Electric fields in matter

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Suppose we apply a constant electric field to a block of material. Then the charges that make up the matter are no longer in equilibrium: the electrons tend to move in the $-\mathbf{E}$ direction and the positive nuclei tend to move in the direction of the field. In the simplest circumstances, this creates a dipole moment within the material, and within the material this dipole field, $\mathbf{E}_{dipole}$ generally opposes the applied field. The *total* electric field is therefore reduced to $\mathbf{E} - \mathbf{E}_{dipole}$ within the material.

Similarly, an electron approaching a spherical atom will alter the atom. In the simplest model of what happens, we have a sphere of negative charge around an equal but positively charged center. As the additional electron approaches, it repels the electron sphere and attracts the nucleus, so the cloud of electrons moves slightly away and the nucleus moves nearer, creating a slight dipole. This electric field points in the direction from the nucleus toward the approaching electron, and therefore attracts the approaching negative charge.

These are highly simplified pictures. What happens in realistic situations may be different in different directions, and may depend nonlinearly on the applied electric field. Here we consider only the linear approximation.

We divide materials into two types: conductors and insulators. Conductors have free charges that rearrange to offset any applied fields, but in insulators the charges stay in their respective atoms or molecules. We call these latter materials *dielectrics*.

1 Effects of an electric field on molecules

1.1 Non-polar molecules

The effect of an electric field on a molecule can be quite simple or highly complex. The simplest situation is for a non-polar atom or molecule, where the charge distribution is nearly spherical. In this case, the positively charged nucleus will experiencen a force in the direction of the field while the electron cloud shifts in the opposite direction. For atoms, this is the linear Stark effect, often computed as a perturbation of the quantum states. The net effect is a dipole in the direction opposite to the field,

$$\mathbf{p} = \alpha \mathbf{E}$$

The constant α is called the atomic polarizability.

If the molecule is not spherically symmetric, the polarizability is likely to be different along different axes. In this case there is a more general linear relationship,

$$p_i = \sum_{i=1}^{3} \alpha_{ij} E_j$$

The *polarizability tensor*, α_{ij} is symmetric. For example, the x component of the polarization due to an electric field in the y direction turns out to be the same as the y component of polarization from an electric field in the x direction. As a result, we may always diagonalize α_{ij} along three principal axes.

For strong electric fields, the dipole moment is no longer linear in the electric field.

We will consider dipole moments produced by a simple linear response, $\mathbf{p} = \alpha \mathbf{E}$.

1.2 Polar molecules

Polar molecules already have a dipole moment, $\mathbf{p}$, and an applied electric field acts directly on the separated plus and minus charges to produce a torque. Write the dipole moment as the charge q at the positive end times an average displacement from the center, $\frac{1}{2}\mathbf{d}$,

$$\mathbf{p} = \frac{1}{2} q\mathbf{d}$$

This charge experiences a force $\mathbf{F}_+ = q\mathbf{E}$. The negative end, at $-\frac{1}{2}\mathbf{d}$ experiences a force $\mathbf{F}_- = -q\mathbf{E}$, so there is a net torque,

$$\begin{aligned}
\boldsymbol{\tau} &= \sum_{a=+,-} \mathbf{r}_a \times \mathbf{F}_a \\
&= \left(\frac{1}{2}\mathbf{d}\right) \times (q\mathbf{E}) + \left(-\frac{1}{2}\mathbf{d}\right) \times (-q\mathbf{E}) \\
&= \mathbf{p} \times \mathbf{E}
\end{aligned}$$

The direction of this torque tends to align $\mathbf{p}$ with $\mathbf{E}$. If the molecule is free to rotate, it will turn until the torque vanishes, when the dipole moment is in the direction of the electric field.

1.3 Net force

If a polar molecule is in a nonuniform electric field, then the ends of the dipole do not experience the same field, so there is a net force on the dipole:

$$\mathbf{F}_{total} = q\mathbf{E}\left(\frac{\mathbf{d}}{2}\right) + (-q)\,\mathbf{E}\left(-\frac{\mathbf{d}}{2}\right)$$

If $\frac{\mathbf{d}}{2}$ is small, we may expand the electric field in a Taylor series,

$$\begin{aligned}
\mathbf{E}\left(\frac{\mathbf{d}}{2}\right) &= \mathbf{E}(0) + \left(\frac{\mathbf{d}}{2} \cdot \boldsymbol{\nabla}\right)\mathbf{E}(0) \\
\mathbf{E}\left(-\frac{\mathbf{d}}{2}\right) &= \mathbf{E}(0) + \left(-\frac{\mathbf{d}}{2} \cdot \boldsymbol{\nabla}\right)\mathbf{E}(0)
\end{aligned}$$

so that substituting into the total force gives

$$\begin{aligned}
\mathbf{F}_{total} &= q\left(\mathbf{E}(0) + \left(\frac{\mathbf{d}}{2} \cdot \boldsymbol{\nabla}\right)\mathbf{E}(0)\right) + (-q)\left(\mathbf{E}(0) + \left(-\frac{\mathbf{d}}{2} \cdot \boldsymbol{\nabla}\right)\mathbf{E}(0)\right) \\
&= q\mathbf{E}(0) + \left(\frac{q\mathbf{d}}{2} \cdot \boldsymbol{\nabla}\right)\mathbf{E}(0) - q\mathbf{E}(0) + \left(\frac{q\mathbf{d}}{2} \cdot \boldsymbol{\nabla}\right)\mathbf{E}(0) \\
&= (\mathbf{p} \cdot \boldsymbol{\nabla})\,\mathbf{E}
\end{aligned}$$

2 The potential produced by a polarized material

We will assume that the polarization of materials is linear in both magnitude and direction, so that each atom in the material acquires a dipole moment $\mathbf{p}$ proportional to the applied field $\mathbf{p} \propto \mathbf{E}$. Suppose the applied electric field varies slowly enough that we can average this dipole moment over an effectively infinitesimal volume (but containing many molecules), d^3x, so that the dipole moment of the small volume is given by $\mathbf{p} = \mathbf{P}d^3x$. It may be that $\mathbf{E}$ and $\mathbf{P}$ depend on position within the material. Then $\mathbf{P}$ is the *dipole moment per unit volume.*

