Separation of variables in spherical coordinates

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We will make important use of the separation of variables in spherical coordinates, because the separation ends up giving us a series in terms of powers of r. For an isolated system, this means the expansion gives a way to approximate the field far from the source.

1 Separation of variables in spherical coordinates

In spherical coordinates the Laplace equation takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=0$$

We apply the same steps as we did for Cartesian coordinates, but the difference in the form of the Laplace equation raises some new issues.

1.1 Step 1: Substitute the separation assumption

We assume a solution of the form

$$V(r,\theta,\varphi)=R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

and substitute.

1.2 Step 2: Find the separated equations

Dividing by $R\Theta\Phi$, we have

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{r^2\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=0$$

This time, nothing separates until we multiply by $r^2\sin^2\theta$,

$$\frac{r^2\sin^2\theta}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=0$$

and even now only the Φ term has completely separated.

Take the partial derivative with respect to φ. Since the first two terms depend only on r and θ, this shows that

$$\frac{\partial}{\partial\varphi}\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}\right) = 0$$

and therefore

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$$

Here we usually choose the constant to be negative because solutions involving the full range of φ from 0 to 2π must be periodic. Lesser ranges may have either sign. The Φ equation is therefore

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0$$

The remainder of the Laplace equation is now

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - m^2 = 0$$

so dividing by $\sin^2\theta$ separates the remaining two variables,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0$$

Differentiating with respect to r and θ respectively show that

$$\begin{aligned}\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) &= a\\ \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} &= -a\end{aligned}$$

where the two constants must add to zero.

1.3 Step 3: Solving the equations

The simplest equation is the one for Φ, with solution

$$\Phi = C\sin m\varphi + D\cos m\varphi$$

or more commonly

$$\Phi(\varphi) = A_m e^{im\varphi}$$

for positive and negative m.

Rewriting the R equation as

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - aR = 0$$

Notice that rescaling r leaves this equation unchanged. This is a clue that powers of r may work. Substituting a power-law solution, $R(r) = r^l$,

$$\begin{aligned}\frac{d}{dr}\left(r^2\frac{d\left(r^l\right)}{dr}\right) - ar^l &= 0\\ l\frac{d}{dr}\left(r^{l+1}\right) - ar^l &= 0\\ \left(l\left(l+1\right) - a\right)r^l &= 0\end{aligned}$$

and with $a = l\left(l+1\right)$ we have a solution for every number l. Notice that there are two values of l that give the same value for $a$since the quadratic equation,

$$l^2 + l - a = 0$$

has two solutions. Let l have some value k, so that $a = k(k+1)$. Then the value $l = -(k+1)$ gives the same value, $a = (-(k+1))(-(k+1)+1) = k(k+1)$.

Setting $a = l(l+1)$, we have only the θ equation remaining,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0 \tag{1}$$

The solutions to the θ equation with $m = 0$ are called the *Legendre polynomials*. Solutions with general m may be found by differentiating the *Legendre polynomials*, giving the *associated Legendre polynomials*. We consider only the $m = 0$ case. To start, we define a new variable,

$$x = \cos\theta$$

Then

$$\begin{aligned}\frac{d}{d\theta} &= \frac{dx}{d\theta}\frac{d}{dx} \\ &= -\sin\theta\frac{d}{dx}\end{aligned}$$

and therefore

$$\frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

Thus, with $m = 0$, and replacing the derivatives, eq.(1) becomes $\frac{d}{dx}\left(\sin^2\theta\frac{d\Theta}{dx}\right) + l(l+1)\Theta = 0$. Replacing $\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$, we have the *Legendre equation*,

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta}{dx}\right) + l(l+1)\Theta = 0 \tag{2}$$

The solutions are polynomials, $P_l(x) = P_l(\cos\theta)$.

