Magnetostatics and the vector potential

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1 The divergence of the magnetic field

Starting with the general form of the Biot-Savart law,

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}) \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} d^3x$$

we take the divergence of both sides with respect to $\mathbf{x}_0$,

$$\boldsymbol{\nabla}_{\mathbf{x}_0} \cdot \mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi}\int \boldsymbol{\nabla}_{\mathbf{x}_0} \cdot \left[\frac{\mathbf{J}(\mathbf{x}) \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}\right] d^3x$$

Now we need to rearrange terms. The divergence of a cross product may be rewritten as

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$$

so we may rewrite the integrand as

$$\boldsymbol{\nabla}_{\mathbf{x}_0} \cdot \left[\mathbf{J}(\mathbf{x}) \times \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}\right] = \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}\left[\boldsymbol{\nabla}_{\mathbf{x}_0} \times \mathbf{J}(\mathbf{x})\right] - \mathbf{J}(\mathbf{x}) \cdot \left[\boldsymbol{\nabla}_{\mathbf{x}_0} \times \frac{\mathbf{x}_0 - \mathbf{x}}{|\mathbf{x}_0 - \mathbf{x}|^3}\right]$$

This is not nearly as bad as it looks! The first term on the right containing $\boldsymbol{\nabla}_{\mathbf{x}_0} \times \mathbf{J}(\mathbf{x})$ vanishes immediately because $\mathbf{J}(\mathbf{x})$ does not depend on the observation point $\mathbf{x}_0$. The second term also vanishes. To see this, remember that $\frac{\mathbf{x}_0 - \mathbf{x}}{|\mathbf{x}_0 - \mathbf{x}|^3}$ is a gradient,

$$\frac{\mathbf{x}_0 - \mathbf{x}}{|\mathbf{x}_0 - \mathbf{x}|^3} = -\boldsymbol{\nabla}_{\mathbf{x}_0}\frac{1}{|\mathbf{x}_0 - \mathbf{x}|}$$

an the curl of a gradient always vanishes. Therefore, the divergence of the magnetic field vanishes:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

It turns out that this law applies even in non-steady state situations.

Integrating the divergence of $\mathbf{B}$ and using the divergence theorem

$$\begin{aligned} 0 &= \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{B} d^3x \\ &= \oint_S \mathbf{B} \cdot \mathbf{n} d^2x \end{aligned}$$

which shows that there is no net magnetic flux across any closed surface. In particular, there are no isolated magnetic charges (monopoles).

2 The equations of magnetostatics

Summarizing the magnetostatic equations, we have

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J}(\mathbf{x}) \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \end{aligned}$$

together with the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

These equations, together with boundary conditions, uniquely determine static magnetic fields.

The integral forms of the magnetostatic laws are therefore,

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I_{\mathcal{S}} \\ \oint_{S} \mathbf{B} \cdot \mathbf{n} d^2 x &= 0 \end{aligned}$$

These integral forms are useful for establishing the boundary conditions.

2.1 Boundary conditions

Consider an interface between two regions in which we have solutions for the magnetic fields. Using the integral forms of the field equations, we find the boundary conditions.

First, use the integral form of Ampère's law. Choose the curve to be a long (length L) narrow (length $l \ll L$) rectangle with the long sides parallel to and on either side of the interface. Then, neglecting the contribution from the short sides that pierce the interface, and choosing the loop small enough that the field is approximately constant along the long side, Ampère's law becomes

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I_{through\, S} \\ \left(\mathbf{B}_{\parallel}^{out} - \mathbf{B}_{\parallel}^{in}\right) L &= \mu_0 I A \end{aligned}$$

where $A = Ll$ is the area of the loop. Setting $IA = K_{surface} = \hat{\mathbf{n}} \cdot \mathbf{K}$ to be the surface current through the loop,

$$\mathbf{B}_{\parallel}^{out} - \mathbf{B}_{\parallel}^{in} = \mu_0 K_{surface}$$

If the loop is parallel to the surface current $\mathbf{K}$, then the right side vanishes and the corresponding component of the magnetic field is continuous.