We need to compute the electric potential produced by this polarization, which will then combine with the applied potential to give a total electric potential in the material. Since a single dipole at the origin produces a potential at $\mathbf{x}_0$ given by

$$V(\mathbf{x}_0) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}_0}{|\mathbf{x}_0|^3}$$

we may shift the origin to see that a single dipole at postion $\mathbf{x}$ produces a potential at $\mathbf{x}_0$ of

$$V(\mathbf{x}_0) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}_0-\mathbf{x})}{|\mathbf{x}_0-\mathbf{x}|^3}$$

Now we can add up the combined contributions of small volumes, $\mathbf{P}d^3x'$, over a finite volume,

$$V(\mathbf{x}_0) = \frac{1}{4\pi\epsilon_0}\int\limits_{\mathcal{V}}\frac{\mathbf{P}(\mathbf{x})\cdot(\mathbf{x}_0-\mathbf{x})}{|\mathbf{x}_0-\mathbf{x}|^3}d^3x$$

where $\mathcal{V}$ is the volume of dielectric.

Now, recall from our discussion of potential that the gradient of $\frac{1}{|\mathbf{x}_0-\mathbf{x}|}$ with respect to the observation point $\mathbf{x}_0$ is

$$\boldsymbol{\nabla}_{\mathbf{x}_0}\frac{1}{|\mathbf{x}_0-\mathbf{x}|} = -\frac{\mathbf{x}_0-\mathbf{x}}{|\mathbf{x}_0-\mathbf{x}|^3}$$

The differentiation with respect to $\mathbf{x}$ gives the same, but with the opposite sign,

$$\boldsymbol{\nabla}_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}_0-\mathbf{x}|}\right) = \frac{\mathbf{x}_0-\mathbf{x}}{|\mathbf{x}_0-\mathbf{x}|^3}$$

This lets us rewrite the potential:

$$\begin{aligned} V(\mathbf{x}_0) &= \frac{1}{4\pi\epsilon_0}\int\limits_{\mathcal{V}}\frac{\mathbf{P}(\mathbf{x})\cdot(\mathbf{x}_0-\mathbf{x})}{|\mathbf{x}_0-\mathbf{x}|^3}d^3x \\ &= \frac{1}{4\pi\epsilon_0}\int\limits_{\mathcal{V}}\mathbf{P}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}_0-\mathbf{x}|}\right)d^3x \end{aligned}$$

Now the identity $\boldsymbol{\nabla}\cdot(f\mathbf{v}) = \boldsymbol{\nabla}f\cdot\mathbf{v} + f\boldsymbol{\nabla}\cdot\mathbf{v}$ lets us replace

$$\mathbf{P}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}_0-\mathbf{x}|}\right) = \boldsymbol{\nabla}_{\mathbf{x}}\cdot\left(\frac{\mathbf{P}}{|\mathbf{x}_0-\mathbf{x}|}\right) - \frac{1}{|\mathbf{x}_0-\mathbf{x}|}\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{P}$$

Then the potential becomes

$$\begin{aligned} V(\mathbf{x}_0) &= \frac{1}{4\pi\epsilon_0}\int\limits_{\mathcal{V}}\left(\boldsymbol{\nabla}_{\mathbf{x}}\cdot\left(\frac{\mathbf{P}}{|\mathbf{x}_0-\mathbf{x}|}\right) - \frac{1}{|\mathbf{x}_0-\mathbf{x}|}\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{P}\right)d^3x \\ &= \frac{1}{4\pi\epsilon_0}\oint\limits_{\mathcal{S}}\frac{\mathbf{P}\cdot\hat{\mathbf{n}}}{|\mathbf{x}_0-\mathbf{x}|} - \frac{1}{4\pi\epsilon_0}\int\limits_{\mathcal{V}}\frac{\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{P}}{|\mathbf{x}_0-\mathbf{x}|}d^3x \end{aligned}$$

The first term on the right has the form of the potential due to a surface charge $\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$, while the second has the form of the potential due to a charge density $\rho_b \equiv \boldsymbol{\nabla}\cdot\mathbf{P}$. We define the *bound surface charge density* and the *bound charge density*,

$$\begin{aligned} \sigma_b &\equiv \mathbf{P}\cdot\hat{\mathbf{n}} \quad (1) \\ \rho_b &\equiv -\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{x}) \quad (2) \end{aligned}$$

In terms of these the final potential is

$$V(\mathbf{x}_0) = \frac{1}{4\pi\epsilon_0}\oint\limits_{\mathcal{S}} \frac{\sigma_b}{|\mathbf{x}_0 - \mathbf{x}|} + \frac{1}{4\pi\epsilon_0}\int\limits_{\mathcal{V}} \frac{\rho_b}{|\mathbf{x}_0 - \mathbf{x}|} d^3x$$

Once we know the dipole moment per unit volume, $\mathbf{P}(\mathbf{x})$ we can compute σ_b and ρ_b and find the electric potential directly.

3 Electric displacement in linear materials

We could like to characterize the electric field inside a dielectric, or in regions which include dielectrics. We write the charge density as the effective bound charge density due to the polarization, together with any other charges ("free charges") in the system, inside or outside the dielectric,

$$\rho = \rho_b + \rho_f$$

The free charges from the familiar charge density. The bound charges arise only from the polarization of the material according to eqs.(1) and (2). Then Gauss's law is

$$\begin{aligned} \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho \\ &= \rho_f - \boldsymbol{\nabla} \cdot \mathbf{P} \end{aligned}$$

Bringing both divergence terms to the left side of the equation so that $\boldsymbol{\nabla} \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$, we defining the *electric displacement*,

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

so that

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$$

The electric field is now replaced by the electric displacement and the charge density is given by the free charge density.

Notice that in general there is no potential for $\mathbf{D}$ since we may have $\boldsymbol{\nabla} \times \mathbf{D} = \boldsymbol{\nabla} \times \mathbf{P} \neq 0$.

