. First, the pole here lies directly on the path of integration. We solve this difficulty with a trick: displace the pole slightly, then do the integral, then take the limit as the displacement vanishes. Specifically, let ε be an arbitrary positive real number and write the integral as

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} g(k)\left[-\frac{1}{2\pi ik}\left(e^{-ikR} - e^{ikR}\right)\right] dk &= \lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty} g(k)\left[\frac{1}{2\pi i(k - i\varepsilon)}\left(e^{ikR} - e^{-ikR}\right)\right] dk \\
&= \lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty} \frac{g(k)\, e^{ikR}}{2\pi i(k - i\varepsilon)} dk - \lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty} \frac{g(k)\, e^{-ikR}}{2\pi i(k - i\varepsilon)} dk
\end{aligned}$$

The second problem is to complete a closed curve without changing the value of the integral. We begin by analytically extending the integration variable k to a complex variable, $k = k_R + ik_I$. The first integral is then

$$\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty} \frac{g(k)\, e^{ikR}}{2\pi i(k - i\varepsilon)} = \lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty} \frac{g(k_R + ik_I)\, e^{ik_R R} e^{-k_I R}}{2\pi i(k_R + ik_I - i\varepsilon)}$$

and we see that if $k_I > 0$ the integrand is suppressed by $e^{-k_I R}$. If we close the contour by adding a semicircle in the upper half plane of radius κ, then at any angle ϕ on the semicircle the imaginary component k_I is given by $k_I = \kappa \sin\phi$. As the radius tends to infinity, $\kappa \to \infty$, this diverges and the exponential factor $e^{-k_I R}$ tends to zero. This means that the integrand vanishes on this upper semicircle and we can integrate over a closed curve C which runs along the entire real k axis and returns on the semicircle, without changing the value of the integral,

$$\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty} \frac{g(k_R + ik_I)\, e^{i(k_R + ik_I)R}}{2\pi i(k_R + ik_I - i\varepsilon)} = \lim_{\varepsilon\to 0}\oint\limits_{C} \frac{g(k)\, e^{ikR}}{2\pi i(k - i\varepsilon)} dk$$

We may now apply the Residue Theorem. The contour is integrated in the positive sense, i.e., counterclockwise, and encloses the simple pole at $k = i\varepsilon$, so the residue is taken there

$$\begin{aligned}
\lim_{\varepsilon\to 0}\oint_C \frac{g(k)\,e^{ikR}}{2\pi i\,(k-i\varepsilon)}dk &= \lim_{\varepsilon\to 0} 2\pi i Res\left(\frac{g(k)\,e^{ikR}}{2\pi i\,(k-i\varepsilon)}\right) \\
&= \lim_{\varepsilon\to 0} 2\pi i\left(\frac{g(i\varepsilon)\,e^{-\varepsilon R}}{2\pi i}\right) \\
&= \lim_{\varepsilon\to 0}\left(g(i\varepsilon)\,e^{-\varepsilon R}\right) \\
&= g(0)
\end{aligned}$$

The second integral is handled in the same way,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(k)\,e^{-ikR}}{2\pi i\,(k-i\varepsilon)}dk = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(k_R+ik_I)\,e^{-ik_R R}e^{+k_I R}}{2\pi i\,(k_R+ik_I-i\varepsilon)}$$

There is one important difference. The exponential factor is now $e^{k_I R}$, which converges only when $k_I < 0$. This means that we must close the contour, C', in the lower half plane. We now have a clockwise contour, running along the entire real k axis then circling back along a semicircle in the lower half plane. We pick up a minus sign because of the direction of the contour, but more importantly, the shifted pole no longer lies inside the contour. Since the integrand lies in a region containing no poles it is analytic and the second integral vanishes.

Returning to the original problem, we have

$$\begin{aligned}
\lim_{R\to\infty}\int_{-\infty}^{\infty} g(k)\left[-\frac{1}{2\pi ik}\left(e^{-ikR}-e^{ikR}\right)\right]dk &= \lim_{R\to\infty}\int_{-\infty}^{\infty} g(k)\,\frac{1}{2\pi ik}e^{ikR}dk - \lim_{R\to\infty}\int_{-\infty}^{\infty} g(k)\,\frac{1}{2\pi ik}e^{-ikR}dk \\
&= \lim_{R\to\infty} g(0) \\
&= g(0)
\end{aligned}$$

and we have established that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dx = \delta(k)$$

This shows the completeness of Fourier integrals.