

Electric potential energy

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1 Conservation of energy in the presence of electrostatic forces

Suppose a particle of mass m and charge Q is in an electric field $\mathbf{E}(\mathbf{x})$. Then it experiences a force, $\mathbf{F}(\mathbf{x}) = Q\mathbf{E}(\mathbf{x})$ and we may write Newton's second law, $\mathbf{F}(\mathbf{x}) = m\mathbf{a}$, to find its motion,

$$Q\mathbf{E}(\mathbf{x}) = m\frac{d\mathbf{v}}{dt}$$

To integrate this, we take the dot product with $\mathbf{v} = \frac{d\mathbf{x}}{dt}$,

$$\begin{aligned} Q\mathbf{E}(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt} &= m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \\ Q\mathbf{E}(\mathbf{x}) \cdot d\mathbf{x} &= m\mathbf{v} \cdot d\mathbf{v} \end{aligned}$$

Integrating, along a given path C , with endpoints at \mathbf{x}_0 and \mathbf{x} ,

$$Q \int_{C, \mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x} = m \int_{C, \mathbf{v}_0}^{\mathbf{v}} \mathbf{v} \cdot d\mathbf{v}$$

we can work the second integral exactly, since

$$\begin{aligned} \int_{\mathbf{v}_0}^{\mathbf{v}} \mathbf{v} \cdot d\mathbf{v} &= \int_{\mathbf{v}_0}^{\mathbf{v}} v_x dv_x + \int_{\mathbf{v}_0}^{\mathbf{v}} v_y dv_y + \int_{\mathbf{v}_0}^{\mathbf{v}} v_z dv_z \\ &= \frac{1}{2} (v_x^2 + v_y^2 + v_z^2) \Big|_{\mathbf{v}_0}^{\mathbf{v}} \\ &= \frac{1}{2} \mathbf{v}^2 - \frac{1}{2} \mathbf{v}_0^2 \end{aligned}$$

We therefore have the change in kinetic energy equal to Q times the line integral of the electric field:

$$Q \int_{C, \mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x} = \frac{1}{2} m\mathbf{v}^2 - \frac{1}{2} m\mathbf{v}_0^2$$

In general, such line integrals depend on the curve C . However, consider any two curves, C_1 and C_2 , between the same endpoints. The difference in the integrals is given by

$$\Delta = \int_{C_2, \mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x} - \int_{C_1, \mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x}$$

$$\begin{aligned}
&= \int_{C_2, \mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x} + \int_{-C_1, \mathbf{x}}^{\mathbf{x}_0} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x} \\
&= \oint_{C_2 - C_1} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x}
\end{aligned}$$

where we integrate from \mathbf{x}_0 to \mathbf{x} along C_2 then backwards from \mathbf{x} to \mathbf{x}_0 along $-C_1$ to form a closed loop. But in the previous section, we showed that all such integrals vanish for the electric field. Therefore, $\Delta = 0$, and the integral from \mathbf{x}_0 to \mathbf{x} is *independent of path*. This allows us to define the *potential energy* of the electric force,

$$U(\mathbf{x}) = -Q \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}) \cdot d\mathbf{x}$$

The potential energy is a function because it has a single, well-defined value at each point \mathbf{x} . We therefore have the energy theorem,

$$\begin{aligned}
-U(\mathbf{x}) + U(\mathbf{x}_0) &= \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2 \\
\frac{1}{2}m\mathbf{v}^2 + U(\mathbf{x}) &= U(\mathbf{x}_0) + \frac{1}{2}m\mathbf{v}_0^2
\end{aligned}$$

since the quantity

$$E = U(\mathbf{x}) + \frac{1}{2}m\mathbf{v}^2$$

retains the same value throughout the motion.

2 Energy of a charge configuration

2.1 Discrete charges

Suppose we bring one charge q_1 close to another, q_2 . Then the energy of the pair is

$$W = q_1 V_2$$

or equivalently

$$W = q_2 V_1$$

Since, for a single point charge, $V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$, both of these give

$$W_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$

where r_{12} is the distance between the charges.