For example, suppose Θ takes the form

$$P_2(x) = a_0 + a_1 x$$

Then substituting,

$$\begin{aligned}0 &= \frac{d}{dx}\left(\left(1-x^2\right)\frac{dP_2(x)}{dx}\right) + l(l+1)P_2(x) \\ &= \frac{d}{dx}\left(\left(1-x^2\right)\frac{d(a_0+a_1x)}{dx}\right) + l(l+1)(a_0+a_1x) \\ &= \frac{d}{dx}\left(a_1\left(1-x^2\right)\right) + l(l+1)(a_0+a_1x) \\ &= -2a_1x + l(l+1)a_0 + l(l+1)a_1x \\ &= l(l+1)a_0 + (l(l+1)a_1 - 2a_1)x\end{aligned}$$

This gives two equations,

$$\begin{aligned}l(l+1)a_0 &= 0 \\ l(l+1)a_1 - 2a_1 &= 0\end{aligned}$$

The first gives either $a_0 = 0$ or $l(l+1) = 0$, while the second gives either $a_1 = 0$ or $l(l+1) = 2$.

This equation for l has two possible solutions, $l = 1$ or $l = -2$. Since we cannot have both a_0 and a_1 vanishing, we get two possibilities:

$$P_0 = a_0,\ l = -1\,or\,0$$

or

$$P_1 = a_1 x,\ l = -2\, or\, 1$$

The remaining constants are chosen so that $P_l(1) = 1$. The presence of two solutions for l for the same Legendre polynomial means that for each P_l there will be two powers of r. The solutions will be of the form

$$V(r,\theta) = \left(r^0 + \frac{1}{r}\right) P_0(\cos\theta) + \left(r + \frac{1}{r^2}\right) P_1(\cos\theta) + \cdots$$

where

$$\begin{aligned} P_0(\cos\theta) &= 1 \\ P_1(\cos\theta) &= \cos\theta \end{aligned}$$

Like the harmonic functions, there are many relationships among the Legendre polynomials. The most important for us now is the orthogonality relationship,

$$\int_0^{\frac{\pi}{2}} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$$

Notice that the right side may be written as either $\frac{2}{2l+1}\delta_{ll'}$ or as $\frac{2}{2l'+1}\delta_{ll'}$ since it is only nonzero when $l = l'$. We may also write this in terms of $x = \cos\theta$. With $dx = -\sin\theta d\theta$, we have

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \frac{2}{2l+1}\delta_{ll'}$$

When m is nonzero the solutions are called associated Legendre polynomials, $P_l^m(x)$. The range of m is from $-l$ to $+l$.

1.4 Step 4: Put it all together

For general m, the full solution is

$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) P_l^m(\cos\theta) e^{im\varphi}$$

We will only consider $m = 0$ cases, which apply to problems with azimuthal symmetry. Setting $A_{l0} = A_l$ and $B_{lm} = B_l$,

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \tag{3}$$

The constants A_{lm} and B_{lm} are uniquely determined by the boundary conditions.

2 Fitting boundary conditions in spherical coordinates

2.1 Example: Piecewise constant potential on hemispheres

Let the region of interest be the interior of a sphere of radius R. Let the potential be V_0 on the upper hemisphere, and $-V_0$ on the lower hemisphere,

$$V(R) = V_0\left(\Theta\left(\frac{\pi}{2} - \theta\right) - \Theta\left(\theta - \frac{\pi}{2}\right)\right)$$

We require the potential to be nonsingular everywhere within the sphere.

To find the potential, we may immediately write

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

and impose the boundary conditions. First, notice that finiteness everywhere means that we cannot have the $\frac{B_l}{r^{l+1}}$ terms because they diverge at $r=0$. Therefore, we set $B_l=0$ for all l. The remaining condition at $r=0$ is

$$V_0\left(\Theta\left(\frac{\pi}{2}-\theta\right)-\Theta\left(\theta-\frac{\pi}{2}\right)\right)=\sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)$$

