Ampère's law

December 7, 2015

1 Ampère's law

Ampère's law relates currents to the curl of the magnetic field. To derive it, we take the curl of the Biot-Savart law.

1.1 The curl of the magnetic field

Beginning again with the Biot-Savart law,

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}) \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} d^3x$$

we take the curle of both sides with respect to $\mathbf{x}_0$,

$$\boldsymbol{\nabla}_{\mathbf{x}_0} \times \mathbf{B}(\mathbf{x}_0) \quad = \quad \frac{\mu_0}{4\pi}\int \boldsymbol{\nabla}_{\mathbf{x}_0} \times \left[\mathbf{J}(\mathbf{x}) \times \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}\right] d^3x$$

Now, use expression for the curl of a cross product,

$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{C} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{C} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{C}) - \mathbf{C}(\boldsymbol{\nabla} \cdot \mathbf{A})$$

with $\mathbf{A} = \mathbf{J}(\mathbf{x})$. Since the derivatives are with respect to $\mathbf{x}_0$ and the current density depends only on $\mathbf{x}_0$, the terms with derivatives of $\mathbf{J}(\mathbf{x})$ drop out, leaving

$$\boldsymbol{\nabla} \times (\mathbf{J} \times \mathbf{C}) = -(\mathbf{J} \cdot \boldsymbol{\nabla})\mathbf{C} + \mathbf{J}(\boldsymbol{\nabla} \cdot \mathbf{C})$$

The vector $\mathbf{C}$ is now replaced by $\frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}$,

$$\boldsymbol{\nabla} \times \left(\mathbf{J} \times \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}\right) = -\left(\mathbf{J}(\mathbf{x}) \cdot \boldsymbol{\nabla}_{\mathbf{x}_0}\right) \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} + \mathbf{J}\left(\boldsymbol{\nabla} \cdot \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}\right)$$

We need to do some work to show that integral of the first term on the right vanishes. To start, notice that

$$\boldsymbol{\nabla}_{\mathbf{x}_0} \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} \quad = \quad -\boldsymbol{\nabla}_{\mathbf{x}} \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}$$

Then, writing the dot product as

$$\mathbf{J}(\mathbf{x}) \cdot \boldsymbol{\nabla}_{\mathbf{x}} = \sum_{i=1}^{3} J_i \frac{\partial}{\partial x^i}$$

the first term becomes

$$\begin{aligned}
\left(\mathbf{J}\left(\mathbf{x}\right)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\right)\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}} &= \sum_{i=1}^{3}J_{i}\frac{\partial}{\partial x^{i}}\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}} \\
&= \sum_{i=1}^{3}\frac{\partial}{\partial x^{i}}\left[J_{i}\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\right]-\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\sum_{i=1}^{3}\frac{\partial J_{i}}{\partial x^{i}}
\end{aligned}$$

But

$$\sum_{i=1}^{3}\frac{\partial J_{i}}{\partial x^{i}}=\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{J}\left(\mathbf{x}\right)$$

and since we are interested in static fields, the continuity equation, eq.(**??**), shows that the second term vanishes. The first term is a total divergence and when we integrate it to find its contribution to the curl of the magnetic field, we may integrate by parts. The integration is clearest if we look at one component at a time. The integrand is

$$\sum_{i=1}^{3}\frac{\partial}{\partial x^{i}}\left[J_{i}\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\right]$$

and its k component is

$$\sum_{i=1}^{3}\frac{\partial}{\partial x^{i}}\left[J_{i}\frac{\left(x_{0k}-x_{k}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\right]=\boldsymbol{\nabla}_{\mathbf{x}}\cdot\left[\frac{\left(x_{0k}-x_{k}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\mathbf{J}\right]$$

Now use the divergence theorem,

$$\begin{aligned}
\left[-\frac{\mu_{0}}{4\pi}\int\left(\mathbf{J}\left(\mathbf{x}\right)\cdot\boldsymbol{\nabla}_{\mathbf{x}_{0}}\right)\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}d^{3}x\right]_{k} &= -\frac{\mu_{0}}{4\pi}\int_{\mathcal{V}}\boldsymbol{\nabla}_{\mathbf{x}}\cdot\left[\frac{\left(x_{0k}-x_{k}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\mathbf{J}\right]d^{3}x \\
&= -\frac{\mu_{0}}{4\pi}\oint_{\mathcal{S}}\sum_{i=1}^{3}\hat{\mathbf{n}}\cdot\left[\frac{\left(x_{0k}-x_{k}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\mathbf{J}\right]d^{2}x
\end{aligned}$$

Since the volume $\mathcal{V}$ is arbitrary, we take it to extend beyond all of the currents into a region of empty space. Then the current $\mathbf{J}$ vanishes on the entire boundary $\mathcal{S}$ and the integral vanishes.