Next, we assume a linear material so that the dipole moment per unit volume is proportional to the electric field in that volume. It is convenient to write the proportionality as

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

The constant χ_e is called the *electric susceptibility*. Clearly this lets us write the electric displacement as

$$\begin{aligned} \mathbf{D} &= \epsilon_0 (1 + \chi_e) \mathbf{E} \\ &= \epsilon \mathbf{E} \end{aligned}$$

where $\epsilon \equiv \epsilon_0 (1 + \chi_e)$. These things all have names:

$$\begin{array}{cc} \epsilon_0 & permitivity\, of\, free\, space \\ \epsilon & permitivity,\, dielectric\, constant \\ \chi_e & susceptibiliby \\ \frac{\epsilon}{\epsilon_0} & dielectric\, constant \end{array}$$

We can also simplify our relations for the bound charge densities in linear materials. Starting from $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, and using Gauss's law,

$$\begin{aligned}
\rho_b &\equiv -\boldsymbol{\nabla} \cdot \mathbf{P} \\
&= -\epsilon_0 \chi_e \boldsymbol{\nabla} \cdot \mathbf{E} \\
&= \epsilon_0 \chi_e \frac{1}{\epsilon_0} \rho \\
&= -\chi_e \rho \\
&= -\chi_e \left(\rho_f + \rho_b\right)
\end{aligned}$$

Solving for ρ_b,

$$\rho_b (\mathbf{x}) = -\frac{\chi_e}{1+\chi_e} \rho_f (\mathbf{x})$$

This holds point by point for all $\mathbf{x}$, so if there is no free charge suspended in the dielectric then there is no bound charge either,

$$\rho_f = 0 \implies \rho_b = 0$$

We may re-express the bound surface charge in terms of the potential. Setting $\mathbf{E} = -\boldsymbol{\nabla} V$ so that $\mathbf{P} = -\epsilon_0 \chi_e \boldsymbol{\nabla} V$ gives

$$\begin{aligned}
\sigma_b &= \hat{\mathbf{n}} \cdot \mathbf{P} \\
&= -\epsilon_0 \chi_e \frac{\partial V}{\partial n}
\end{aligned}$$

where $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} V \equiv \frac{\partial V}{\partial n}$ is the directional derivative of V in the outward normal direction. The bound surface charge is therefore proportional to the normal derivative of the potential at the boundary.

4 Boundary conditions

Recall the discussion of boundary conditions when we introduced Maxwell's equations for electrostatics. We now modify those results to include the presence of a linear material.

Integrating the divergence of the displacement over a volume, and using the divergence theorem, we have

$$\begin{aligned}
\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{D}\, d^3x &= \int_{\mathcal{V}} \rho_f d^3x \\
\oint_{\mathcal{S}} \mathbf{D} \cdot \hat{\mathbf{n}}\, d^2x &= Q_f
\end{aligned}$$

and we can find the boundary condition at an interface by integrating over a small cylinder that pierces the surface. Making the sides of the cylinder vanishingly small so that the charge density reduces to the surface density and the surface integral is essentially the contribution of the top and bottom of the cylinder, we have

$$\left(\mathbf{D}_{top} \cdot \hat{\mathbf{n}} - \mathbf{D}_{bottom} \cdot \hat{\mathbf{n}}\right) A = \sigma_f A$$

and therefore the change in the normal component of the electric displacement is

$$D_\perp^{out} - D_\perp^{in} = \sigma_f$$

For the tangential component of the displacement, we take the curl of $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$,

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{D} &= \boldsymbol{\nabla} \times \left(\epsilon_0 \mathbf{E} + \mathbf{P}\right) \\
&= \boldsymbol{\nabla} \times \mathbf{P}
\end{aligned}$$

since the curl of $\mathbf{E}$ vanishes. Choose an elongated rectangle with one long side of length L and direction $\hat{\mathbf{m}}$ parallel to the surface on the outside and one long side parallel to the surface on the inside. Let the short sides have length $d \ll L$. Now take the dot product of the normal to this surface with the curl, and integrate over the surface,

$$\iint \mathbf{n} \cdot (\boldsymbol{\nabla} \times \mathbf{D})\, d^2x = \iint \mathbf{n} \cdot (\boldsymbol{\nabla} \times \mathbf{P})\, d^2x$$

Using Stokes' theorem this becomes the line integral around the boundary,

$$\oint \mathbf{D} \cdot d\mathbf{l} = \oint \mathbf{P} \cdot d\mathbf{l}$$

Neglecting any contribution from the short sides (you may take the limit as $d \longrightarrow 0$),

$$L \left(\mathbf{D}^{out} \cdot \hat{\mathbf{m}} - \mathbf{D}^{in} \cdot \hat{\mathbf{m}} \right) = L \left(\mathbf{P}^{out} \cdot \hat{\mathbf{m}} - \mathbf{P}^{in} \cdot \hat{\mathbf{m}} \right)$$

Cancel L; now, since this relation holds for any $\hat{\mathbf{m}}$ parallel to the surface, we must have

$$\mathbf{D}^{out}_{\parallel} - \mathbf{D}^{in}_{\parallel} = \mathbf{P}^{out}_{\parallel} - \mathbf{P}^{in}_{\parallel}$$

Recall that the same computation with $\boldsymbol{\nabla} \times \mathbf{E} = 0$ gave us $\mathbf{E}^{out}_{\parallel} - \mathbf{E}^{in}_{\parallel} = 0$. In linear materials, we may simply use the equality of parallel components of the electric field, and compute $\mathbf{D}$ from $\mathbf{E}$.

5 Summary

In the presence of linear materials we have

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$$

Since there is always a potential for the electric field, we may write

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ &= -\epsilon \boldsymbol{\nabla} V \end{aligned}$$

so that we have the Poisson equation in terms of the free charge density as before, but with ϵ_0 replaced by ϵ:

$$\nabla^2 V = -\frac{\rho_f}{\epsilon}$$

The boundary conditions change to

$$\begin{aligned} D^{out}_{\perp} - D^{in}_{\perp} &= \sigma_f \\ \mathbf{E}^{out}_{\parallel} - \mathbf{E}^{in}_{\parallel} &= 0 \end{aligned}$$

With free charges absent, $\rho_f = \rho_b = 0$ and $\sigma_f = 0$, so for the interior or exterior regions we are solving only the Laplace equation

$$\begin{aligned} \nabla^2 V_{in} &= 0 \\ \nabla^2 V_{out} &= 0 \end{aligned}$$

and the boundary conditions are simple continuity of the normal component of $\mathbf{D}$ and the parallel component of $\mathbf{E}$:

$$\begin{aligned} D^{out}_{\perp} - D^{in}_{\perp} &= 0 \\ \mathbf{E}^{out}_{\parallel} - \mathbf{E}^{in}_{\parallel} &= 0 \end{aligned}$$

If we have azimuthal symmetry, the solutions may be written as

$$\begin{aligned} V_{in}(r,\theta) &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \\ V_{out}(r,\theta) &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \end{aligned}$$

and we need only impose the boundary conditions.