We multiply this by $P_k(\cos\theta)$ and integrate from 0 to π,

$$\begin{aligned}
\int_0^{\pi}V_0\left(\Theta\left(\frac{\pi}{2}-\theta\right)-\Theta\left(\theta-\frac{\pi}{2}\right)\right)P_k(\cos\theta)\sin\theta d\theta &= \sum_{l=0}^{\infty}A_l R^l\int_0^{\pi}P_l(\cos\theta)P_k(\cos\theta)\sin\theta d\theta\\
V_0\int_0^{\frac{\pi}{2}}P_k(\cos\theta)\sin\theta d\theta-V_0\int_{\frac{\pi}{2}}^{\pi}P_k(\cos\theta)\sin\theta d\theta &= \sum_{l=0}^{\infty}\frac{2A_l R^l}{2l+1}\delta_{kl}\\
V_0\int_0^1 P_k(x)\,dx-V_0\int_{-1}^0 P_k(x)\,dx &= \frac{2A_k R^k}{2k+1}
\end{aligned}$$

First, for $k=0$, $P_0(x)=1$ and the left side vanishes and $A_0=0$.

For the $k>0$ integrals on the left we use the symmetry of the Legendre polynomials. Change variable from x to $-x$ in the second integral to write

$$\begin{aligned}
V_0\left(\int_0^1 P_k(x)\,dx-\int_{-1}^0 P_k(x)\,dx\right) &= V_0\left(\int_0^1 P_k(x)\,dx+\int_1^0 P_k(-x)\,dx\right)\\
&= V_0\left(\int_0^1 P_k(x)\,dx-\int_0^1 P_k(-x)\,dx\right)\\
&= V_0\int_0^1\left(P_k(x)\,dx-P_k(-x)\right)dx
\end{aligned}$$

We know that the odd Legendre polynomials are polynomials in odd powers of x, and the even are even. Therefore,

$$\begin{aligned}
P_k(-x) &= P_k(x) \qquad k\,even\\
P_k(-x) &= -P_k(x) \qquad k\,odd
\end{aligned}$$

and the difference we need vanishes for all even polynomials. For the odd cases, the left side becomes

$$V_0\int_0^1\left(P_k(x)\,dx-P_k(-x)\right)dx=2V_0\int_0^1 P_k(x)\,dxdx$$

Now, among many identities for the Legendre polynomials, we find

$$\int P_k(x)\,dx=\frac{P_{k+1}(x)-P_{k-1}(x)}{2k+1}$$

so the integral we need becomes

$$\begin{aligned}
2V_0 \int_0^1 P_k(x)\,dx &= 2V_0 \left.\frac{P_{k+1}(x) - P_{k-1}(x)}{2k+1}\right|_0^1 \\
&= \frac{2V_0}{2k+1}\left[\left(P_{k+1}(1) - P_{k-1}(1)\right) - \left(P_{k+1}(0) - P_{k-1}(0)\right)\right] \\
&= -\frac{2V_0}{2k+1}\left(P_{k+1}(0) - P_{k-1}(0)\right)
\end{aligned}$$

since $P_k(1) = 1$ for all k. This still requires some work to determine the value of Legendre polynomials at $x = 0$. For now we just leave the answer in terms of these values, so for all odd k,

$$\begin{aligned}
-\frac{2V_0}{2k+1}\left(P_{k+1}(0) - P_{k-1}(0)\right) &= \frac{2A_k R^k}{2k+1} \\
A_k &= \frac{V_0}{R^k}\left(P_{k-1}(0) - P_{k+1}(0)\right)
\end{aligned}$$

The potential at all points inside the sphere is therefore,

$$V(r,\theta) = V_0 \sum_{l=0}^{\infty}\left(P_{l-1}(0) - P_{l+1}(0)\right)\frac{r^l}{R^l} P_l(\cos\theta)$$

2.2 Example: Varying potential on a sphere

Let a sphere of radius R have potential

$$V(r = R, \theta, \varphi) = V_0 \cos^2\theta$$

Find the potential everywhere inside and outisde the sphere.

