Multipole moments

November 9, 2015

1 The far field expansion

Suppose we have a localized charge distribution, confined to a region near the origin with $r < R$. Then for values of $r > R$, the electric field must be described by solutions to the Laplace equation with vanishing boundary condition at infinity. This requires $A_l = 0$ for an expansion in Legendre polynomials and the potential for any azimuthally symmetric source is

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} B_l P_l(\cos\theta)$$

For large values of r, the lowest nonvanishing term will dominate this approximation, giving an excellent approximation for the potential

Now consider the full solution for *any* localized source near the origin. Let the charge density be $\rho(\mathbf{x})$, and suppose we with to know the potential at a point $\mathbf{x}_0$ far from the source. The general solution for the potential is

$$V(\mathbf{x}_0) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x})\, d^3x}{|\mathbf{x}_0 - \mathbf{x}|}$$

Let the angle between $\mathbf{x}_0 = r_0\hat{\mathbf{r}}_0$ and $\mathbf{x} = r\hat{\mathbf{r}}$ be θ, so the denominator is

$$\begin{aligned} |\mathbf{x}_0 - \mathbf{x}| &= \sqrt{x_0^2 + x^2 - 2xx_0\cos\theta} \\ &= r_0\sqrt{1 + \frac{r^2 - 2rr_0\cos\theta}{r_0^2}} \end{aligned}$$

Since $r_0 \gg r$, we could expand the square root in a Taylor series and collect the resulting powers of $\cos\theta$ into Legendre polynomials. Fortunately, there is an easier way.

We would like the Legendre series for $\frac{1}{|\mathbf{x}_0 - \mathbf{x}|}$, and in order to vanish at infinity it must take the form

$$\frac{1}{|\mathbf{x}_0 - \mathbf{x}|} = \sum_{l=0}^{\infty} \frac{1}{r_0^{l+1}} B_l P_l(\cos\theta)$$

Consider what happens at $\theta = 0$. The Legendre polynomials all give 1 since$P_l(\cos 0) = P_l(1) = 1$, so when $\mathbf{x}_0$ and $\mathbf{x}$ are parallel the series reduces to

$$\frac{1}{|r_0 - r|} = \sum_{l=0}^{\infty} \frac{1}{r_0^{l+1}} B_l$$

But

$$\frac{1}{|r_0 - r|} = \frac{1}{r_0}\left(1 - \frac{r}{r_0}\right)^{-1}$$

and we know the Taylor series for $(1-x)^{-1}$,

$$\begin{aligned}
f(x) &= (1-x)^{-1} \\
f'(x) &= (1-x)^{-2} \\
f''(x) &= 2(1-x)^{-3} \\
&\vdots \\
f^{(n)}(x) &= n!\,(1-x)^{-n-1}
\end{aligned}$$

Evaluating these at $x=0$ gives $\left.\frac{d^n f}{dx^n}\right|_{x=0} = n!$ so the Taylor series is

$$\begin{aligned}
(1-x)^{-1} &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} n! x^n \\
&= \sum_{n=0}^{\infty} x^n
\end{aligned}$$

Letting $x = \frac{r}{r_0}$ and substituting into our expansion, this means that

$$\begin{aligned}
\sum_{l=0}^{\infty} \frac{1}{r_0^{l+1}} B_l &= \frac{1}{|r_0 - r|} \\
&= \frac{1}{r_0}\left(1 - \frac{r}{r_0}\right)^{-1} \\
&= \frac{1}{r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n
\end{aligned}$$

Comparing like powers of r_0 shows that

$$B_l = r^l$$

This gives all of the coefficients in the original series! Therefore,

$$\frac{1}{|\mathbf{x}_0 - \mathbf{x}|} = \sum_{l=0}^{\infty} \frac{r^l}{r_0^{l+1}} P_l(\cos\theta)$$

Now, returning to the potential,

$$\begin{aligned}
V(\mathbf{x}_0) &= \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\mathbf{x}_0 - \mathbf{x}|} \rho(\mathbf{x})\, d^3x \\
&= \frac{1}{4\pi\epsilon_0} \int \sum_{l=0}^{\infty} \frac{r^l}{r_0^{l+1}} P_l(\cos\theta)\, \rho(\mathbf{x})\, d^3x \\
&= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r_0^{l+1}} \int r^l \rho(\mathbf{x})\, P_l(\cos\theta)\, d^3x
\end{aligned}$$

where the angle θ is the angle between $\mathbf{x}_0$ and $\mathbf{x}$.