6 Solution to the Laplace equation in cylindrical coordinates

First, we need the solution to the Laplace equation in cylindrical coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Using separation of variables, we write $V = R(\rho)\Phi(\varphi)Z(z)$, substitute, then divide by V,

$$\begin{aligned} \Phi Z\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + RZ\frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + R\Phi\frac{d^2 Z}{dz^2} &= 0 \\ \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} &= 0 \end{aligned}$$

The only z-dependence is in the final term, so we must have

$$\begin{aligned} \frac{1}{Z}\frac{d^2 Z}{dz^2} &= \alpha^2 \\ \frac{d^2 Z}{dz^2} - \alpha^2 Z &= 0 \end{aligned}$$

so that for $\alpha^2 > 0$, $Z = Ae^{\alpha z} + Be^{-\alpha z}$. Then, multiplying by ρ^2 we have

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + \alpha^2\rho^2 = 0$$

leaving the φ dependence isolated in the middle term. Therefore, since we need periodicity in φ, we set

$$\begin{aligned} \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} &= -\beta^2 \\ \frac{d^2\Phi}{d\varphi^2} + \beta^2\Phi &= 0 \end{aligned}$$

to get $\Phi = C\cos\beta\varphi + D\sin\beta\varphi$. Since we require the full azimuthal angle, $\varphi : 0 \to 2\pi$, we must have $\beta = m$ for some integer m.

Finally, we are left with

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(\alpha^2\rho^2 - \beta^2\right)R = 0$$

This is the Bessel equation. We will only solve a simpler version of this here.

We do not need the full solution to the Bessel equation, since the current problem has no z dependence. Setting $\alpha^2 = 0$ and $\beta = m$ gives a simpler equation,

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - m^2 R = 0$$

When $m \neq 0$, powers of ρ work. Let $R = \rho^{\lambda}$. Then

$$\begin{aligned}
\rho \frac{d}{d\rho}\left(\rho \lambda \rho^{a-1}\right) - m^2 \rho^a &= 0 \\
\lambda^2 \rho^a - m^2 \rho^a &= 0
\end{aligned}$$

so that $\lambda = \pm m$. Finally, when $m = 0$, we have

$$\begin{aligned}
\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) &= 0 \\
\rho \frac{dR}{d\rho} &= b \\
dR &= \frac{b}{\rho} d\rho
\end{aligned}$$

and integrating,

$$R = a + b \ln \rho$$

The full solution when $V = V(\rho, \varphi)$ only is therefore

$$V(\rho, \varphi) = a + b \ln \rho + \sum_{m=1}^{\infty} \left(a_m \rho^m + \frac{b_m}{\rho^m}\right)(c_m \cos m\varphi + d_m \sin m\varphi)$$

7 Examples

7.1 Infinite half-space

Let the $x > 0$ half of space be filled with a uniform linear dielectric with permitivity ϵ, while the region $x < 0$ remains empty. Suppose there is a constant electric field, $\mathbf{E}^{out} = E_x^{out}\hat{\mathbf{i}} + E_y^{out}\hat{\mathbf{j}}$ in the free space region outside the dielectric ($x < 0$). Find the electric field in the dielectric.

The boundary conditions are:

$$\begin{aligned}
D_\perp^{out} - D_\perp^{in} &= 0 \\
\mathbf{E}_\parallel^{out} - \mathbf{E}_\parallel^{in} &= 0
\end{aligned}$$

Let the field outside the dielectric, $\mathbf{E} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}$, be $\mathbf{E}^{out}$, and let

$$\mathbf{E}^{in} = E_x^{in}\hat{\mathbf{i}} + E_y^{in}\hat{\mathbf{j}} + E_k^{in}\hat{\mathbf{k}}$$

be the field inside the dielectric. Then

$$\begin{aligned}
\mathbf{E}_\parallel^{out} &= E_y^{out}\hat{\mathbf{j}} \\
E_\perp^{out} &= E_x^{out}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}_\parallel^{in} &= E_y^{in}\hat{\mathbf{j}} + E_k^{in}\hat{\mathbf{k}} \\
E_\perp^{in} &= E_x^{in}
\end{aligned}$$

Imposing the boundary conditions, with $\mathbf{D}^{out} = \epsilon_0 \mathbf{E}^{out}$ and $\mathbf{D}^{in} = \epsilon \mathbf{E}^{in}$

$$\begin{aligned}
\epsilon_0 E_x^{out} - \epsilon E_x^{in} &= 0 \\
E_y^{out}\hat{\mathbf{j}} - E_y^{in}\hat{\mathbf{j}} - E_k^{in}\hat{\mathbf{k}} &= 0
\end{aligned}$$

and therefore

$$\begin{aligned}
\epsilon E_x^{in} &= \epsilon_0 E_x^{out} \\
E_y^{in} &= E_y^{out} \\
E_z^{in} &= 0
\end{aligned}$$

and the field inside the dielectric is

$$\mathbf{E}^{in} = \frac{\epsilon_0}{\epsilon} E_x^{out} \hat{\mathbf{i}} + E_y^{out} \hat{\mathbf{j}}$$

Suppose we write this in terms of angles. The normal to the surface is $\hat{\mathbf{i}}$. Let the incoming electric field make an angle θ with this normal. Then

$$\mathbf{E}^{out} = E_0^{out} \left(\hat{\mathbf{i}} \sin\theta + \hat{\mathbf{j}} \cos\theta \right)$$

so that

$$\begin{aligned}
E_x^{out} &= E_0 \cos\theta \\
E_y^{out} &= E_0 \sin\theta
\end{aligned}$$