This leaves us with only one term,

$$\boldsymbol{\nabla}_{\mathbf{x}_{0}}\times\mathbf{B}\left(\mathbf{x}_{0}\right) = \frac{\mu_{0}}{4\pi}\int\mathbf{J}\left(\boldsymbol{\nabla}\cdot\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}\right)d^{3}x$$

Again using the fact that

$$\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}=-\boldsymbol{\nabla}\frac{1}{\left|\mathbf{x}_{0}-\mathbf{x}\right|}$$

the term in parenthesis becomes a Laplacian,

$$\boldsymbol{\nabla}\cdot\frac{\left(\mathbf{x}_{0}-\mathbf{x}\right)}{\left|\mathbf{x}_{0}-\mathbf{x}\right|^{3}}=-\nabla^{2}\frac{1}{\left|\mathbf{x}_{0}-\mathbf{x}\right|}$$

Finally, we use the Poisson equation for a point charge. In general, we have

$$\nabla^{2}V=-\frac{1}{\epsilon_{0}}\rho$$

But when ρ is the charge density for a single point charge, $\rho=q\delta^{3}\left(\mathbf{x}_{0}-\mathbf{x}\right)$ we have Coulomb's law and the potential is

$$V\left(\mathbf{x}_{0}\right)=\frac{1}{4\pi\epsilon_{0}}\frac{q}{\left|\mathbf{x}_{0}-\mathbf{x}\right|}$$

Therefore,

$$\nabla^2 \left(\frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{x}_0 - \mathbf{x}|} \right) = -\frac{1}{\epsilon_0} q \delta^3 (\mathbf{x}_0 - \mathbf{x})$$

$$\nabla^2 \frac{1}{|\mathbf{x}_0 - \mathbf{x}|} = -4\pi \delta^3 (\mathbf{x}_0 - \mathbf{x})$$

The curl of the magnetic field becomes

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathbf{x}_0} \times \mathbf{B}(\mathbf{x}_0) &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}) \left(\boldsymbol{\nabla} \cdot \frac{(\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} \right) d^3x \\
&= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}) \nabla^2 \frac{1}{|\mathbf{x}_0 - \mathbf{x}|} d^3x \\
&= \mu_0 \int \mathbf{J}(\mathbf{x}) \delta^3 (\mathbf{x}_0 - \mathbf{x}) d^3x \\
&= \mu_0 \mathbf{J}(\mathbf{x}_0)
\end{aligned}$$

and therefore,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

This is the static form of *Ampère's law.*

Notice that Ampère's law in this form is inconsistent when the fields are time dependent. This follows from the full continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

If the charge density changes in time, $\frac{\partial \rho}{\partial t}$ is nonzero and we must have

$$\boldsymbol{\nabla} \cdot \mathbf{J} \neq 0$$

But the divergence of the static form of Ampère's law requires this to vanish, since we have

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J}(\mathbf{r}) \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) &= \mu_0 \boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{r})
\end{aligned}$$

On the left, the divergence of a curl always vanishes, leaving us with

$$0 = \mu_0 \boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{r}),$$

a contradiction.

1.2 The integral form of Ampère's law

The integral form of Ampère's law is easily found. Integrating normal component of the curl over an arbitrary surface and using Stokes' theorem, we have

$$\begin{aligned}
\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{n} d^2x &= \mu_0 \int_{\mathcal{S}} \mathbf{J} \cdot \mathbf{n} d^2x \\
\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} &= \mu_0 \int_{\mathcal{S}} \mathbf{J} \cdot \mathbf{n} d^2x
\end{aligned}$$

and since the integral of the current density over a surface gives the current, I_S, across that surface,

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathcal{S}}$$

This expression is useful when a current distribution has sufficient symmetry.

2 Examples

2.1 Magnetic field of a long straight wire

Above, we found the magnetic field near a long, straight wire using the Biot-Savart law. Now we solve the same problem using Ampère's law.