2 Multipole moments

The integrals

$$Q_l \equiv \frac{1}{4\pi\epsilon_0}\int r^l \rho(\mathbf{x}) P_l(\cos\theta)\ d^3x$$

still depend on the angle between the charge increment at $\mathbf{x}$ and the observation point $\mathbf{x}_0$. For the first few values of l, we now separate Q_l into a part depending on the observation direction and another part which depends only on the charge distribution. These latter expressions are the multipole moments of the charge distribution. Once we know all of the multipole moments that characterize a charge distribution, we have the complete potential:

$$V(\mathbf{x}_0) = \sum_{l=0}^{\infty} \frac{Q_l}{r_0^{l+1}}$$

It is important to be able to recognize and understand the properties of the first few multipole moments. They are extremely useful for describing arbitrary charge distributions.

3 Total charge

The zeroth moment is proportional to the total charge,

$$\begin{aligned} Q_0 &= \frac{1}{4\pi\epsilon_0}\int r^0 \rho(\mathbf{x}) P_0(\cos\alpha)\, d^3x \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x})\, d^3x \\ &= \frac{1}{4\pi\epsilon_0} Q_{total} \end{aligned}$$

Notice that the Q_{total} is independent of the observation point $\mathbf{x}_0$. It characterizes only the source.

The potential of a pure monopole is given by Coulomb's law for a single point charge,

$$V(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r}$$

and the resulting electric field is

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla} V \\ &= -\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0 r}\right) \\ &= \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \end{aligned}$$

as we know.

For *any* source, at large distances $r_0 \gg r$ the dominant term in the potential is the Coulomb potential of the total charge,

$$V(\mathbf{x}_0) \approx \frac{Q_{total}}{4\pi\epsilon_0 r_0}$$

Higher multipole moments give corrections to this.

4 Dipole moment

The next term depends on the dipole moment,

$$\begin{aligned}
Q_1 &\equiv \frac{1}{4\pi\epsilon_0}\int r\rho(\mathbf{x})\,P_1(\cos\theta)\;d^3x \\
&= \frac{1}{4\pi\epsilon_0}\int r\rho(\mathbf{x})\cos\theta\,d^3x \\
&= \frac{1}{4\pi\epsilon_0}\int r\rho(\mathbf{x})\cos\theta d^3x \\
&= \frac{1}{4\pi\epsilon_0}\hat{\mathbf{r}}_0\cdot\int \mathbf{x}\rho(\mathbf{x})\,d^3x
\end{aligned}$$

We define the dipole moment to be

$$\mathbf{p}\equiv\int \mathbf{x}\rho(\mathbf{x})\,d^3x$$

A dipole moment may be thought of as a pair of equal and opposite charges, slightly displaced. If we take charges $\pm q$ displaced by a vector $d\hat{\mathbf{k}}$, then the charge density is

$$\rho(\mathbf{r}') = q\left(\delta^3\left(\mathbf{r}'-\frac{d}{2}\hat{\mathbf{k}}\right)-\delta^3\left(\mathbf{r}'+\frac{d}{2}\hat{\mathbf{k}}\right)\right)$$

so the dipole moment is

$$\begin{aligned}
\mathbf{p} &\equiv \int \mathbf{x}\rho(\mathbf{x})\,d^3x \\
&= \int \mathbf{x}q\left(\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right)-\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right)\right)d^3x \\
&= q\left(\int \mathbf{x}\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right)d^3r-\int \mathbf{x}\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right)d^3x\right) \\
&= q\left(\frac{d}{2}\hat{\mathbf{k}}+\frac{d}{2}\hat{\mathbf{k}}\right) \\
&= qd\hat{\mathbf{k}}
\end{aligned}$$

that is, the vector displacement between the two charges times one charge. The dipole moment $\mathbf{p}$ depends only on the charge distribution, not on the observation point.

To see if this is a pure dipole, we check the higher moments of the distribution,

$$\begin{aligned}
Q_l &= \frac{1}{4\pi\epsilon_0}\int r^l\rho(\mathbf{x})\,P_l(\cos\theta)\,d^3x \\
&= \frac{q}{4\pi\epsilon_0}\int r^l\left(\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right)-\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right)\right)P_l(\cos\theta)\,d^3x \\
&= \frac{q}{4\pi\epsilon_0}\int r^l\left(\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right)-\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right)\right)P_l(\cos\theta)\,r^2dr\sin\theta d\theta d\varphi \\
&= \frac{q}{4\pi\epsilon_0}\left(\int\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right)P_l(\cos\theta')\,r^{l+2}dr\sin\theta d\theta d\varphi-\int\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right)P_l(\cos\theta)\,r^{l+2}dr\sin\theta d\theta d\varphi\right)
\end{aligned}$$