Similarly, if the electric field in the dielectric makes an angle θ' with $\hat{\mathbf{i}}$ then

$$\begin{aligned}
\mathbf{E}^{in} &= E_0^{in} \left(\hat{\mathbf{i}} \cos\theta' + \hat{\mathbf{j}} \sin\theta' \right) \\
&= \frac{\epsilon_0}{\epsilon} E_x^{out} \hat{\mathbf{i}} + E_y^{out} \hat{\mathbf{j}}
\end{aligned}$$

Equating components,

$$\begin{aligned}
E_0^{in} \cos\theta' &= \frac{\epsilon_0}{\epsilon} E_x^{out} \\
&= \frac{\epsilon_0}{\epsilon} E_0 \cos\theta \\
E_0^{in} \sin\theta' &= E_0 \sin\theta
\end{aligned}$$

we have

$$\begin{aligned}
\frac{E_0^{in} \sin\theta'}{E_0^{in} \cos\theta'} &= \frac{E_0 \sin\theta}{\frac{\epsilon_0}{\epsilon} E_0 \cos\theta} \\
\frac{\sin\theta'}{\cos\theta'} &= \frac{\epsilon \sin\theta}{\epsilon_0 \cos\theta} \\
\epsilon_0 \tan\theta' &= \epsilon \tan\theta
\end{aligned}$$

This is not the same as Snell's law, which holds for electromagnetic waves at a similar interface and depends on the index of refraction, n, and not just the dielectric constant.

7.2 Infinite half-space with a charge

7.2.1 The induced bound charge density

Let the $z < 0$ half of space be filled with a uniform linear dielectric with permitivity ϵ, while the region $x > 0$ remains empty except for a single charge Q on the z axis at a distance $d > 0$. Find the electric field on the positive z axis and compute the force on the charge Q.

The solution for the potential in the upper region is a linear combination of the potential of the given charge and the response of the dielectric. Since $\rho_b = -\frac{\chi_e}{1+\chi_e}\rho_f$ and there is no free charge inside the dielectric, $\rho_b = 0$. However, there will be a surface charge density given by

$$\begin{aligned}\sigma_b &= -\epsilon_0 \chi_e \frac{\partial V}{\partial n} \\ &= -\epsilon_0 \chi_e \frac{\partial V}{\partial z}\end{aligned}$$

at the boundary, where the potential at the boundary is:

$$V(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} + V_b(\mathbf{x})$$

To take the derivative to find σ_b, we need to find the potential V_b due to σ_b. This isn't as circular as it seems, since sufficiently near the surface the field is the same as for an infinite plane with charge per unit area σ_b ,

$$E_b|_{above\,surface} = \frac{\sigma_b}{2\epsilon_0}\hat{\mathbf{n}}$$

Integrating gives the electric field,

$$V_b(z) = \frac{\sigma_b |z|}{2\epsilon_0}$$

for small z. The surface charge therefore satisfies

$$\begin{aligned}\sigma_b &= -\epsilon_0 \chi_e \frac{\partial}{\partial z}\left(\frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{\sigma_b z}{2\epsilon_0}\right)\Bigg|_{z=0} \\ &= \frac{\chi_e Q}{4\pi}\frac{z-d}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{1}{2}\chi_e\sigma_b\Bigg|_{z=0} \\ &= -\frac{\chi_e Q}{4\pi}\frac{d}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{1}{2}\chi_e\sigma_b\Bigg|_{z=0}\end{aligned}$$

Solving for σ_b,

$$\begin{aligned}\left(1+\frac{\chi_e}{2}\right)\sigma_b &= -\frac{\chi_e Q}{4\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}} \\ \sigma_b &= -\frac{\chi_e}{2\pi(2+\chi_e)}\frac{Qd}{(x^2+y^2+d^2)^{3/2}}\end{aligned}$$

Once we know this, it is possible to find the potential using the method of images. The result follows by placing a charge q at $-d\hat{\mathbf{k}}$ and matching the boundary conditions.

7.2.2 Doing it the hard way (optional!)

It is also possible to find the full potential without the method of images, while still avoiding elliptic integrals.

First, notice that given σ_b, we may compute V_b everywhere by direct integration, but the form of the integral is daunting.

$$V_b(\mathbf{x}_0) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{|\mathbf{x}_0 - \mathbf{x}|}$$

$$
\begin{aligned}
&= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \frac{\sigma_b}{|\mathbf{x}_0 - \mathbf{x}|}\Big|_{z=0} \\
&= -\frac{\chi_e}{2\pi(2+\chi_e)} \frac{Qd}{4\pi\epsilon_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \frac{1}{\left((x_0-x)^2 + (y_0-y)^2 + z_0^2\right)^{1/2}} \frac{1}{(x^2+y^2+d^2)^{3/2}}
\end{aligned}
$$

Instead of tackling this straight on, we first restrict to the positive z-axis. From there we can use a Legendre series to write the potential for all $z > 0$. Finally, we recognize the expansion as the sum of two Coulomb potentials. To begin, we rewrite

$$
V_b(z_0) = -\frac{\chi_e}{2\pi(2+\chi_e)} \frac{Qd}{4\pi\epsilon_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \frac{1}{(x^2+y^2+z_0^2)^{1/2}} \frac{1}{(x^2+y^2+d^2)^{3/2}}
$$

in cylindrical coordinates. With the observation point at

$$
\mathbf{x}_0 = z_0\hat{\mathbf{k}}
$$

the potential becomes

$$
\begin{aligned}
V_b(z_0) &= -\frac{\chi_e}{2\pi(2+\chi_e)} \frac{Qd}{4\pi\epsilon_0} \int_0^{\infty}\int_0^{2\pi} d\varphi\rho d\rho \frac{1}{(\rho^2+z_0^2)^{1/2}} \frac{1}{(\rho^2+d^2)^{3/2}} \\
&= -\frac{\chi_e}{2+\chi_e} \frac{Qd}{4\pi\epsilon_0} \int_0^{\infty} \rho d\rho \frac{1}{(\rho^2+z_0^2)^{1/2}} \frac{1}{(\rho^2+d^2)^{3/2}}
\end{aligned}
$$