For the interior solution, we again must have the potential finite at $r = 0$, so we immediately set $B_l = 0$ and write the potential as

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

To fit the boundary condition at $r = R$, we require

$$V_0 \cos^2\theta = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

or, with $x = \cos\theta$,

$$V_0 x^2 = \sum_{l=0}^{\infty} A_l R^l P_l(x)$$

Because the left hand side is a simple, low order polynomial, it is easiest to write the left side in Legendre polynomials. We only need polynomials up to order x^2 and only even ones. The two relevant Legendre polynomials are therefore

$$\begin{aligned}
P_0(x) &= 1 \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right)
\end{aligned}$$

Solving for x^2,

$$\begin{aligned} x^2 &= \frac{2}{3}P_2(x) + \frac{1}{3} \\ &= \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) \end{aligned}$$

The boundary condition therefore requires

$$\frac{2V_0}{3}P_2(x) + \frac{V_0}{3}P_0(x) = \sum_{l=0}^{\infty} A_l R^l P_l(x)$$

and orthogonality of the polynomials implies, matching terms,

$$\begin{aligned} A_0 &= \frac{V_0}{3} \\ A_2 R^2 &= \frac{2V_0}{3} \end{aligned}$$

The potential everywhere inside the sphere is

$$\begin{aligned} V(r,\theta) &= \frac{V_0}{3}r^0 P_0(\cos\theta) + \frac{2V_0}{3R^2}r^2 P_2(\cos\theta) \\ &= \frac{V_0}{3} + \frac{V_0}{3}\frac{2r^2}{R^2}\frac{1}{2}\left(3\cos^2\theta - 1\right) \\ &= \frac{V_0}{3}\left(1 + \frac{r^2}{R^2}\left(3\cos^2\theta - 1\right)\right) \end{aligned}$$

The potential outside the sphere has the same boundary condition at $r = R$, but requires $A_l = 0$. Therefore, the boundary condition at R will be

$$\frac{2V_0}{3}P_2(x) + \frac{V_0}{3}P_0(x) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(x)$$

so that

$$\begin{aligned} \frac{B_0}{R} &= \frac{V_0}{3} \\ \frac{B_2}{R^3} &= \frac{2V_0}{3} \end{aligned}$$

and the potential becomes

$$\begin{aligned} V(r,\theta) &= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(x) \\ &= \frac{B_0}{r}P_0(x) + \frac{B_2}{r^3}P_2(x) \\ &= \frac{V_0 R}{3r}\left(1 + \frac{R^2}{r^2}\left(3\cos^2\theta - 1\right)\right) \end{aligned}$$

Notice that the interior and exterior solutions agree at $r = R$.

2.3 Example: Varying potential on a sphere (a more challenging example)

Let a sphere of radius R have potential

$$V(r = R, \theta, \varphi) = V_0 \sin^4\theta$$

Find the potential everywhere inside and outisde the sphere.

Inside:

Since we have spherical boundary conditions, it is easiest to use the spherical separation, and since the problem has azimuthal symmetry, we may use the solution to the Laplace equation in the form

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Our boundary conditions are

$$\begin{aligned} V(0) &= finite \\ V(R,\theta,\varphi) &= V_0\sin^4\theta \end{aligned}$$

The first of these conditions shows that $B_l=0$ for all l. The outer boundary condition becomes

$$V_0\sin^4\theta=\sum_{l=0}^{\infty}A_l R^l P_l(x)$$

Essentially, we must express $V_0\sin^4\theta$ in terms of Legendre polynomials.

Let $x=\cos\theta$. Then noting that we may write

$$\begin{aligned} V_0\sin^4\theta &= V_0\left(1-\cos^2\theta\right)^2 \\ &= V_0\left(1-x^2\right)^2 \\ &= V_0\left(1-2x^2+x^4\right) \end{aligned}$$

we should be able to write the potential on the sphere in terms of Legendre polynomials of even order less than or equal to x^4. Using higher order polynomials would introduce undesired higher powers of x. The three relevant polynomials are:

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= \frac{1}{2}\left(3x^2-1\right) \\ P_4(x) &= \frac{1}{8}\left(35x^4-30x^2+3\right) \end{aligned}$$