The divergence equation may be evaluated over a small cylinder piercing the surface. As we have seen before, the surface integral reduces to the normal component of the fields times the area of the circular cross-section. Since the magnetic flux across any closed surface vanishes,

$$B_{\perp}^{out} - \mathbf{B}_{\perp}^{in} = 0$$

so the normal component of the magnetic field is continuous across the boundary.

2.2 The vector potential for the magnetic field

As we found for the electric field, it simplifies calculations to write the magnetic field in terms of a potential. However, while the electric field has vanishing curl and a source for its divergence, the magnetic field has the opposite: a source for the curl and a vanishing divergence. Therefore, we make use of the vanishing divergence of $\mathbf{B}$ to write $\mathbf{B}$ as a curl.

Clearly, if we set

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

for some vector field $\mathbf{A}$, the divergence will vanish automatically because the divergence of a curl is identically zero. Conversely, the Helmholz theorem shows that if the divergence vanishes then $\mathbf{B}$ may be written as a curl of some vector. The vector field $\mathbf{A}$ is called the *vector potential.*

Writing $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ automatically ensures that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. We now substitute this into Ampère's law and use the identity for a double curl:

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \mu_0 \mathbf{J}(\mathbf{r}) \\ \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= \mu_0 \mathbf{J}(\mathbf{r}) \end{aligned}$$

We can eliminate the first term by using *gauge freedom.*

Gauge freedom arises because there is more than one allowed vector potential. If $\mathbf{A}$ is any vector field satisfying $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, and we set $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f$ for any function f, then it is also true that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}'$. The choice of the function f is called the gauge, and this choice has no effect on the measurable magnetic field, must like our freedom to add a constant to a scalar potential.

To use the gauge freedom to simplify the form of Ampère's law, first suppose we have any vector potential $\mathbf{A}_0(\mathbf{x})$ satisfying $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}_0$. Furthermore, it will be the case that the divergence of $\mathbf{A}_0$ is some function $g(\mathbf{x})$,

$$\boldsymbol{\nabla} \cdot \mathbf{A}_0 = g(\mathbf{x})$$

Let $\mathbf{A}$ be another allowed vector potential related to $\mathbf{A}_0$ by

$$\mathbf{A} = \mathbf{A}_0 + \boldsymbol{\nabla} f$$

where f is a function of our choosing. We would like to choose the function f so that the divergence of the new potential vanishes. This requires

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \mathbf{A} \\ &= \boldsymbol{\nabla} \cdot (\mathbf{A}_0 + \boldsymbol{\nabla} f) \\ &= g + \nabla^2 f \end{aligned}$$

so that f must satisfy the Poisson equation,

$$\nabla^2 f = -g$$

where g is the known divergence of our original vector potential $\mathbf{A}_0$. We have techniques for solving the Poisson equation, so we can always find the required function f.

We now have a vector potential satisfying two conditions:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{A} &= \mathbf{B} \\ \boldsymbol{\nabla} \cdot \mathbf{A} &= 0 \end{aligned}$$

Substituting into Ampère's law now gives the simpler result,

$$\begin{aligned} \mu_0 \mathbf{J}(\mathbf{r}) &= \boldsymbol{\nabla} \times \mathbf{B} \\ &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) \\ &= \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ &= -\nabla^2 \mathbf{A} \end{aligned}$$

and vanish, and Ampère's law is

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}(\mathbf{r})$$

This is just three copies of the Poisson equation, one for each component, so in principal we know how to solve the equations of magnetostatics.

For fields vanishing at infinity, the solution is

$$\mathbf{A}(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|} d^3x$$

For other cases, we know that the solutions are unique, once we specify boundary conditions.