Letting $x=\cos\theta$, the delta functions may be written as

$$\begin{aligned}
\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right) &= \frac{4}{2\pi d^2}\delta\left(r-\frac{d}{2}\right)\delta(x-1) \\
\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right) &= \frac{4}{2\pi d^2}\delta\left(r-\frac{d}{2}\right)\delta(x+1)
\end{aligned}$$

so that

$$
\begin{aligned}
Q_l &= \frac{q}{4\pi\epsilon_0}\frac{4}{2\pi d^2}\left(\int_0^\infty\int_{-1}^1\int_0^{2\pi}\delta\left(r-\frac{d}{2}\right)\delta(x-1)P_l(x)r^{l+2}drdxd\varphi - \int_0^\infty\int_{-1}^1\int_0^{2\pi}\delta\left(r-\frac{d}{2}\right)\delta(x+1)P_l(x)r^{l+2}drdxd\varphi\right) \\
&= \frac{q}{\pi\epsilon_0 d^2}\left(\int_0^\infty\delta\left(r-\frac{d}{2}\right)r^{l+2}dr\int_{-1}^1\delta(x-1)P_l(x)dx - \int_0^\infty\delta\left(r-\frac{d}{2}\right)r^{l+2}dr\int_{-1}^1\delta(x+1)P_l(x)dx\right) \\
&= \frac{q}{\pi\epsilon_0 d^2}\left(\left(\frac{d}{2}\right)^{l+2}P_l(1)-\left(\frac{d}{2}\right)^{l+2}P_l(-1)\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{d}{2}\right)^l\left(1-(-1)^l\right) \\
&= \begin{cases} \frac{2q}{4\pi\epsilon_0}\left(\frac{d}{2}\right)^l & l\ odd \\ 0 & l\ even \end{cases}
\end{aligned}
$$

This means that the distribution is not a "pure" dipole, but contains higher moments as well. To obtain a pure dipole, we take the limit as the separation of charges vanishes, $d \to 0$, in such a way that the product $p = qd$ remains constant,

$$
\rho(\mathbf{x}) = \lim_{d\to 0}\frac{p}{d}\left(\delta^3\left(\mathbf{x}-\frac{d}{2}\hat{\mathbf{k}}\right)-\delta^3\left(\mathbf{x}+\frac{d}{2}\hat{\mathbf{k}}\right)\right)
$$

In this case, the odd moments become

$$
\begin{aligned}
Q_{2k+1} &= \lim_{d\to 0}\frac{2p}{4\pi\epsilon_0 d}\left(\frac{d}{2}\right)^{2k+1} \\
&= \lim_{d\to 0}\frac{p}{4\pi\epsilon_0}\left(\frac{d}{2}\right)^{2k} \\
&= \frac{p}{4\pi\epsilon_0}\lim_{d\to 0}\left(\frac{d}{2}\right)^{2k} \\
&= \begin{cases} \frac{p}{4\pi\epsilon_0} & k=0 \\ 0 & k>0 \end{cases}
\end{aligned}
$$

and we have a pure dipole with dipole moment $\mathbf{p} = p\hat{\mathbf{k}}$

For an arbitrary pure dipole with dipole moment $\mathbf{p}$, the potential is therefore

$$
V(\mathbf{r}_0) = \frac{\hat{\mathbf{r}}_0\cdot\mathbf{p}}{4\pi\epsilon_0 r_0^2}
$$

The electric field of a dipole lying along the z-axis, $\mathbf{p} = p\hat{\mathbf{k}}$, is found by taking the gradient. For simplicity, we drop the subscript 0 on the observation point. Then

$$
\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}V(\mathbf{x}) \\
&= -\boldsymbol{\nabla}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2}\right) \\
&= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right) \\
&= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\hat{\mathbf{r}}\left(-\frac{2p\cos\theta}{4\pi\epsilon_0 r^3}\right) - \hat{\boldsymbol{\theta}}\left(\frac{p\sin\theta}{4\pi\epsilon_0 r^3}\right)\right) \\
&= \frac{1}{4\pi\epsilon_0 r^3}\left(2\left(\hat{\mathbf{r}}\cdot\mathbf{p}\right)\hat{\mathbf{r}} + p\sin\theta\hat{\boldsymbol{\theta}}\right)
\end{aligned}
$$

