The method of images

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In a given limited region, the solution for the potential is unique once we know the boundary conditions. By placing additional charges *outside* the region of interest we can sometimes reproduce the desired boundary condition, allowing us to easily write the solution.

1 Charge near an infinite plane conductor

Consider the problem of finding the electric potential due to a single charge Q held a distance d from an infinite conducting plane. We examine two cases. In the first, the plane is grounded so it is at zero potential. In the second case, we add a net area charge density, σ, to the plane.

In these two examples, the region of interest is the region on the same side (the $+x$ side) of the plane as the charge Q. We are not interested in fields on the $(-x)$ side.

1.1 Grounded plane

Suppose we have an infinite conducting plane held (by grounding) at zero potential, then place a single charge Q a distance d away. Let the position of Q be on the x-axis at $\mathbf{x}_1 = d\hat{\mathbf{i}}$.

In this case, the boundary condition follows from the properties of a conductor. Charges in the conductor move until there is no electric field tangent to the surface of the conductor. This means that the conductor itself is an equipotential. Since the conducting plane is grounded, the entire plane will be at zero potential.

The crucial information about the field to the $+x$ side of the xy-plane is:

1. There is a single charge Q at $\mathbf{x}_1 = d\hat{\mathbf{i}}$.
2. The xy-plane is at zero potential.

This information uniquely determines the field. The statement of the problem gives one way to enforce those conditions – an infinite conducting plane. However, it is easy to see that there is a second way. If we place a charge $-Q$ at a position $\mathbf{x}_2 = -d\hat{\mathbf{i}}$, the combined potential of the two charges is

$$\begin{aligned} V &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\left|\mathbf{x}-d\hat{\mathbf{i}}\right|}-\frac{1}{\left|\mathbf{x}+d\hat{\mathbf{i}}\right|}\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-d)^2+y^2+z^2}}-\frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right) \end{aligned}$$

Setting the x-coordinate to zero, $x = 0$, shows that $V(0, y, z) = 0$ for all y and z, that is, the entire yz-plane. Since this is condition 2 above, and there is only a single charge Q *in the region of interest*, the electric potential must be the same as for the infinite plane. Therefore, the solution for the plane is

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-d)^2+y^2+z^2}}-\frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right) \qquad x > 0$$

We do not care about the potential for negative x where the two solutions differ.

1.2 Charged plane

Now consider the same problem, but with a charge density σ on the conducting plane. It may seem like the additional charge will change the solution dramatically, but the answer follows easily by superposition. We have the solution for both a uniformly charged plane,

$$V_{CP} = \frac{\sigma\,|x|}{\epsilon_0}$$

and for a plane at potential 0 in the presence of the charge Q. The full potential is just the sum of these,

$$V_{Total} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right) + \frac{\sigma x}{\epsilon_0} \qquad x > 0$$

2 Exterior potential for a grounded sphere in the presence of a point charge

Consider a grounded sphere (potential $V(R) = 0$) of radius R centered on the origin, and a charge q brought to within a distance $a > R$. We find the potential using the method of images.

Let the z-axis pass through the charge q. The system then has azimuthal symmetry, so if the problem can be solved using images, any image charge must also lie on the z-axis. Let there be a charge q' at position $b < R$. This is allowed since we are interested in the potential *outside* the sphere. The question is whether we can choose q' and b so that the sphere is an equipotential.

The potential for the charge pair is

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\left|\mathbf{x} - a\hat{\mathbf{k}}\right|} + \frac{q'}{\left|\mathbf{x} - b\hat{\mathbf{k}}\right|}\right)$$

Now,

$$\begin{aligned}\left|\mathbf{x} - a\hat{\mathbf{k}}\right| &= \sqrt{\left(\mathbf{x} - a\hat{\mathbf{k}}\right)\cdot\left(\mathbf{x} - a\hat{\mathbf{k}}\right)} \\ &= \sqrt{r^2 + a^2 - 2ar\cos\theta}\end{aligned}$$

so that

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2br\cos\theta}}\right)$$

For the sphere to be at zero potential, the term in parentheses must vanish when $r = R$ for all θ. This means that

$$\begin{aligned}\sqrt{R^2+a^2-2aR\cos\theta} &= -\frac{q}{q'}\sqrt{R^2+b^2-2bR\cos\theta} \\ &= -\frac{q}{q'}\frac{b}{R}\sqrt{\frac{R^2}{b^2}\left(R^2+b^2-2bR\cos\theta\right)} \\ &= -\frac{q}{q'}\frac{b}{R}\sqrt{\frac{R^4}{b^2}+R^2-2R\frac{R^2}{b}\cos\theta}\end{aligned}$$

The two roots will be equal if $\frac{R^2}{b} = a$, so that

$$b = \frac{R^2}{a}$$

and the charges must be related by

$$\begin{aligned} 1 &= -\frac{q}{q'}\frac{b}{R} \\ q' &= -q\frac{b}{R} \\ &= -\frac{qR}{a} \end{aligned}$$

The potential is then zero on the sphere, and the potential anywhere outside the sphere is given by

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\left|\mathbf{x} - a\hat{\mathbf{k}}\right|} - \frac{R}{a\left|\mathbf{x} - \frac{R^2}{a}\hat{\mathbf{k}}\right|}\right)$$

3 Exercises

3.1 Grounded plane with two charges

Let a grounded, conducting plane occupy the yz-plane, and two charges, q_1 and q_2, be brought to positions $\mathbf{x}_1 = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ and $\mathbf{x}_2 = c\hat{\mathbf{i}} - b\hat{\mathbf{j}}$. Use the method of images and superposition to write the potential on the $+x$ side of the plane.

3.2 Grounded corner

Two grounded conducting planes make a right angle along the x-axis, so the planes occupy positions $(x, y > 0, z = 0)$ and $(x, y = 0, z > 0)$. Let a charge Q be placed at $\mathbf{x} = a\hat{\mathbf{j}} + b\hat{\mathbf{k}}$. Use the method of images to write the potential in the region $(x, y > 0, z > 0)$.

3.3 Infinite cylinder and wire (more challenging)

An infinite, grounded, conducting cylindrical shell of radius R lies centered on the z-axis and a wire with uniform charge per unit length λ lies parallel at a distance $d > R$. Use the method of images to find the potential everywhere outside the cylinder by considering a second wire with charge per unit length $-\lambda$ at a distance $a < R$ from the $z$axis. Find $a$in terms of d, then write the potential outisde the cylinder. Include an additive constant in the potential to make it zero on the cylinder. A careful diagram in the xy-plane will help.