We start from the integral form of Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{encl}$$

where I_{encl} is the total current passing through the closed contour of integration. Like a Gaussian surface, this closed contour may be chosen arbitrarily. For the long straight wire, the magnetic field can depend only on the distance ρ from the wire. We can get the direction from the Biot-Savart numerator, $d\mathbf{l} \times (\mathbf{x}_0 - \mathbf{x})$. Since $d\mathbf{l}$ is parallel to $\mathbf{x}$ and $d\mathbf{l} \times \mathbf{x}_0$ is in the $\hat{\boldsymbol{\varphi}}$ direction, the resultant field must be in the $\hat{\boldsymbol{\varphi}}$ direction.

Alternatively we may get the direction from the right-hand rule: placing the right thumb in the direction of the current, the fingers curl in the direction of the magnetic field. For a current in the z-direction the curl of our fingers is counter-clockwise in the xy-plane, again the $\hat{\boldsymbol{\varphi}}$ direction.

Therefore, we write

$$\mathbf{B} = B(\rho)\,\hat{\boldsymbol{\varphi}}$$

and choose a circular path around the wire at distance ρ. Then $d\mathbf{l} = \rho d\varphi \hat{\boldsymbol{\varphi}}$ and the law becomes

$$\begin{aligned} \oint B(\rho)\,\hat{\boldsymbol{\varphi}} \cdot \rho d\varphi \hat{\boldsymbol{\varphi}} &= \mu_0 I \\ B(\rho)\,\rho \oint d\varphi &= \mu_0 I \\ B(\rho)\, 2\pi\rho &= \mu_0 I \end{aligned}$$

Solving, we have

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}$$

Ampère's law considerably simplifies the calculation.

2.2 Magnetic field of a straight wire with non-uniform current

Suppose a long straight cylindrical wire of radius R carries total current I which is distributed throughout its cross-section so that it increases with radius as ρ^2. Find the magnetic field both inside and outside the wire.

Let the current flow in the z-direction. The current density is

$$\mathbf{J} = Ar^2\Theta(R-\rho)\,\hat{\mathbf{k}}$$

where A is determined by the total current,

$$\begin{aligned} I &= \int_0^\infty \int_0^{2\pi} r dr d\varphi A r^2 \Theta(R-r)\,\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \\ &= 2\pi A \int_0^R r^3 dr \\ &= 2\pi A \frac{R^4}{4} \\ A &= \frac{2I}{\pi R^4} \end{aligned}$$

so that

$$\mathbf{J} = \frac{2I\rho^2}{\pi R^4}\Theta\left(R-\rho\right)\hat{\mathbf{k}}$$

Now use Ampère's law. Since the problem is symmetric in the z and φ directions, the magnitude of $\mathbf{B}$ can only depend on ρ. Again, we find the direction from either the Biot-Savart law, or the right-hand rule. Either way, we see that $\mathbf{B}$ must be in the φ-direction,

$$\mathbf{B} = B\left(\rho\right)\hat{\boldsymbol{\varphi}}$$

Now imagine a circular loop in a plane orthogonal to the wire, lying at constant ρ, so $d\mathbf{l} = \rho d\varphi\hat{\boldsymbol{\varphi}}$.

For $\rho < R$, Ampère's law gives

$$\begin{aligned}
\oint \mathbf{B}\cdot d\mathbf{l} &= \mu_0 I_{encl} \\
\oint B\left(\rho\right)\hat{\boldsymbol{\varphi}}\cdot\rho d\varphi\hat{\boldsymbol{\varphi}} &= \mu_0\int\limits_0^\rho\int\limits_0^{2\pi}\rho d\rho d\varphi\frac{2I\rho^2}{\pi R^4}\Theta\left(R-\rho\right)\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}
\end{aligned}$$

Notice that the right side differs from the total current because the limit of the ρ integration is no longer R when we are interested in the magnetic field inside the wire. On the left side, we can see that $\mathbf{B}$ must have a $\hat{\boldsymbol{\varphi}}$ component, since the right side of the equation is nonzero. (On the other hand, a loop chosen in the ρz-plane encloses no current, so the field integral must vanish. We could use this to prove that $\mathbf{B}$ is in the φ-direction).

