Separation of variables: Cartesian coordinates

October 30, 2015

1 Separation of variables in Cartesian coordinates

The *separation of variables* technique is more powerful than the methods we have studied so far. The approach begins with a simplifying assumption, that the potential may be written as a product (or in some cases, the sum) of some simpler functions. The procedure is the same for other coordinate systems. For the case of Cartesian coordinates, the solution is particularly simple.

The Laplace equation in Cartesian coordinates is

$$\begin{aligned} \nabla^2 V(x,y,z) &= 0 \\ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= 0 \end{aligned}$$

1.1 Step 1:

We assume a solution of the form

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

Substitution gives

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

where the partial derivatives now become ordinary.

1.2 Step 2:

Next, divide by $V = XYZ$. This gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

and each terms is now a function of only one variable. Taking the partial derivative of the whole expression with respect to x gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\left(\frac{1}{X}\frac{d^2X}{dx^2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{Y}\frac{d^2Y}{dy^2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{Z}\frac{d^2Z}{dz^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{X}\frac{d^2X}{dx^2}\right) + 0 + 0 \end{aligned}$$

showing that $\frac{1}{X}\frac{d^2X}{dx^2}$ must be independent of x, hence constant. The same holds if we take y- or z-derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{1}{Y}\frac{d^2Y}{dy^2}\right) &= 0 \\
\frac{\partial}{\partial z}\left(\frac{1}{Z}\frac{d^2Z}{dz^2}\right) &= 0
\end{aligned}
$$

This means that each of the three terms must be constant,

$$
\begin{aligned}
\frac{1}{X}\frac{d^2X}{dx^2} &= a \\
\frac{1}{X}\frac{d^2X}{dx^2} &= b \\
\frac{1}{Z}\frac{d^2Z}{dz^2} &= c
\end{aligned}
$$

and the constants must add to zero:

$$a + b + c = 0$$

1.3 Step 3:

Solve the equations. It is easiest to choose the constants to be in the form $\pm\alpha^2$ since the solutions are either exponential or sinusoidal. If we choose $a = -\alpha^2$ then

$$
\begin{aligned}
\frac{1}{X}\frac{d^2X}{dx^2} &= -\alpha^2 \\
\frac{d^2X}{dx^2} + \alpha^2 X &= 0
\end{aligned}
$$

and we immediately have the familiar solution,

$$X = A\sin\alpha x + B\cos\alpha x$$

If we choose $a = +\alpha^2$ instead then the equation is

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0$$

and the solution is

$$X = Ae^{\alpha x} + Be^{-\alpha x}$$

Because $a + b + c = 0$, both signs must occur. If two of the constants are negative (giving oscillating solutions) then the third must be positive and will have exponential solutions. If two of the signs are positive then there are two directions with exponential solutions. Then the third constant is negative and has oscillating solutions.

Choose the signs so that the solutions match your boundary conditions.

Suppose we choose

$$
\begin{aligned}
a &= -\alpha^2 \\
b &= -\beta^2 \\
c &= \alpha^2 + \beta^2
\end{aligned}
$$

Then

$$X = A\sin\alpha x + B\cos\alpha x$$

and

$$Y = C \sin \beta y + D \cos \beta y$$

while Z satisfies

$$\frac{d^2 Z}{dz^2} - \left(\alpha^2 + \beta^2\right) Z = 0$$

For convenience, define $\gamma^2 = \alpha^2 + \beta^2$, so that

$$\frac{d^2 Z}{dz^2} - \gamma^2 Z = 0$$

It is easy to see that $e^{\pm\gamma z}$ both give solutions, so the general solution is the arbitrary linear combination

$$Z = E' e^{\gamma z} + F' e^{-\gamma z}$$

It is often more useful to use the symmetric and antisymmetric combinations,

$$\begin{aligned} \cosh \gamma z &= \frac{e^{\gamma z} + e^{-\gamma z}}{2} \\ \sinh \gamma z &= \frac{e^{\gamma z} - e^{-\gamma z}}{2} \end{aligned}$$

Then

$$\begin{aligned} e^{\gamma z} &= \cosh \gamma z + \sinh \gamma z \\ e^{-\gamma z} &= \cosh \gamma z - \sinh \gamma z \end{aligned}$$

and our solution is

$$\begin{aligned} Z &= E' \left(\cosh \gamma z + \sinh \gamma z\right) + F' \left(\cosh \gamma z - \sinh \gamma z\right) \\ &= \left(E' + F'\right) \cosh \gamma z + \left(E' - F'\right) \sinh \gamma z \end{aligned}$$