Now let $\xi = \rho^2$ so that $d\xi = 2\rho d\rho$. Then

$$
\begin{aligned}
V_b(z_0) &= -\frac{\chi_e}{2+\chi_e} \frac{Qd}{8\pi\epsilon_0} \int_0^{\infty} \frac{d\xi}{(\xi+z_0^2)^{1/2}(\xi+d^2)^{3/2}} \\
\left[\frac{(\xi+z_0^2)^{1/2}}{(\xi+d^2)^{1/2}}\right]' &= \frac{1}{2}\frac{1}{(\xi+z_0^2)^{1/2}(\xi+d^2)^{1/2}} - \frac{1}{2}\frac{(\xi+z_0^2)^{1/2}}{(\xi+d^2)^{3/2}} \\
&= \frac{1}{2}\left[\frac{\xi+d^2}{(\xi+z_0^2)^{1/2}(\xi+d^2)^{3/2}} - \frac{\xi+z_0^2}{(\xi+z_0^2)^{1/2}(\xi+d^2)^{3/2}}\right] \\
&= \frac{1}{2}\frac{d^2-z_0^2}{(\xi+z_0^2)^{1/2}(\xi+d^2)^{3/2}}
\end{aligned}
$$

With a little help from Wolfram integrator, we have

$$
\begin{aligned}
V_b(z_0) &= -\frac{\chi_e}{2+\chi_e} \frac{Qd}{8\pi\epsilon_0} \left[\frac{2}{d^2-z_0^2} \frac{(\xi+z_0^2)^{1/2}}{(\xi+d^2)^{1/2}}\right]_0^{\infty} \\
&= -\frac{\chi_e}{2+\chi_e} \frac{Qd}{8\pi\epsilon_0} \left(\frac{2}{d^2-z_0^2} - \frac{2z_0}{d(d^2-z_0^2)}\right) \\
&= -\frac{\chi_e}{2+\chi_e} \frac{Q}{4\pi\epsilon_0(d^2-z_0^2)}(d-z_0) \\
&= -\frac{\chi_e}{2+\chi_e} \frac{Q}{4\pi\epsilon_0(d+z_0)}
\end{aligned}
$$

Therefore, the potential on the $z$axis is (dropping the subscript on z_0)

$$V(z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{(z-d)^2}} - \frac{\chi_e}{2+\chi_e}\frac{Q}{4\pi\epsilon_0(d+z)}$$

Notice that this already has the form of two Coulomb potentials restricted to the z axis.

We can show rigorously that the full solution is exactly two Coulomb potentials. For $z > d$, we may expand in Taylor series,

$$\begin{aligned}
V(z) &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{z-d} - \frac{\chi_e}{2+\chi_e}\frac{1}{d+z}\right) \\
&= \frac{Q}{4\pi\epsilon_0 z}\left(\frac{1}{1-\frac{d}{z}} - \frac{\chi_e}{2+\chi_e}\frac{1}{1+\frac{d}{z}}\right) \\
&= \frac{Q}{4\pi\epsilon_0 z}\sum_{n=0}^{\infty}\left(\left(\frac{d}{z}\right)^n - \frac{\chi_e}{2+\chi_e}(-1)^n\left(\frac{d}{z}\right)^n\right) \\
&= \frac{Q}{4\pi\epsilon_0 z}\sum_{n=0}^{\infty}\left(1 - \frac{\chi_e}{2+\chi_e}(-1)^n\right)\left(\frac{d}{z}\right)^n \\
&= \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(1 - \frac{\chi_e}{2+\chi_e}(-1)^n\right)\left(\frac{d^n}{z^{n+1}}\right)
\end{aligned}$$

If we compare this to the expansion in Legendre polynomials when $r = z$ and $\theta = 0$,

$$\begin{aligned}
V(z) &= \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(A_l z^l + \frac{B_l}{z^{l+1}}\right)P_l(1) \\
\frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(1 - \frac{\chi_e}{2+\chi_e}(-1)^n\right)\left(\frac{d^n}{z^{n+1}}\right) &= \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(A_l z^l + \frac{B_l}{z^{l+1}}\right)
\end{aligned}$$

we see that

$$\begin{aligned}
A_l &= 0 \\
B_l &= \left(1 - \frac{\chi_e}{2+\chi_e}(-1)^n\right)d^n
\end{aligned}$$

and this determines the whole series:

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(1 - \frac{\chi_e}{2+\chi_e}(-1)^n\right)\left(\frac{d^l}{z^{l+1}}\right)P_l(\cos\theta)$$

But we know that we can expand

$$\frac{1}{|\mathbf{x}_0 - \mathbf{x}|} = \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(\frac{r^l}{r_0^{l+1}}\right)P_l(\cos\theta)$$

so we recognize $V(r,\theta)$ as

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\left|\mathbf{x}_0 - d\hat{\mathbf{k}}\right|} - \frac{\chi_e}{2+\chi_e}\frac{1}{\left|\mathbf{x}_0 + d\hat{\mathbf{k}}\right|}\right)$$

and this is just the potential of the original charge Q together with that of an image charge $-\frac{\chi_e}{2+\chi_e}Q$ at $-d\hat{\mathbf{k}}$.

The same reasoning lets us show the result for $z < d$, or we can show that this V satisfies the correct boundary conditions.

7.3 Dielectric sphere in a uniform field

A sphere of radius R, made of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting field inside and outside the sphere. The susceptibility is χ_e.

First, choose the z-axis in the direction of the uniform electric field. Then the boundary condition at infinity is

$$V_\infty = -E_0 r\cos\theta$$

Solving the Laplace equation for the potential outside the sphere, we have

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

As $r\to\infty$, the B_l terms become negligibly small. The r^l terms grow large, but we must nevertheless have a term linear in r,

$$V_\infty(r,\theta) = -E_0 r\cos\theta = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

so $A_1 = -E_0$, with all other $A_l = 0$. Therefore, putting in the single nonzero A_l term explicitly, we have

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

for the exterior solution. To find the constants B_l, we must match this to the interior solution.