3 Multipole expansion of the vector potential

Suppose we restrict the current density a curve carrying a current loop I, so that

$$\begin{aligned} \mathbf{J}(\mathbf{x}) &= I\delta(x_1)\,\delta(x_2)\,\mathbf{l} \\ d^3x &= dx_1 dx_2 d\mathbf{l} \end{aligned}$$

where x_1 and x_2 are coordinates perpendicular to the direction $\mathbf{l}$ of the current. Then the vector potential takes the simpler form

$$\mathbf{A}(\mathbf{x}_0) = \frac{\mu_0 I}{4\pi} \oint\limits_C \frac{d\mathbf{l}}{|\mathbf{x}_0 - \mathbf{x}|} d^3x$$

Circuitry often takes this form, and it is sometimes useful to find the approximate magnetic field far from the circuit in the form of a multipole expansion.

We may use the Legendre expansion

$$\frac{1}{|\mathbf{x}_0 - \mathbf{x}|} = \sum_{n=0}^{\infty} \frac{r^l}{r_0^{l+1}} P_l(\cos\theta)$$

where $|\mathbf{x}| = r$ and $|\mathbf{x}_0| = r_0$ and θ is the angle between $\mathbf{x}$ and $\mathbf{x}_0$, to expand the vector potential when $r_0 > r$. The expansion becomes

$$\begin{aligned} \mathbf{A}(\mathbf{x}_0) &= \frac{\mu_0 I}{4\pi} \oint\limits_C \frac{d\mathbf{l}}{|\mathbf{x}_0 - \mathbf{x}|} d^3x \\ &= \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \oint\limits_C \frac{r^l}{r_0^{l+1}} P_l(\cos\theta)\, d\mathbf{l} \end{aligned}$$

We consider the first two multipoles.

3.1 Magnetic monopole

The $l = 0$ term of the vector potential becomes

$$\begin{aligned} \mathbf{A}_{monopole}(\mathbf{x}_0) &= \frac{\mu_0 I}{4\pi r_0} \oint\limits_C P_0(\cos\theta)\, d\mathbf{l} \\ &= \frac{\mu_0 I}{4\pi} \oint\limits_C d\mathbf{l} \end{aligned}$$

But the integral of $d\mathbf{l}$ around any closed loop vanishes, so the monopole contribution is always zero.

3.2 Magnetic dipole

The magnetic dipole is usually the dominant term of the far field. It is given by

$$\begin{aligned}
\mathbf{A}(\mathbf{x}_0) &= \frac{\mu_0 I}{4\pi r_0^2}\oint_C rP_1(\cos\theta)\,d\mathbf{l} \\
&= \frac{\mu_0 I}{4\pi r_0^2}\oint_C r\cos\theta d\mathbf{l} \\
&= \frac{\mu_0 I}{4\pi r_0^2}\oint_C (\hat{\mathbf{x}}_0\cdot\mathbf{x})\,d\mathbf{l}
\end{aligned}$$

We could extract the observation direction as we did before, writing $\mathbf{A}(\mathbf{x}_0) = \frac{\mu_0 I}{4\pi r_0^2}\sum \hat{x}_{0i}\cdot\oint_C x_i d\mathbf{l}$, but the remaining integral describes a matrix, $K_{ij} = \oint_C x_i dl_j$. With a little effort, we can express the magnetic dipole moment as a vector.

We need an identity to get the desired vector form. Shifting to index notation, the i^{th} component of the integral is

$$\begin{aligned}
\left[\oint_C d\mathbf{l}\,(\mathbf{x}\cdot\hat{\mathbf{x}}_0)\right]_i &= \sum_{j=1}^{3}\oint_C dx_i x_j \hat{x}_{0j} \\
&= \sum_{j,k=1}^{3}\oint_C [x_j\hat{x}_{0j}\delta_{ik}]\,dx_k
\end{aligned}$$