Renaming the constants, $E = E' + F'$ and $F = E' - F'$, this is just

$$Z = E \cosh \gamma z + F \sinh \gamma z$$

1.4 Step 4:

Write the full solution as a linear combination. For any particular values of α and β the solution is

$$V_{\alpha,\beta}(x,y,z) = (A \sin \alpha x + B \cos \alpha x)(C \sin \beta y + D \cos \beta y)(E \cosh \gamma z + F \sinh \gamma z)$$

Since the Laplace equation is linear, we may build the general solution for V as a sum of such solutions for different α and β. The leading constants may be different for each choice of α and β, so we have

$$V(x,y,z) = \sum_{\alpha,\beta} \left(A_{\alpha,\beta} \sin \alpha x + B_{\alpha,\beta} \cos \alpha x\right)\left(C_{\alpha,\beta} \sin \beta y + D_{\alpha,\beta} \cos \beta y\right)\left(E_{\alpha,\beta} \cosh \gamma z + F_{\alpha,\beta} \sinh \gamma z\right)$$

However, this latter form is too general for most problems. It is simpler to choose some of the constants to satisfy the boundary conditions automatically.

2 Fitting boundary conditions in Cartesian coordinates

Suppose we wish to find the potential everywhere inside a conducting cube of side L. Let the boundary be held at zero potential except for the top of the box, which is held at V_0. This means that in the x-direction, the potential starts and ends at zero:

$$\begin{aligned} V(0,y,z) &= 0 \\ V(L,y,z) &= 0 \end{aligned}$$

2.1 Satisfying the first five conditions

The easiest way to satisfy this condition is to choose the sinusoidal solutions in the x-direction,

$$X(x) = A\sin\alpha x + B\cos\alpha x$$

Then the boundary conditions require

$$\begin{aligned} V(0,y,z) &= X(0)Y(y)Z(z) \\ V(L,y,z) &= X(L)Y(y)Z(z) \end{aligned}$$

for all y and z. Therefore,

$$\begin{aligned} 0 &= X(0) \\ &= A\sin 0 + B\cos 0 \\ &= B \end{aligned}$$

With $B = 0$ we look at the second condition,

$$\begin{aligned} 0 &= X(L) \\ &= A\sin\alpha L + B\cos\alpha L \\ &= A\sin\alpha L \end{aligned}$$

Since $A = 0$ would make the entire potential vanish, we must restrict the possible values of α instead. The boundary condition is satisfied if and only if

$$\begin{aligned} \alpha L &= n\pi \\ \alpha &= \frac{n\pi}{L} \end{aligned}$$

for any integer n. We conclude

$$X_n(x) = A_n \sin\frac{n\pi x}{L}$$

Notice that the constant A_n may depend on n.

The y-direction is completely analogous. We require vanishing potential on both sides,

$$\begin{aligned} V(x,0,z) &= 0 \\ V(x,L,z) &= 0 \end{aligned}$$

for all x and z, and this is achieved only if

$$\begin{aligned} Y(0) &= 0 \\ Y(L) &= 0 \end{aligned}$$

Again, we choose an oscillating solution and we find that

$$Y_m(y) = C_m \sin\frac{m\pi y}{L}$$

The integer m is independent of the integer n.

Finally, we fit the boundary condition for $Z(z)$. Remembering that the solution in this direction must be exponential, we write

$$Z = E\cosh\gamma z + F\sinh\gamma z$$

where $\gamma = +\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2} = +\frac{\pi}{L}\sqrt{n^2+m^2}$. With the potential on the top of the box equal to V_0, the boundary conditions for z are

$$\begin{aligned} V(x, y, 0) &= 0 \\ V(x, y, L) &= V_0 \end{aligned}$$

Since these relations hold for all x and y, they must be satisfied by our choice of $Z(0)$ and $Z(L)$. The $z = 0$ condition is completely satisfied by

$$\begin{aligned} 0 &= Z(0) \\ &= E \end{aligned}$$

This leaves our complete solution in the form

$$\begin{aligned} V(x, y, z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n \sin\frac{n\pi x}{L} C_m \sin\frac{m\pi y}{L} E_{m,n} \sinh\gamma z \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_n C_m E_{m,n}) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L} \sinh\gamma z \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L} \sinh\gamma z \end{aligned}$$

where it is sufficient to set $A_{nm} = A_n C_m E_{m,n}$ since we only have a single overall constant for each pair m, n.