We may rewrite the second term using

$$
\begin{aligned}
\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\mathbf{p}\right) &= \left(\hat{\mathbf{r}}\cdot\mathbf{p}\right)\hat{\mathbf{r}} - \mathbf{p}\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right) \\
&= \left(\hat{\mathbf{r}}\cdot\mathbf{p}\right)\hat{\mathbf{r}} - \mathbf{p}
\end{aligned}
$$

together with

$$
\begin{aligned}
\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\mathbf{p}\right) &= -\hat{\mathbf{r}}\times\hat{\boldsymbol{\varphi}}p\sin\theta \\
&= \hat{\boldsymbol{\theta}}p\sin\theta
\end{aligned}
$$

The electric field becomes

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0 r^3}\left(2\left(\hat{\mathbf{r}}\cdot\mathbf{p}\right)\hat{\mathbf{r}} + \left(\hat{\mathbf{r}}\cdot\mathbf{p}\right)\hat{\mathbf{r}} - \mathbf{p}\right) \\
\mathbf{E} &= \frac{3\left(\hat{\mathbf{r}}\cdot\mathbf{p}\right)\hat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0 r^3}
\end{aligned}
$$

5 Quadrupole

For the quadrupole moment, we need to compute

$$
Q_2 \equiv \frac{1}{4\pi\epsilon_0}\int d^3x\left(r^2\rho\left(\mathbf{x}\right)P_2\left(\cos\theta\right)\right)
$$

where θ is the angle with the observation direction. Then substituting for $P_2(x)$, we have

$$
Q_2 = \frac{1}{8\pi\epsilon_0}\int d^3x\,\rho\left(\mathbf{x}\right)r^2\left(3x^2-1\right)
$$

We need to rewrite this to separate the direction $\hat{\mathbf{r}}_0$ at which we wish to know the field from the integration variables. First, write the final terms in vector notation,

$$
\begin{aligned}
\left(3x^2-1\right)r^2 &= 3r^2\cos^2\theta - r^2 \\
&= 3\left(\hat{\mathbf{r}}_0\cdot\mathbf{r}\right)^2 - \mathbf{r}\cdot\mathbf{r}
\end{aligned}
$$

Now, using index notation, we insert

$$
\hat{\mathbf{r}}_0\cdot\hat{\mathbf{r}}_0 = 1 = \sum_{i,j=1}^{3}\delta_{ij}\hat{r}_{0i}\hat{r}_{0j}
$$

into the second term and separate the observation direction from the integration variables,

$$
\begin{aligned}
3\left(\hat{\mathbf{r}}_0\cdot\mathbf{r}\right)^2 - \mathbf{r}\cdot\mathbf{r} &= 3\sum_{i=1}^{3}\hat{r}_{0i}x_i\sum_{j=1}^{3}\hat{r}_{0j}x_j - r^2\sum_{i,j=1}^{3}\delta_{ij}\hat{r}_{0i}\hat{r}_{0j} \\
&= \sum_{i=1}^{3}\hat{r}_{0i}\sum_{j=1}^{3}\hat{r}_{0j}\left(3x_ix_j - r^2\delta_{ij}\right)
\end{aligned}
$$

Now we may write the quadrupole moment in a way that is independent of where we choose the direction of $\hat{\mathbf{r}}$:

$$\begin{aligned} Q_2 &= \frac{1}{8\pi\epsilon_0}\sum_{i,j=1}^{3}\hat{r}_{0i}\hat{r}_{0j}\left[\int d^3x\,\rho\left(\mathbf{x}\right)\left(3x_ix_j - r^2\delta_{ij}\right)\right] \\ &\equiv \frac{1}{8\pi\epsilon_0}\sum_{i,j=1}^{3}\hat{r}_{0i}Q_{ij}\hat{r}_{0j} \end{aligned}$$

where we define the quadrupole moment tensor,

$$Q_{ij} \equiv \int d^3x\,\rho\left(\mathbf{x}\right)\left(3x_ix_j - r^2\delta_{ij}\right)$$

Notice that Q_{ij} describes only the charge distribution, and is independent of the observation point.