Rearranging and combining to produce x^4-2x^2+1, we start with

$$\begin{aligned} x^4 &= \frac{1}{35}\left(8P_4(x)+30x^2-3\right) \\ x^2 &= \frac{1}{3}\left(2P_2(x)+1\right) \\ x^4-2x^2+1 &= \frac{1}{35}\left(8P_4(x)+30x^2-3\right)-2x^2+1 \\ &= \frac{8}{35}P_4(x)+\left(\frac{30}{35}-2\right)x^2+\left(1-\frac{3}{35}\right) \\ &= \frac{8}{35}P_4(x)+\left(\frac{6}{7}-\frac{14}{7}\right)x^2+\frac{32}{35} \\ &= \frac{8}{35}P_4(x)-\frac{8}{7}\left(\frac{1}{3}\left(2P_2(x)+1\right)\right)+\frac{32}{35} \\ &= \frac{8}{35}P_4(x)-\frac{16}{21}P_2(x)-\frac{8}{21}+\frac{32}{35} \end{aligned}$$

$$
\begin{aligned}
&= \frac{8}{35}P_4(x) - \frac{16}{21}P_2(x) + \frac{8}{7}\left(\frac{4}{5} - \frac{1}{3}\right) \\
&= \frac{8}{35}P_4(x) - \frac{16}{21}P_2(x) + \frac{8}{7}\left(\frac{7}{15}\right) \\
&= \frac{8}{35}P_4(x) - \frac{16}{21}P_2(x) + \frac{8}{15}P_0(x)
\end{aligned}
$$

Therefore, we must solve

$$
V_0\left(\frac{8}{35}P_4(x) - \frac{16}{21}P_2(x) + \frac{8}{15}P_0(x)\right) = \sum_{l=0}^{\infty} A_l R^l P_l(x)
$$

for the coefficients A_l. To do this, we use the orthogonality relation,

$$
\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \frac{2}{2l+1}\delta_{ll'}
$$

Multiply our equation by an arbitrary $P_k(x)$ for any fixed k, and integrate over all x,

$$
\begin{aligned}
V_0 \int_{-1}^{1}\left(\frac{8}{35}P_4(x) - \frac{16}{21}P_2(x) + \frac{8}{15}P_0(x)\right) P_k(x)\, dx &= \sum_{l=0}^{\infty} A_l R^l \int_{-1}^{1} P_l(x)\, P_k(x)\, dx \\
\frac{8V_0}{35}\int_{-1}^{1} P_4(x)\, P_k(x)\, dx - \frac{16V_0}{21}\int_{-1}^{1} P_2(x)\, P_k(x)\, dx + \frac{8V_0}{15}\int_{-1}^{1} P_0(x)\, P_k(x)\, dx &= \sum_{l=0}^{\infty} A_l R^l \frac{2}{2l+1}\delta_{lk} \\
\frac{8V_0}{35}\frac{2}{2k+1}\delta_{k4} - \frac{16V_0}{21}\frac{2}{2k+1}\delta_{k2} + \frac{8V_0}{15}\frac{2}{1}\delta_{0k} &= A_k R^k \frac{2}{2k+1} \\
\frac{8V_0}{35}\frac{2}{9}\delta_{k4} - \frac{16V_0}{21}\frac{2}{5}\delta_{k2} + \frac{8V_0}{15}\frac{2}{1}\delta_{0k} &= A_k R^k \frac{2}{2k+1}
\end{aligned}
$$

The left side vanishes unless $k = 0, 2 \ or \ 4$, and in these cases we have

$$
\begin{aligned}
A_0 &= \frac{8V_0}{15} \\
A_2 &= -\frac{16V_0}{21R^2} \\
A_4 &= \frac{8V_0}{35R^4}
\end{aligned}
$$

The full solution everywhere inside the sphere is found by putting these coefficients back into the general form, giving the final potential as

$$
\begin{aligned}
V(r,\theta) &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \\
&= A_0 r^0 P_0(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_4 r^4 P_4(\cos\theta) \\
&= \frac{8V_0}{15} r^0 P_0(\cos\theta) - \frac{16V_0}{21R^2} r^2 P_2(\cos\theta) + \frac{8V_0}{35R^4} r^4 P_4(\cos\theta) \\
&= 8V_0\left(\frac{1}{15} - \frac{2r^2}{21R^2}P_2(\cos\theta) + \frac{r^4}{35R^4}P_4(\cos\theta)\right) \\
&= V_0\left(\frac{8}{15} - \frac{16r^2}{21R^2}P_2(\cos\theta) + \frac{8r^4}{35R^4}P_4(\cos\theta)\right)
\end{aligned}
$$

We immediately verify that if we set $r = R$ and use the form of $\sin^4\theta$ in terms of Legendre polynomials, we recover the boundary condition. At the center of the sphere, we find (setting $r = 0$) that the potential is $V(0) = \frac{8}{15}V_0$.