2.2 The final solution

Now comes the tricky part. We have one final boundary condition, $V(x, y, L) = V_0$, and we no longer have the freedom to satisfy it by choosing γ. Instead, we must choose the constants A_{mn} so that

$$V_0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{nm} \sinh\gamma L) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L}$$

for all x and all y. This is a double Fourier series. To solve, we use the orthogonality property of the sine function (see below for the explicit integrations),

$$\int_0^L \sin\frac{n_1\pi x}{L} \sin\frac{n_2\pi x}{L}\, dx = \frac{L}{2}\delta_{n_1 n_2}$$

Consider the x-direction first. Multiply both sides of the boundary condition by $\sin\frac{k\pi x}{L}$ and integrate x from 0 to L,

$$\int_0^L V_0 \sin\frac{k\pi x}{L}\, dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{nm} \sinh\gamma L) \sin\frac{m\pi y}{L} \int_0^L \sin\frac{n\pi x}{L} \sin\frac{k\pi x}{L}\, dx$$

The left side is easy to integrate (in this example!), while on the right we use orthogonality:

$$-\frac{LV_0}{k\pi} \cos\frac{k\pi x}{L}\Big|_0^L = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{nm} \sinh\gamma L) \sin\frac{m\pi y}{L} \frac{L}{2}\delta_{kn}$$

Performing the sum over all n, only one term survives,

$$\begin{aligned}
-\frac{LV_0}{k\pi}\cos k\pi + \frac{LV_0}{k\pi}\cos 0 &= \frac{L}{2}\sum_{m=1}^{\infty}\left(A_{km}\sinh\sqrt{k^2+m^2}L\right)\sin\frac{m\pi y}{L} \\
\frac{LV_0}{k\pi}\left(1-(-1)^k\right) &= \frac{L}{2}\sum_{m=1}^{\infty}\left(A_{km}\sinh\sqrt{k^2+m^2}L\right)\sin\frac{m\pi y}{L}
\end{aligned}$$

Now repeat the procedure for the y direction. Multiply by $\sin\frac{j\pi y}{L}$ and integrate,

$$\begin{aligned}
\frac{LV_0}{k\pi}\left(1-(-1)^k\right)\int_0^L \sin\frac{j\pi y}{L}dy &= \frac{L}{2}\sum_{m=1}^{\infty}\left(A_{km}\sinh\sqrt{k^2+m^2}L\right)\int_0^L \sin\frac{m\pi y}{L}\sin\frac{j\pi y}{L}dy \\
\frac{LV_0}{k\pi}\left(1-(-1)^k\right)\left[-\frac{L}{j\pi}\cos\frac{j\pi y}{L}\right]_0^L &= \left(\frac{L}{2}\right)^2\sum_{m=1}^{\infty}\left(A_{km}\sinh\sqrt{k^2+m^2}L\right)\delta_{jm} \\
\frac{LV_0}{k\pi}\left(1-(-1)^k\right)\left(\frac{L}{j\pi}-\frac{L}{j\pi}\cos j\pi\right) &= \left(\frac{L}{2}\right)^2 A_{kj}\sinh\sqrt{k^2+j^2}L \\
\frac{4V_0}{jk\pi^2}\left(1-(-1)^k\right)\left(1-(-1)^j\right) &= A_{kj}\sinh\sqrt{k^2+j^2}L
\end{aligned}$$

Since we can carry out these steps for any values of j and k, we have found *all* of the coefficients A_{kj},

$$A_{kj} = \frac{4V_0}{jk\pi^2\sinh\sqrt{k^2+j^2}L}\left(1-(-1)^k\right)\left(1-(-1)^j\right)$$

Substituting these (constant!) values into the potential,

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{4V_0}{mn\pi^2\sinh\left(\sqrt{n^2+m^2}L\right)}\left(1-(-1)^n\right)\left(1-(-1)^m\right)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L}\sinh\gamma z$$

This simplifies to sums over the odd terms since

$$\left(1-(-1)^n\right) = \begin{cases} 0 & n\ even \\ 2 & n\ odd \end{cases}$$

and we may write

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=odd}^{\infty}\sum_{m=odd}^{\infty}\frac{1}{mn}\frac{\sinh\sqrt{n^2+m^2}z}{\sinh\sqrt{n^2+m^2}L}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L}$$

This is our final potential.

2.3 Orthogonality of sine functions

We evaluate the integrals

$$\int_0^L \sin\frac{n_1\pi x}{L}\,\sin\frac{n_2\pi x}{L}\,dx$$

First, when n_1 and n_2 are different, we use the sum and difference formulas to write

$$\begin{aligned}
\int_0^L \sin \frac{n_1 \pi x}{L} \sin \frac{n_2 \pi x}{L} \, dx &= \frac{1}{2} \int_0^L \left[\cos \left(\frac{n_1 \pi x}{L} - \frac{n_2 \pi x}{L} \right) - \cos \left(\frac{n_1 \pi x}{L} + \frac{n_2 \pi x}{L} \right) \right] dx \\
&= \frac{1}{2} \left[\frac{L}{\pi (n_1 - n_2)} \sin \left(\frac{\pi x}{L} (n_1 - n_2) \right) - \frac{L}{\pi (n_1 + n_2)} \sin \left(\frac{\pi x}{L} (n_1 + n_2) \right) \right]_0^L \\
&= \frac{1}{2} \left(\frac{L}{\pi (n_1 - n_2)} \sin (\pi (n_1 - n_2)) - \frac{L}{\pi (n_1 + n_2)} \sin (\pi (n_1 + n_2)) - \frac{L}{\pi (n_1 - n_2)} \sin (0) - \frac{L}{\pi \left(n_1 + \right.} \right. \\
&= 0
\end{aligned}$$