We can develop a pure quadrupole moment tensor in the same way as we built a dipole moment vector, by placing four charges of alternating sign at the corners of a square of side $\frac{d}{2}$ in the xy plane. Then the charge density is

$$\rho\left(\mathbf{x}\right) = \lim_{d\to 0} q\left(\delta^3\left(\mathbf{x} - \frac{d}{2}\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{j}}\right) - \delta^3\left(\mathbf{x} - \frac{d}{2}\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{j}}\right) + \delta^3\left(\mathbf{x} + \frac{d}{2}\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{j}}\right) - \delta^3\left(\mathbf{x} + \frac{d}{2}\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{j}}\right)\right)$$

and the quadrupole moment may be evaluated easily in Cartesian coordinates. The Dirac delta functions become, for example,

$$\delta^3\left(\mathbf{x} - \frac{d}{2}\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{j}}\right) = \delta\left(z\right)\delta\left(x - \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right)$$

and so on, changing signs appropriately. Then, noticing that the quadrupole tensor is a symmetric matrix, $Q_{ij} = Q_{ji}$, we need to compute $Q_{xx}, Q_{xy}, Q_{xz}, Q_{yy}, Q_{yz}, Q_{zz}$. We can avoid one of thes by noticing that the trace of the matrix is

$$\begin{aligned} tr\left(Q_{ij}\right) &= \sum_{i=1}^{3} Q_{ii} \\ &= \int d^3x\rho\left(\mathbf{x}\right)\sum_{i=1}^{3}\left(3x_ix_i - r^2\delta_{ii}\right) \\ &= \int d^3x\rho\left(\mathbf{x}\right)\left(3r^2 - 3r^2\right) \\ &= 0 \end{aligned}$$

Therefore, $Q_{zz} = -Q_{xx} - Q_{yy}$.

In Cartesian coordinates, it is easy to write the components of the matrix $3x_ix_j - r^2\delta_{ij}$. For $i = 1, 2, 3$, we just replace $(x_1, x_2, x_3) = (x, y, z)$. For example if $i = 1, j = 2$,

$$\left(3x_1x_2 - r^2\delta_{12}\right) = 3xy$$

since $\delta_{12} = 0$. For $i = j = 1$,

$$\begin{aligned} \left(3x_1x_2 - r^2\delta_{12}\right) &= 3x^2 - r^2\delta_{11} \\ &= 3x^2 - \left(x^2 + y^2 + z^2\right) \\ &= 2x^2 - y^2 - z^2 \end{aligned}$$

Now we compute one component at a time. For the first element,

$$\begin{aligned}
Q_{11} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \rho\left(\mathbf{x}\right)\left(2x^2 - y^2 - z^2\right) \\
&= q \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\delta\left(x - \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right) - \delta\left(x - \frac{d}{2}\right)\delta\left(y + \frac{d}{2}\right)\right)\left(2x^2 - y^2\right) \\
&\quad + q \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\delta\left(x + \frac{d}{2}\right)\delta\left(y + \frac{d}{2}\right) - \delta\left(x + \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right)\right)\left(2x^2 - y^2\right) \\
&= q\left(\left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right) \\
&= 0
\end{aligned}$$

and for Q_{22},

$$\begin{aligned}
Q_{22} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \rho\left(\mathbf{x}\right)\left(2y^2 - x^2 - z^2\right) \\
&= q \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\delta\left(x - \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right) - \delta\left(x - \frac{d}{2}\right)\delta\left(y + \frac{d}{2}\right)\right)\left(2y^2 - x^2\right) \\
&\quad + q \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\delta\left(x + \frac{d}{2}\right)\delta\left(y + \frac{d}{2}\right) - \delta\left(x + \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right)\right)\left(2y^2 - x^2\right) \\
&= q\left(\left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right) \\
&= 0
\end{aligned}$$

and therefore $Q_{33} = -Q_{11} - Q_{22} = 0$ as well.

For the off-diagonal components,

$$\begin{aligned}
Q_{12} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \rho\left(\mathbf{x}\right)\left(3xy\right) \\
&= 3q \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\delta\left(x - \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right) - \delta\left(x - \frac{d}{2}\right)\delta\left(y + \frac{d}{2}\right)\right) xy \\
&\quad + 3q \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\delta\left(x + \frac{d}{2}\right)\delta\left(y + \frac{d}{2}\right) - \delta\left(x + \frac{d}{2}\right)\delta\left(y - \frac{d}{2}\right)\right) xy \\
&= 3q\left(\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2\right) \\
&= 3qd^2
\end{aligned}$$

The remaining two,

$$Q_{13} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \rho\left(\mathbf{x}\right)\left(3xz\right)$$

$$Q_{23} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \rho(\mathbf{x})(3yz)$$

both depend on the z-integral

$$\int_{-\infty}^{\infty} dz\, z\delta(z) = 0$$

and therefore vanish.