Aside: Verify Theorem 3.1.4 If we integrate the potential over the surface of the sphere (using Wolfram integrator),

$$\begin{aligned}
\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi V_0 \sin^4\theta &= 2\pi V_0 \int_0^{\pi} \sin^5\theta d\theta \\
&= 2\pi V_0 \left(-\frac{5}{8}\cos\theta + \frac{5}{48}\cos^3\theta - \frac{1}{80}\cos 5\theta \right)\Big|_0^{\pi} \\
&= 2\pi V_0 \left(\left(\frac{5}{8} - \frac{5}{48} + \frac{1}{80} \right) - \left(-\frac{5}{8} + \frac{5}{48} - \frac{1}{80} \right) \right) \\
&= 2\pi V_0 \left(\frac{5}{8} - \frac{5}{48} + \frac{1}{80} + \frac{5}{8} - \frac{5}{48} + \frac{1}{80} \right) \\
&= 2\pi V_0 \left(\frac{5}{4} - \frac{5}{24} + \frac{1}{40} \right) \\
&= 2\pi V_0 \left(\frac{150}{120} - \frac{25}{120} + \frac{3}{120} \right) \\
&= 2\pi V_0 \left(\frac{128}{120} \right) \\
&= \frac{32\pi}{15} V_0
\end{aligned}$$

so that the potential at the center of the sphere is

$$V_{center} = \frac{8}{15} V_0 = \frac{1}{4\pi R^2} \int_0^{\pi} \int_0^{2\pi} V(R,\theta,\varphi) R^2 \sin\theta d\theta d\varphi$$

as required by the theorem of section 3.1.4 in Griffiths.

Outside:

The exterior solution is similar, but this time the boundary conditions are

$$\begin{aligned}
V(\infty) &= 0 \\
V(R,\theta,\varphi) &= V_0 \sin^4\theta
\end{aligned}$$

which means that we must have $A_l = 0$ for all l. The second condition now reads

$$V_0 \left(\frac{8}{35} P_4(x) - \frac{16}{21} P_2(x) + \frac{8}{15} P_0(x) \right) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(x)$$

so we need only replace $A_l R^l$ of the interior solution by $\frac{B_l}{R^{l+1}}$. This gives

$$V(r,\theta) = 8V_0 \left(\frac{R}{15r} - \frac{2R^3}{21r^3} P_2(\cos\theta) + \frac{R^5}{35r^5} P_4(\cos\theta) \right)$$

2.4 Example: Extending the solution for a disk

Here is a somewhat different example of the use of the series solution and uniqueness. Although this is not a boundary condition in the usual sense, it is still enough information to find all of the coefficients and construct the potential everywhere. We have found previously that the potential on the z-axis above a circular disk of radius R lying in the xy-plane is

$$V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$

Since we have axial symmetry, the full solution

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

must agree with this when we set $z = r$ and $\theta = 0$:

$$\begin{aligned}
\frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z\right)\Big|_{z=r} &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)\big|_{\theta=0} \\
\frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right) &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)
\end{aligned}$$

where we have used $P_l(\cos 0) = P_l(1) = 1$, for all l.