so the integral vanishes. However, when $n_1 = n_2$,

$$\begin{aligned}
\int_0^L \sin \frac{n_1 \pi x}{L} \sin \frac{n_2 \pi x}{L} \, dx &= \int_0^L \sin^2 \frac{n_1 \pi x}{L} \, dx \\
&= \frac{1}{2} \int_0^L \left(1 - \cos^2 \frac{n_1 \pi x}{L} + \sin^2 \frac{n_1 \pi x}{L} \right) dx \\
&= \frac{1}{2} \int_0^L \left(1 - \cos \frac{2 n_1 \pi x}{L} \right) dx \\
&= \frac{1}{2} \left[x - \frac{L}{2 n_1 \pi} \sin \frac{2 n_1 \pi x}{L} \right]_0^L \\
&= \frac{L}{2}
\end{aligned}$$

We summarize these integrals by writing

$$\int_0^L \sin \frac{n_1 \pi x}{L} \sin \frac{n_2 \pi x}{L} \, dx = \frac{L}{2} \delta_{n_1 n_2}$$

3 Example

Consider the same cube as in the example above, and let $V = 0$ on the sides and bottom as before, but suppose the potential on the top is

$$V(x, y, L) = V_0 \left(\sin \frac{\pi x}{L} - \frac{4}{3} \sin^3 \frac{\pi x}{L} \right)$$

The steps are the same as above up to the final integration. At $z = L$, we must satisfy

$$V_0 \left(\sin \frac{\pi x}{L} - \frac{4}{3} \sin^3 \frac{\pi x}{L} \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (A_{nm} \sinh \gamma L) \sin \frac{n \pi x}{L} \sin \frac{m \pi y}{L}$$

and the Fourier series in the y-direction is found as above to reduce this to

$$\frac{2 V_0}{k \pi} \left(1 - (-1)^k \right) \left(\sin \frac{\pi x}{L} - \frac{4}{3} \sin^3 \frac{\pi x}{L} \right) = \sum_{n=1}^{\infty} (A_{nk} \sinh \gamma L) \sin \frac{n \pi x}{L}$$

At this point we could multiply by $\sin lx$ and integrate. However, in this case we may write the left hand side using trigonometric identities as

$$\sin\frac{\pi x}{L} - \frac{4}{3}\sin^3\frac{\pi x}{L} = \frac{1}{3}\sin\frac{3\pi x}{L}$$

Since the functions $\sin\frac{n\pi x}{L}$ are mutually orthogonal, we immediately see that

$$\frac{2V_0}{3k\pi}\left(1-(-1)^k\right)\sin\frac{3\pi x}{L} = \sum_{n=1}^{\infty}\left(A_{nk}\sinh\gamma L\right)\sin\frac{n\pi x}{L}$$

requires all of the A_{nk} to vanish except for $n = 3$. For A_{3k}, we have

$$\begin{aligned}\frac{2V_0}{3k\pi}\left(1-(-1)^k\right)\sin\frac{3\pi x}{L} &= A_{3k}\sinh\gamma L\sin\frac{3\pi x}{L}\\ A_{3k} &= \frac{2V_0}{3k\pi\sinh\gamma L}\left(1-(-1)^k\right)\end{aligned}$$

and the unique solution for the potential is

$$V(x,y,z) = \sum_{m\ odd}\frac{4V_0}{3m\pi}\frac{\sinh\gamma z}{\sinh\gamma L}\sin\frac{3\pi x}{L}\sin\frac{m\pi y}{L}$$

This simplification works any time the potential on the left can be easily written in terms of Fourier series.

The same approach works in other coordinate systems when the potential on a given boundary can be written in terms of the orthogonal functions of the solution.

4 Exercise

Repeat the problem of a cube, but this time center the cube in the z-direction and let both the top and the bottom have potential V_0. Now the boundary conditions are:

$$\begin{aligned}V(0,y,z) &= 0\\ V(L,y,z) &= 0\\ V(x,0,z) &= 0\\ V(x,L,z) &= 0\\ V\left(x,y,-\frac{L}{2}\right) &= V_0\\ V\left(x,y,+\frac{L}{2}\right) &= V_0\end{aligned}$$

Notice that $\cosh\gamma z$ is symmetrical in z, so that $\cosh(\gamma z) = \cosh(-\gamma z)$. Suggestion: work through all the details rather than just copying the calculations above.