Thinking of $M_{ik} = \sum_{j=1}^{3} x_j\hat{x}_{0j}\delta_{ik}$ as a vector for each value of i allows us to use Stokes' theorem,

$$\begin{aligned}
\left[\oint_C d\mathbf{l}\,(\mathbf{x}\cdot\hat{\mathbf{x}}_0)\right]_i &= \sum_{k=1}^{3}\oint_C M_{ik}dx_k \\
&= \oint_C \mathbf{M}_i\cdot d\mathbf{l} \\
&= \int \hat{\mathbf{n}}\cdot(\nabla\times\mathbf{M}_i)\,d^2x
\end{aligned}$$

Now, recalling that the i^{th} component of the curl may be written as

$$[\nabla\times\mathbf{v}]_i = \sum_{j,k=1}^{3}\varepsilon_{ijk}\frac{\partial}{\partial x_j}v_k$$

we compute

$$\begin{aligned}
\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{M}_i) &= \sum_{k,l,m=1}^{3}\varepsilon_{klm}n_k\nabla_l M_{im} \\
&= \sum_{j,k,l,m=1}^{3}\varepsilon_{klm}n_k\nabla_l\,[x_j\hat{x}_{0j}\delta_{im}] \\
&= \sum_{j,k,l,m=1}^{3}\varepsilon_{klm}n_k\delta_{lj}\hat{x}_{0j}\delta_{im}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,j=1}^{3} \varepsilon_{kji} n_k \hat{x}_{0j} \\
&= \sum_{k,j=1}^{3} \varepsilon_{ikj} n_k \hat{x}_{0j} \\
&= [\hat{\mathbf{n}} \times \hat{\mathbf{x}}_0]_i
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\oint_C d\mathbf{l}\,(\mathbf{x} \cdot \hat{\mathbf{x}}_0) &= \int \hat{\mathbf{n}} \times \hat{\mathbf{x}}_0 d^2x \\
&= -\hat{\mathbf{x}}_0 \times \int \hat{\mathbf{n}} d^2x
\end{aligned}
$$

The remaining integral is a vector weighted area called the *vector area*:

$$
\mathbf{a} \equiv \int \hat{\mathbf{n}} d^2x
$$

Returning to the vector potential, we have

$$
\begin{aligned}
\mathbf{A}_{dipole}(\mathbf{x}_0) &= \frac{\mu_0 I}{4\pi r_0^2} \oint_C (\hat{\mathbf{x}}_0 \cdot \mathbf{x})\, d\mathbf{l} \\
&= \frac{\mu_0}{4\pi r_0^2} I \int \hat{\mathbf{n}} d^2x \times \hat{\mathbf{x}}_0
\end{aligned}
$$

We define the magnetic moment to be

$$
\mathbf{m} \equiv I \int \hat{\mathbf{n}} d^2x = I\mathbf{a}
$$

Then (dropping subscripts) the vector potential for a magnetic dipole is

$$
\mathbf{A}_{dipole}(\mathbf{x}) = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \hat{\mathbf{x}}
$$

3.3 Dipole moment of a current loop

For a simple current loop of radius R in the xy-plane, carrying a steady current I, the magnetic dipole moment is

$$
\begin{aligned}
\mathbf{m} &\equiv I \int \hat{\mathbf{n}} d^2x \\
&= I\hat{\mathbf{k}} \int d^2x \\
&= \pi R^2 I \hat{\mathbf{k}} \\
&= m\hat{\mathbf{k}}
\end{aligned}
$$

The dipole contribution to the vector potential is

$$
\begin{aligned}
\mathbf{A}_{dipole}(\mathbf{x}) &= \frac{\mu_0 m}{4\pi r^2} \hat{\mathbf{k}} \times \hat{\mathbf{x}} \\
&= \frac{\mu_0 R^2 I}{4r^2} \sin\theta \hat{\boldsymbol{\varphi}}
\end{aligned}
$$

and the (dominant) dipole contribution to the magnetic field is

$$\begin{aligned}
\mathbf{B}_{dipole} &= \frac{\mu_0}{4\pi} m \boldsymbol{\nabla} \times \left(\frac{1}{r^2} \sin\theta \hat{\boldsymbol{\varphi}} \right) \\
&= \frac{\mu_0}{4\pi} m \left(\frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{1}{r^2} \sin^2\theta \right) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \sin\theta \right) \hat{\boldsymbol{\theta}} \right) \\
&= \frac{\mu_0}{4\pi} m \left(\frac{2}{r^3} \cos\theta \hat{\mathbf{r}} + \frac{1}{r^3} \sin\theta \hat{\boldsymbol{\theta}} \right) \\
&= \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}} \right)
\end{aligned}$$