In the interior of the sphere, we have the free and bound charge densities related by

$$\rho_b = \frac{\chi_e}{1-\chi_e}\rho_f$$

but there is no free charge so both vanish, and we again solve the Laplace equation,

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos\theta)$$

This time, regularity at the origin tells us that $D_l = 0$ for all l, so

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta)$$

The boundary conditions at $r = R$ are:

$$\begin{aligned} D_\perp^{out} - D_\perp^{in} &= \sigma_f \\ \mathbf{E}_\parallel^{out} - \mathbf{E}_\parallel^{in} &= 0 \end{aligned}$$

Now, compute the electric field:

$$\begin{aligned}
\mathbf{E}_{out} &= -\boldsymbol{\nabla} V_{out}(r,\theta) \\
&= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(-E_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l\right) \\
&= \hat{\mathbf{r}}\left(E_0\cos\theta + \sum_{l=0}^{\infty}\frac{(l+1)B_l}{r^{l+2}}P_l\right) - \hat{\boldsymbol{\theta}}\left(E_0\sin\theta + \sum_{l=1}^{\infty}\frac{B_l}{r^{l+2}}\frac{\partial P_l}{\partial\theta}\right) \\
\mathbf{E}_{in} &= -\boldsymbol{\nabla} V_{in}(r,\theta) \\
&= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\sum_{l=0}^{\infty} C_l r^l P_l \\
&= -\hat{\mathbf{r}}\sum_{l=1}^{\infty} l C_l r^{l-1} P_l - \hat{\boldsymbol{\theta}}\sum_{l=1}^{\infty} C_l r^{l-1}\frac{\partial P_l}{\partial\theta}
\end{aligned}$$

where we use $\frac{\partial P_0}{\partial\theta} = 0$ in the second sum. The parallel components are those in the $\hat{\boldsymbol{\theta}}$ direction. For the parallel components at $r = R$

$$
\begin{aligned}
\mathbf{E}_{\parallel}^{out} &= \mathbf{E}_{\parallel}^{in} \\
-\hat{\boldsymbol{\theta}}\left(E_0 \sin\theta + \sum_{l=1}^{\infty} \frac{B_l}{R^{l+2}} \frac{\partial P_l}{\partial\theta}\right) &= -\hat{\boldsymbol{\theta}} \sum_{l=1}^{\infty} C_l R^{l-1} \frac{\partial P_l}{\partial\theta} \\
\left(-E_0 \frac{\partial P_1}{\partial\theta} + \frac{B_1}{R^3} \frac{\partial P_1}{\partial\theta}\right) + \sum_{l=2}^{\infty} \frac{B_l}{R^{l+2}} \frac{\partial P_l}{\partial\theta} &= C_1 \frac{\partial P_1}{\partial\theta} + \sum_{l=2}^{\infty} C_l R^{l-1} \frac{\partial P_l}{\partial\theta}
\end{aligned}
$$

where we separate out the $l = 1$ terms.

The derivative terms are all independent (see optional section below), so each coefficient must vanish. For each $l \geq 2$, this implies

$$
C_l = \frac{B_l}{R^{2l+1}}
$$

while for the $l = 1$ term,

$$
C_1 = -E_0 + \frac{B_1}{R^3}
$$

This fixes all of the C_l in terms of the B_l.

Now we look at the normal components of $\mathbf{D}$. Again, there is no *free* surface charge, so we have continuity of the normal components at R

$$
\begin{aligned}
D_{\perp}^{out} &= D_{\perp}^{in} \\
\epsilon_0 E_0 P_1 + \epsilon_0 \sum_{l=0}^{\infty} \frac{(l+1) B_l}{R^{l+2}} P_l &= -\epsilon \sum_{l=1}^{\infty} l C_l R^{l-1} P_l
\end{aligned}
$$

where we have written P_1 for $\cos\theta$. For $l = 0$,

$$
\begin{aligned}
\epsilon_0 \frac{B_0}{R^2} &= 0 \\
B_0 &= 0
\end{aligned}
$$

For $l = 1$,

$$
\begin{aligned}
\epsilon_0 E_0 P_1 + \epsilon_0 \frac{2B_1}{R^3} P_1 &= -\epsilon C_1 P_1 \\
\epsilon_0 E_0 + \epsilon_0 \frac{2B_1}{R^3} &= -\epsilon C_1 \\
\epsilon_0 E_0 + \epsilon_0 \frac{2B_1}{R^3} &= -\epsilon\left(-E_0 + \frac{B_1}{R^3}\right) \\
\left(\frac{2\epsilon_0 + \epsilon}{R^3}\right) B_1 &= (\epsilon - \epsilon_0) E_0 \\
B_1 &= \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon}\right) E_0 R^3
\end{aligned}
$$

Finally, for the remaining l,

$$
\epsilon_0 \frac{(l+1) B_l}{R^{l+2}} = -\epsilon l C_l R^{l-1}
$$

Substituting the values for C_l,

$$\begin{aligned}
\epsilon_0 \frac{(l+1)\, B_l}{R^{l+2}} &= -\epsilon l \frac{B_l}{R^{2l+1}} R^{l-1} \\
B_l \left(\epsilon_0 \frac{(l+1)}{R^{l+2}} + \epsilon l \frac{1}{R^{2l+1}} R^{l-1} \right) &= 0 \\
\frac{1}{R^{l+2}} B_l \left(\epsilon_0 (l+1) + \epsilon l \right) &= 0 \\
B_l &= 0
\end{aligned}$$

and therefore $B_l = 0$ and $C_l = 0$ for all $l \geq 2$, while for the $l = 1$ term,

$$\begin{aligned}
C_1 &= -E_0 + \frac{B_1}{R^3} \\
&= -E_0 + \frac{1}{R^3} \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} \right) E_0 R^3 \\
&= -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0
\end{aligned}$$

The potential is

$$\begin{aligned}
V_{out}(r, \theta) &= -E_0 r \cos\theta + \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} \right) \frac{R^3}{r^2} E_0 \cos\theta \\
V_{in}(r, \theta) &= -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 r \cos\theta
\end{aligned}$$

At the boundary, we check continuity:

$$\begin{aligned}
V_{out}(R, \theta) &= -E_0 R \cos\theta + \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} \right) E_0 R \cos\theta \\
&= -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 R \cos\theta \\
&= V_{in}(R, \theta)
\end{aligned}$$

The final electric fields are

$$\begin{aligned}
\mathbf{E}_{out} &= \left(1 + \frac{2R^3}{r^3} \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} \right) \right) E_0 \left(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}} \right) \\
&= \left(1 + \frac{2R^3}{r^3} \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} \right) \right) E_0 \hat{\mathbf{k}} \\
\mathbf{E}_{in} &= \frac{3\epsilon_0 E_0}{2\epsilon_0 + \epsilon} \left(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}} \right) \\
\mathbf{E}_{in} &= \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 \hat{\mathbf{k}}
\end{aligned}$$

Notice that the polarization density is in the same direction as the electric field.