The potential of the pair of charges is now

$$V(x) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{1x}} + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_{2x}}$$

where r_{1x} and r_{2x} are the distances from each of the charges to the point where we want to know the potential. Therefore, bringing in a third charge, the energy will be the energy to bring in the first two, together with $q_3 V(x_3)$, so

$$W_{123} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}}$$

We can iterate this procedure, in a proof by induction. Noticing that we have a sum over all pairs of charges, we suppose, after we have brought together $n - 1$ charges, the energy is

$$W_{n-1} = \frac{1}{4\pi\epsilon_0} \sum_{\text{pairs}}^{n-1} \frac{q_i q_j}{r_{ij}}$$

while the potential will be

$$V_{n-1}(x) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n-1} \frac{q_i}{r_{ix}}$$

so we have

$$\begin{aligned} W_n &= \frac{1}{4\pi\epsilon_0} \sum_{\text{pairs}}^{n-1} \frac{q_i q_j}{r_{ij}} + q_n V_{n-1}(x) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{\text{pairs}}^{n-1} \frac{q_i q_j}{r_{ij}} + \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n-1} \frac{q_i q_n}{r_{in}} \\ &= \frac{1}{4\pi\epsilon_0} \sum_{\text{pairs}}^n \frac{q_i q_j}{r_{ij}} \end{aligned}$$

which is the form we assumed for $n - 1$, extended to n . This is therefore the form for all n .

Notice that the sum is over all pairs of charges. We can write this as a double sum,

$$\sum_{\text{pairs}}^n \frac{q_i q_j}{r_{ij}} = \sum_{i=1}^n \sum_{j=1}^{i-1} \frac{q_i q_j}{r_{ij}}$$

We don't sum over an $i = j$ term. Since we would have the same thing if we summed over all $j > i$ instead of $j < i$,

$$\sum_{\text{pairs}}^n \frac{q_i q_j}{r_{ij}} = \sum_{i=1}^n \sum_{j=i+1}^n \frac{q_i q_j}{r_{ij}}$$

we can take both together, divide by 2, and just skip $i = j$,

$$\sum_{\text{pairs}}^n \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{\text{all } i} \sum_{\text{all } j \neq i} \frac{q_i q_j}{r_{ij}}$$

and the energy is

$$W_n = \frac{1}{8\pi\epsilon_0} \sum_{\text{all } i} \sum_{\text{all } j \neq i} \frac{q_i q_j}{r_{ij}}$$

2.2 Continuum distributions of charge

The continuum case is completely analogous. Rewrite W_n as

$$W_n = \frac{1}{2} \sum_{\text{all } j \neq i} q_j \sum_{\text{all } i} \frac{q_i}{4\pi\epsilon_0 r_{ij}}$$

and take the limit as the distribution becomes continuous, with $q_i \rightarrow \rho(\mathbf{x}') d^3 x'$ and $q_j \rightarrow \rho(\mathbf{x}) d^3 x$. Then, taking the limit of one sum at a time,

$$W = \frac{1}{2} \sum_{\text{all } j \neq i} q_j \int \frac{\rho(\mathbf{x}') d^3 x'}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}'|}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{\text{all } j \neq i} q_j V(\mathbf{x}) \\
&= \frac{1}{2} \int \rho(\mathbf{x}) V(\mathbf{x}) d^3x
\end{aligned}$$

Now take the limit as the second sum becomes continuous.