We may write this for an arbitrary dipole moment $\mathbf{m}$ by noticing that

$$\begin{aligned}
2m\cos\theta \hat{\mathbf{r}} &= \frac{2}{r} \mathbf{m} \cdot \mathbf{x} \\
m \sin\theta \hat{\boldsymbol{\theta}} &= -\frac{1}{r^2} \mathbf{x} \times (\mathbf{m} \times \mathbf{x}) \\
&= -\frac{1}{r^2} \left(r^2 \mathbf{m} - \mathbf{x} (\mathbf{m} \cdot \mathbf{x}) \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{B}_{dipole} &= \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}} \right) \\
&= \frac{\mu_0}{4\pi r^3} \left(\frac{2}{r} (\mathbf{m} \cdot \mathbf{x}) \hat{\mathbf{r}} - \frac{1}{r^2} \left(r^2 \mathbf{m} - \mathbf{x} (\mathbf{m} \cdot \mathbf{x}) \right) \right) \\
&= \frac{\mu_0}{4\pi r^3} \left(2 (\mathbf{m} \cdot \mathbf{x}) \hat{\mathbf{r}} - \mathbf{m} + \hat{\mathbf{r}} (\mathbf{m} \cdot \hat{\mathbf{r}}) \right)
\end{aligned}$$

so that

$$\mathbf{B}_{dipole} = \frac{\mu_0}{4\pi r^3} \left(3 (\mathbf{m} \cdot \mathbf{x}) \hat{\mathbf{r}} - \mathbf{m} \right)$$

This holds regardless of the orientation of the dipole moment.

4 Summary of magnetostatics

We have the following basic equations for magnetostatics.

Ampère's law and the absence of magnetic charges:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{J}(\mathbf{r})\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\end{aligned}$$

The Lorentz force law,

$$\mathbf{F} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

The vector potential:

$$\begin{aligned}\nabla^2\mathbf{A} &= -\mu_0\mathbf{J}\\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}\\ \boldsymbol{\nabla}\cdot\mathbf{A} &= 0\end{aligned}$$

Boundary conditions:

$$\begin{aligned}\mathbf{B}_{\parallel}^{out}-\mathbf{B}_{\parallel}^{in} &= \mu_0 K_{surface}\\ B_{\perp}^{out}-B_{\perp}^{in} &= 0\end{aligned}$$

Additional useful equations include integral form of the basic field equations,

$$\begin{aligned}\oint_{\mathcal{C}}\mathbf{B}\cdot d\mathbf{l} &= \mu_0 I_{through\,S}\\ \oint_{S}\mathbf{B}\cdot\mathbf{n}d^2x &= 0\end{aligned}$$

the Biot-Savart law

$$\mathbf{B}\left(\mathbf{x}_0\right) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\left(\mathbf{x}\right)\times\left(\mathbf{x}_0-\mathbf{x}\right)}{\left|\mathbf{x}_0-\mathbf{x}\right|^3}d^3x$$

the Biot-Savart law for circuits,

$$\mathbf{B}\left(\mathbf{x}_0\right) = \frac{\mu_0}{4\pi}\int\frac{Id\mathbf{l}\times\left(\mathbf{x}_0-\mathbf{x}\right)}{\left|\mathbf{x}_0-\mathbf{x}\right|^3}$$

and the force on a current carrying wire in a magnetic field

$$\mathbf{F} = I\int\left(d\mathbf{l}\times\mathbf{B}\right)$$