The problem is simplest for the region where $r > R$. In this case, we may expand the square root in a Taylor series as

$$\sqrt{r^2+R^2} = r\sqrt{1+\frac{R^2}{r^2}}$$

We need the Taylor series for $(1+x)^{1/2}$ for small x. Looking at the first few derivatives

$$\begin{aligned}
f(x) &= (1+x)^{1/2} \\
f^{(1)}(x) &= \frac{1}{2}(1+x)^{-1/2} \\
f^{(2)}(x) &= \left(-\frac{1}{2}\right)\frac{1}{2}(1+x)^{-3/2} \\
f^{(3)}(x) &= \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\frac{1}{2}(1+x)^{-5/2}
\end{aligned}$$

This is enough to see the pattern. After k differentiations, we will have

$$\begin{aligned}
f^{(k)}(x) &= -\frac{(-1)^k}{2^k}(2k-3)(2k-5)\cdots 1\,(1+x)^{-(2k-1)/2} \\
&= -\frac{(-1)^k}{2^k}\frac{(2k-3)!}{(2k-2)(2k-4)(2k-6)\cdots 2}(1+x)^{-(2k-1)/2} \\
&= -\frac{(-1)^k}{2^k}\frac{(2k-3)!}{2^{k-1}(k-1)(k-2)(k-3)\cdots 1}(1+x)^{-(2k-1)/2} \\
&= -\frac{(-1)^k}{2^k}\frac{(2k-3)!}{2^{k-1}(k-1)!}(1+x)^{-(2k-1)/2}
\end{aligned}$$

so the full Taylor series is

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k-1}}\frac{(2k-3)!}{(k-1)!}x^k$$

Setting $x = \frac{R^2}{r^2}$, the right side of our equality therefore becomes

$$\begin{aligned}
\frac{\sigma}{2\epsilon_0}\left(r\left(1+\frac{1}{2}\frac{R^2}{r^2}-\sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k-1}}\frac{(2k-3)!}{(k-1)!}\frac{R^{2k}}{r^{2k}}\right)-r\right) &= \sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right) \\
\frac{\sigma}{\epsilon_0}\left(\frac{R^2}{4r}-\sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k}}\frac{(2k-3)!}{(k-1)!}\frac{R^{2k}}{r^{2k-1}}\right) &= \sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)
\end{aligned}$$

This means that $A_l = 0$ for all l, and for the B_l we match term by term to find:

$$\begin{aligned}
B_0 &= \frac{\sigma R^2}{4\epsilon_0} \\
-\frac{\sigma}{\epsilon_0}\sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k}}\frac{(2k-3)!}{(k-1)!}\frac{R^{2k}}{r^{2k-1}} &= \sum_{l=1}^{\infty}\frac{B_l}{r^{l+1}}
\end{aligned}$$

Note that we have only odd powers of r on the left, so only even l will occur on the right. Let $l = 2k-2$ on the right. Then for all $k > 1$

$$\begin{aligned}
-\frac{\sigma}{\epsilon_0}\sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k}}\frac{(2k-3)!}{(k-1)!}\frac{R^{2k}}{r^{2k-1}} &= \sum_{k=2}^{\infty}\frac{B_{2k-2}}{r^{2k-1}} \\
-\frac{\sigma}{\epsilon_0}\frac{(-1)^k}{2^{2k}}\frac{(2k-3)!}{(k-1)!}R^{2k} &= B_{2k-2}
\end{aligned}$$

All odd B_l vanish. The potential everywhere for $r > R$ (including off axis!) is therefore

$$V(r,\theta) = \frac{\sigma R^2}{4\epsilon_0 r} - \frac{\sigma R}{\epsilon_0}\sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k}}\frac{(2k-3)!}{(k-1)!}\left(\frac{R}{r}\right)^{2k-1} P_{2k}(\cos\theta)$$

The solution for $r < R$ may be found in a similar way, although it requires two subcases.

3 Exercises

3.1 Separation of variables

Separate the Laplace equation in cylindrical coordinates to find the differential equations for three functions. Solve the equations for the φ and z directions. The radial equation gives Bessel functions.

3.2 Legendre polynomials

Suppose $\Theta(\theta)$ takes the form

$$P_l(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Substitute into the Legendre equation, eq.(2) and find all allowed combinations of solutions for a_1, a_2, a_3 and l. This will lead you to P_0, P_1, P_2 and P_3. Choose the normalizations so that $P_l(1) = 1$.

3.3 Boundary conditions on a sphere

Consider a sphere of radius R held at a potential

$$V(R,\theta) = V_0 \cos\theta \sin^2\theta$$

Find the potential everywhere, both inside and outside the sphere.