We can write this entirely in terms of the field. Since $\rho(\mathbf{x})$ vanishes where there is no charge, we may take the integral over all space. Then, using the differential form of Gauss' law, replace the charge density,

$$W = \frac{\epsilon_0}{2} \int_V (\nabla \cdot \mathbf{E}) V(\mathbf{x}) d^3x$$

Now remember that the divergence of a function times a vector is $\nabla \cdot (V\mathbf{E}) = \nabla V \cdot \mathbf{E} + V\nabla \cdot \mathbf{E}$ so we can write

$$V\nabla \cdot \mathbf{E} = \nabla \cdot (V\mathbf{E}) - \nabla V \cdot \mathbf{E}$$

Therefore

$$\begin{aligned}
W &= \frac{\epsilon_0}{2} \int_V (\nabla \cdot \mathbf{E}) V(\mathbf{x}) d^3x \\
&= \frac{\epsilon_0}{2} \int_V (\nabla \cdot (V\mathbf{E}) - \nabla V \cdot \mathbf{E}) d^3x \\
&= \frac{\epsilon_0}{2} \int_V \nabla \cdot (V\mathbf{E}) d^3x - \frac{\epsilon_0}{2} \int_V \nabla V \cdot \mathbf{E} d^3x
\end{aligned}$$

Using the divergence theorem on the first term,

$$\frac{\epsilon_0}{2} \int_V \nabla \cdot (V\mathbf{E}) d^3x = \frac{\epsilon_0}{2} \oint_S V\mathbf{E} \cdot \mathbf{n} d^3x$$

where the surface integral is at infinity. Assuming the field drops to zero at infinite distance, this term vanishes and we are left with

$$\begin{aligned}
W &= -\frac{\epsilon_0}{2} \int_V \nabla V \cdot \mathbf{E} d^3x \\
&= \frac{\epsilon_0}{2} \int_V \mathbf{E} \cdot \mathbf{E} d^3x
\end{aligned}$$

We interpret the integrand,

$$w = \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E}$$

as the energy density of the electric field. Once again, we are able to associate measurable properties directly with the electric field.

3 Force on a charged surface

Suppose we have a charged surface with charge density $\sigma(\mathbf{x})$. Then we know that

$$\mathbf{E}_{out} - \mathbf{E}_{in} = \frac{\sigma}{\epsilon_0}$$

The surface cannot put a net force on itself, so we need to find the *external* electric field – the field that would be present without the surface. Treating the surface in a small region as planar, we know that the charge density produces a field $\mathbf{E}_\sigma = \pm \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$ above and below the surface, so

$$\begin{aligned}\mathbf{E}_{out} &= \mathbf{E}_{ext} + \frac{\sigma}{\epsilon_0} \\ \mathbf{E}_{in} &= \mathbf{E}_{ext} - \frac{\sigma}{\epsilon_0}\end{aligned}$$

Solving for \mathbf{E}_{ext} , we have

$$\mathbf{E}_{ext} = \frac{1}{2}(\mathbf{E}_{out} + \mathbf{E}_{in})$$

The force on an infinitesimal area A of the surface is then

$$\begin{aligned}\mathbf{F} &= Q\mathbf{E}_{ext} \\ &= \sigma A \cdot \frac{1}{2}(\mathbf{E}_{out} + \mathbf{E}_{in})\end{aligned}$$

and the force per unit area is

$$\mathbf{f} = \frac{\sigma}{2}(\mathbf{E}_{out} + \mathbf{E}_{in})$$

For a conductor, we know that $\mathbf{E}_{out} = \frac{\sigma}{\epsilon_0} \mathbf{n}$ and $\mathbf{E}_{in} = 0$ so this becomes $\mathbf{f} = \frac{\sigma^2}{2\epsilon_0} \mathbf{n}$. This gives an outward pressure on the conductor.

Notice that this force can be quite large. For a spherical conductor with surface area of 1 square meter, charged with 1 Coulomb, the outward force is

$$\begin{aligned}\mathbf{F} &= \frac{\sigma^2 A}{2\epsilon_0} \hat{\mathbf{r}} \\ &= \frac{1}{2 \cdot 8.85 \times 10^{-12}} \hat{\mathbf{r}} \left(\frac{C^2}{m^4} m^2 \frac{Nm^2}{C^2} \right) \\ &= 5.65 \times 10^{10} N \hat{\mathbf{r}}\end{aligned}$$

4 Capacitance

Capacitance can be defined for any system of conductors. Here we define the capacitance of a pair of conductors held at equal but opposite charges, $\pm Q$. Essentially, the capacitance characterizes those conductors' ability to hold charge.