Evaluating the integrals,

$$\begin{aligned}
\oint B\left(\rho\right)\hat{\boldsymbol{\varphi}}\cdot\rho d\varphi\hat{\boldsymbol{\varphi}} &= \mu_0\int\limits_0^\rho\int\limits_0^{2\pi}\rho d\rho d\varphi\frac{2I\rho^2}{\pi R^4}\Theta\left(R-\rho\right)\hat{\mathbf{k}}\cdot\hat{\mathbf{k}} \\
2\pi B\left(\rho\right)\rho &= 2\pi\mu_0\int\limits_0^\rho\frac{2I\rho^3}{\pi R^4}d\rho \\
2\pi B\left(\rho\right)\rho &= 2\pi\mu_0\frac{2I\rho^4}{4\pi R^4} \\
B\left(\rho\right) &= \frac{\mu_0 I\rho^3}{2\pi R^4}
\end{aligned}$$

so the field inside the wire is

$$\mathbf{B} = \frac{\mu_0 I\rho^3}{2\pi R^4}\hat{\boldsymbol{\varphi}}$$

For $\rho > R$, the only difference is that the full current is enclosed (i.e., we take the upper ρ limit to be R), so the right hand side of the equation is simply $\mu_0 I$, and the field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\varphi}}$$

Notice that the interior and exterior solutions agree when $\rho = R$.

2.3 Magnetic field of a long solenoid

Let a solenoid of total length L consist of n turns per unit length of tightly wound wire carrying a current I in each turn. We can use Ampère's law if we can figure out the effect of the symmetry. Let the solenoid lie centered along the z-axis.

The first point is that we require $nLI \gg I$ so that the field due to I is negligible compared to the full field. We may treat each loop as essentially a circle, and neglect the overall flow of current in the z-direction.

With this understanding, we have symmetry in the z-direction, so there can be no z dependence to *magnitude* of the magnetic field.

Rotational symmetry about the z axis tells us there can be no φ-dependence of the magnitude.

Suppose $\mathbf{B}$ takes the form

$$\mathbf{B} = B_z(\rho)\,\hat{\mathbf{k}} + B_\varphi(\rho)\,\hat{\boldsymbol{\varphi}} + B_r(\rho)\,\hat{\mathbf{r}}$$

and choose paths along which to evaluate Ampère's law. First, let C be circle in the $\hat{\boldsymbol{\varphi}}$-direction. No such circle inside the solenoid encloses any current, so for $r < R$,

$$\begin{aligned} \oint \mathbf{B} \cdot \hat{\boldsymbol{\varphi}} \rho d\varphi &= 0 \\ B_\varphi(\rho)\,2\pi\rho &= 0 \end{aligned}$$

so $B_\varphi = 0$. If $\rho > R$, the same argument applies since we are neglecting the overall current in the z-direction. Therefore, $B_\varphi = 0$ everywhere.

For the radial component, notice from the Biot-Savart law that $\mathbf{B}$ depends linearly on the curren I. Therefore, if we change the direction of the current, the direction of the field must reverse. Now, flip the solenoid end for end. This flip cannot change the direction of radial component of the field, but it returns the system to one identical to the initial one. However, the radial component has flipped sign. The only possibility is $B_r(\rho) = 0$.

We conclude that as long as we can neglect the slight movement of current ($I \ll nLI$), the magnetic field is along the axis of the solenoid:

$$\mathbf{B} = B_z(r)\,\hat{\mathbf{k}}$$

This we evaluate using three rectangular loops. Let the first loop lie entirely inside the solenoid with one side along the z axis for a distance L, two radial sides being of length $\rho < R$, and the second long side parallel to the first at radius ρ. Since $\mathbf{B}$ has no radial component, and this first curve encloses no current, we have

$$B_z(0)\,L - B_z(\rho)\,L = 0$$

showing that the magnitude of $B_z(\rho)$ is constant within the solenoid. For the second loop, let both long sides of the loop lie outside the solenoid. We get a similar result, $B_z(\rho_2) - B_z(\rho_1) = 0$, showing that any field outside the solenoid must be constant as well. This outer constant must be zero, however, since otherwise a finite current would produce infinite field energy, which as we shall see depends on the integral of $\mathbf{B}^2$. Alternatively, we may argue that for a long but finite solenoid, the current from a sufficiently great distance looks like only the total z-flow that we have neglected, and this would produce a field that falls off as $\frac{1}{\rho}$, in contradiction to the constancy we have just shown.

We conclude that the field outside the solenoid vanishes.

Finally, choose a third rectangle with one side lying on the z-axis and the other be at any $\rho > R$. Now the loops encloses nL turns of the wire, hence a total current nLI. The integrals along the two radial sides vanish, and there is no field outside, so the only contribution is from the side along the z-axis, giving

$$\begin{aligned} B_z(0)\,L &= \mu_0 nLI \\ B_z(0) &= \mu_0 nI \end{aligned}$$

The magnetic field is confined to the interior and is given by,

$$\mathbf{B} = \mu_0 nI\hat{\mathbf{k}}$$