7.4 Proof of the orthogonality of derivatives of Legendre polynomials (optional)

In solving this, we had the equality

$$E_0 \sin\theta - \frac{B_1}{R^3} \sin\theta + \frac{B_2}{R^4} \frac{\partial}{\partial\theta} P_2(\cos\theta) + \sum_{l=3}^{\infty} \frac{B_l}{R^{l+2}} \frac{\partial}{\partial\theta} P_l(\cos\theta) = -A_1' \sin\theta + A_2' R \frac{\partial}{\partial\theta} P_2(\cos\theta) + \sum_{l=0}^{\infty} A_l' R^{l-1} \frac{\partial}{\partial\theta} P_l(\cos\theta)$$

It's not hard to guess that

$$E_0 \sin\theta = -A_1' \sin\theta$$

but let's look in detail at more general derivative terms. For $P_2 = \frac{1}{2}\left(3x^2 - 1\right)$ we have

$$\begin{aligned} \frac{\partial}{\partial\theta} P_2(\cos\theta) &= \frac{\partial}{\partial\theta}\frac{1}{2}\left(3\cos^2\theta - 1\right) \\ &= -3\cos\theta\sin\theta \end{aligned}$$

Notice two things:

1. The derivative is of the same order as P_2 in trig functions
2. The expression involves $\sin\theta$, which is indepenent of $\cos\theta$

These derivatives must be expressed in terms the *associated Legendre polynomials*, which give the θ-dependence of the *spherical harmonics* (which also depend on φ). All of these are orthogonal to the rest. To show the independence of two of the derivative terms,

$$\frac{\partial}{\partial\theta} P_l(\cos\theta), \frac{\partial}{\partial\theta} P_{l'}(\cos\theta)$$

we may integrate,

$$\int_0^\pi \frac{\partial}{\partial\theta} P_l(\cos\theta) \frac{\partial}{\partial\theta} P_{l'}(\cos\theta)\sin\theta d\theta$$

To change to $x = \cos\theta$, we need

$$\begin{aligned} \frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} \\ &= -\sin\theta\frac{\partial}{\partial x} \end{aligned}$$

and the integral becomes

$$\begin{aligned} \int_0^\pi \frac{\partial}{\partial\theta} P_l(\cos\theta) \frac{\partial}{\partial\theta} P_{l'}(\cos\theta)\sin\theta d\theta &= \int_{-1}^1 \sin^2\theta \frac{\partial}{\partial x} P_l(x) \frac{\partial}{\partial x} P_{l'}(x)\, dx \\ &= \int_{-1}^1 \frac{\partial}{\partial x} P_l(x) \frac{\partial}{\partial x} P_{l'}(x)\left(1 - x^2\right) dx \end{aligned}$$

Without loss of generality, suppose $l' < l$. Then we may integrate by parts

$$\int_{-1}^1 \frac{\partial P_l}{\partial x}\frac{\partial P_{l'}}{\partial x}\left(1 - x^2\right) dx = \int_{-1}^1 \frac{\partial}{\partial x}\left(P_l \frac{\partial P_{l'}}{\partial x}\left(1 - x^2\right)\right) dx - \int_{-1}^1 P_l \frac{\partial}{\partial x}\left(\frac{\partial P_{l'}}{\partial x}\left(1 - x^2\right)\right) dx$$

But $\frac{\partial P_{l'}}{\partial x}$ is a polynomial of order $x^{l'-1}$ so that $\frac{\partial}{\partial x}\left(\frac{\partial P_{l'}}{\partial x}\left(1 - x^2\right)\right)$ is a polynomial of order $x^{l'-1+2-1} = x^{l'}$ and therefore expressible in terms of Legendre polynomials of order l' or less. Since this linear combination,

$$\frac{\partial}{\partial x}\left(\frac{\partial P_{l'}}{\partial x}\left(1 - x^2\right)\right) = \sum_{k=0}^{l'} \alpha_k P_k(x)$$

is multiplied by P_l with $l > k$, each term in the sum is orthogonal and the integral vanishes.

This proves that $\frac{\partial}{\partial\theta} P_l(\cos\theta)$ and $\frac{\partial}{\partial\theta} P_{l'}(\cos\theta)$ are orthogonal if $l \neq l'$.

You may assume this when working these problems.

8 Exercises

8.1 Parallel plate capacitor

Consider two parallel rectangular conducting plates of area A and with separation d, with $d \ll A$.

1. Neglecting edge effects, compute the capacitance when there is no material between the plates.
2. Recompute the capacitance if a uniform, linear dielectric material of permitivity ϵ fills the region between the plates.

8.2 Plane and dielectric

The region below the xy-plane is filled with a dielectric material of permitivity ϵ. An infinite conducting plane with surface charge density σ is held at $z = d$ parallel to the surface of the dielectric. Find the bound surface charge density of the dielectric.

8.3 Spherical shell

A hollow dielectric sphere extends from $r = a$ to $r = b$. The region with $r < a$ and the region with $r > b$ are empty except for a charge Q placed at the origin.

1. Find the electric potential and electric field everywhere.
2. Find the bound surface charge on both spherical surfaces (at a and at b).
3. Integrate your answers to part 2 over each surface to find the total bound charge on each surface.

8.4 Charge outside a dielectric half-space

Repeat the example above of a single charge Q a distance d from an infinite half-space filled with a material of permitivity ϵ using the method of images.

8.5 Dielectric cylinder

A solid dielectric cylinder of radius $\rho = R$ and infinitely long lies centered along the z-axis. There is a uniform electric field in the x-direction, $\mathbf{E} = E_0\hat{\mathbf{i}}$, perpendicular to the axis of the cylinder. Find the electric potential everywhere.