We put together three facts:

1. Each conductor is an equipotential
2. The potential between the two conductors is

$$\Delta V(\mathbf{x}) = - \int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$

This won't depend on the curve of integration, as long as it starts on the $-Q$ conductor and ends on the $+Q$. Since the field points from the positive to the negative, this ΔV is always positive.

3. The electric field is proportional to Q . To see this, remember superposition. Consider duplicating the original system and overlaying the pair to get the same two conductors with twice the charge. The electric field will also double.

Since ΔV is proportional to \mathbf{E} and \mathbf{E} is proportional to Q , the potential difference between the conductors is proportional to Q ,

$$\Delta V = aQ$$

where $a = \frac{1}{C}$ is the proportionality constant. The amount of charge the system can hold for a given potential difference is then

$$C = \frac{1}{a} = \frac{Q}{\Delta V}$$

and this is called the *capacitance*.

4.1 Example: parallel plate capacitor

Consider two square plane conductors of side L held a distance d apart. The distance d is much less than L so we may neglect the slight deviation of the potential at the edges of the planes. Find the capacitance.

To find the capacitance, we need to find how the electric potential between the plates varies with the charge on the plates.

We have found that the electric field near a (nearly) infinite plane coincident with the xy -plane is

$$\mathbf{E}_+(z) = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\mathbf{k}} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{k}} & z < 0 \end{cases}$$

Therefore, if we have a second plane at $z = d$ and charge per unit area $-\sigma$, the electric field of the second plane will be

$$\mathbf{E}_-(z) = \begin{cases} -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{k}} & z > d \\ \frac{\sigma}{2\epsilon_0} \hat{\mathbf{k}} & z < d \end{cases}$$

and because of superposition we may simply add these fields to get the total field at any point, $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$. This results in three regions:

$$\mathbf{E}(z) = \begin{cases} 0 & z > d \\ \frac{\sigma}{\epsilon_0} \hat{\mathbf{k}} & 0 < z < d \\ 0 & z < 0 \end{cases}$$

For the large but finite planes of the problem, we assume this same result, with the total charge on each plate given by $Q = \pm\sigma L^2$, so that

$$\mathbf{E}(z) = \begin{cases} 0 & z > d \\ \frac{Q}{\epsilon_0 L^2} \hat{\mathbf{k}} & 0 < z < d \\ 0 & z < 0 \end{cases}$$

Now integrate the electric field to find the potential,

$$V(z) = - \int \mathbf{E} \cdot d\mathbf{l}$$

Since the potential does not depend on the path chosen, we may choose any convenient path. The easiest is to let $d\mathbf{l} = \hat{\mathbf{k}} dz$. We integrate from the negatively charged plate and integrate to the positively charged plate,

$$\begin{aligned} \Delta V &= - \int_d^0 (\mathbf{E} \cdot \hat{\mathbf{k}}) dz \\ &= - \frac{Q}{\epsilon_0 L^2} \int_d^0 (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}) dz \\ &= \frac{Qd}{\epsilon_0 L^2} \end{aligned}$$

Notice that the result is linear in the charge on the plates, as expected. The charge per unit potential is therefore

$$C = \frac{\epsilon_0 L^2}{d}$$

This is a measure of the charge carrying capability of the plates. It increases as the plates become larger and closer together.

5 Exercises

For the following two problems use any method you like to find the potential difference between the shells. Which method is easiest?

5.1 Cylindrical capacitor

Find the capacitance of a pair of coaxial cylindrical shells of length L and of radii R_1 and R_2 . Let $R_2 > R_1$ and $L \gg R_2$. Neglect edge effects.

5.2 Spherical capacitor

Find the capacitance of a pair of concentric spherical shells of radii R_1 and R_2 with $R_2 > R_1$.