The limit as $d \to 0$ in such a way that $Q = qd^2$ remains constant gives a pure quadrupole. The quadrupole matrix is then simply

$$Q_{ij} = 3Q \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The second moment is therefore

$$\begin{aligned} Q_2 &= \frac{1}{8\pi\epsilon_0} \sum_{i,j=1}^{3} \hat{x}_{0i} Q_{ij} \hat{x}_{0j} \\ &= \frac{3Q}{8\pi\epsilon_0} \hat{x}_{01} \hat{x}_{02} \end{aligned}$$

where we may write the components of the unit vector in the observation direction $\mathbf{x}_0 = (x_0, y_0, z_0)$ as

$$\hat{x}_{0i} = \frac{1}{r_0}(x_0, y_0, z_0)$$

The potential due for a quadrupole is therefore

$$\begin{aligned} V(\mathbf{x}_0) &= \sum_{l=0}^{\infty} \frac{Q_2}{r_0^{l+1}} \\ &= \frac{Q_2}{r_0^3} \\ &= \frac{3Q}{8\pi\epsilon_0 r_0^3} \hat{x}_{01} \hat{x}_{02} \\ &= \frac{3Q}{8\pi\epsilon_0 r_0^5} x_0 y_0 \end{aligned}$$

or, dropping the subscripts,

$$V(\mathbf{x}) = \frac{3Q}{8\pi\epsilon_0} \frac{xy}{r^5}$$

Writing this in spherical coordinates,

$$\begin{aligned} V(\mathbf{x}) &= \frac{3Q}{8\pi\epsilon_0 r^3} \sin^2\theta \cos\varphi \sin\varphi \\ &= \frac{3Q}{16\pi\epsilon_0 r^3} \sin^2\theta \sin 2\varphi \end{aligned}$$

6 Potentials of the first three multipoles

Collecting our results, the potentials of the particular monopole, dipole and quadrupole we studied are

$$V_M(\mathbf{x}) = \frac{q}{4\pi\epsilon_0 r}$$

$$\begin{aligned} V_D(\mathbf{x}) &= \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \\ V_Q(\mathbf{x}) &= \frac{3Q}{16\pi\epsilon_0 r^3}\sin^2\theta\sin 2\varphi \end{aligned}$$

where q is the total charge, $p = \lim(qd)$ and $Q = \lim\left(qd^2\right)$. Notice that the strength of each successive multipole falls off with a higher power of r.

The dipole and quadrupole formulas above apply only to dipoles and quadrupoles oriented as in our examples. For a dipole oriented in a general direction, $\mathbf{p} = p\hat{\mathbf{n}}$, the potential is

$$V(\mathbf{x}) = \frac{p\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

where the observation point is $\mathbf{x} = r\hat{\mathbf{r}}$. The dipole moment is given by

$$\mathbf{p} \equiv \int \mathbf{x}\rho(\mathbf{x})\, d^3x$$

The potential of a general quadrupole moment tensor has the form

$$Q_2 = \frac{1}{8\pi\epsilon_0 r^5}\sum_{i,j=1}^{3} Q_{ij}x_ix_j$$

where

$$Q_{ij} \equiv \int d^3x\,\rho(\mathbf{x})\left(3x_ix_j - r^2\delta_{ij}\right)$$

In general, an arbitrary charge distribution will have all multipole moments nozero. The potential is then the sum of potentials like those above, but at large enough distances it is only the lowest multipole that accounts for the bulk of the field. For example, a charge distribution with no net total charge, but all other multipoles nonzero is well approximated by

$$V(\mathbf{x}) \approx \frac{\mathbf{p}\cdot\hat{\mathbf{x}}}{4\pi\epsilon_0 r^2}$$

7 Exercises (required)

1. Find the total charge and dipole moments of a charged ring of radius R lying in the xy plane, with (constant) charge density given by

$$\rho(\mathbf{x}) = \frac{q}{2\pi R}\delta(z)\,\delta(\rho - R)$$

2. Find the total charge and dipole moments of a charged ring of radius R lying in the xy plane, with charge density given by

$$\rho(\mathbf{x}) = \frac{q\sin\varphi}{2R}\delta(z)\,\delta(\rho - R)$$

3. Three charges lie in the xy plane: $+2q$ at the origin, $-q$ on the x-axis at $d\hat{\mathbf{i}}$, and $-q$ on the y-axis at $d\hat{\mathbf{j}}$. Without computing, what is the direction of the dipole moment?

4. Compute the quadrupole moment tensor of the three charges in the previous exercise.