Maxwell's equations for electrostatics

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1 The differential form of Gauss's law

Starting from the integral form of Gauss's law, we treat the charge as a continuous distribution, $\rho(\mathbf{x})$. Then, letting V be the volume enclosed by the arbitrary closed surface S, the total charge in V is

$$Q_{total\ enclosed} = \int\limits_V \rho(\mathbf{x})\, d^3x$$

This allows us to write Gauss's Law in differential form. Substituting the integral for Q_{total},

$$\oint\limits_S \mathbf{E} \cdot \hat{\mathbf{n}}\, d^2x = \frac{1}{\epsilon_0} \int\limits_V \rho(\mathbf{x})\, d^3x$$

we apply the divergence theorem to the left side to get

$$\int\limits_V \boldsymbol{\nabla} \cdot \mathbf{E}\, d^3x = \frac{1}{\epsilon_0} \int\limits_V \rho(\mathbf{x})\, d^3x$$

Combining the integrals, we have

$$\int\limits_V \left[\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} \right] d^3x = 0$$

where V is an *arbitrary* volume.

We now prove by contradiction that the integrand must be zero everywhere in V. Suppose there is some point $\mathcal{P}$ in V where $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} > 0$. Then, since we expect $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0}$ to be continuous, there must be a region around $\mathcal{P}$ over which $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0}$ remains positive. Take the arbitrary volume V to be this region. Then the integral is necessarily positive, and we have a contradition. A similar argument holds if $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} < 0$, so we must have $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} = 0$ at $\mathcal{P}$. Since this argument holds for any point in the region, the integrand must vanish everywhere, and we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{x})$$

This is the differential form of Gauss's law. It will be extremely useful once we also know about the curl of $\mathbf{E}$.

2 The curl of the electric field

2.1 The curl using Stokes' theorem

Again consider the electric field of a point charge at the origin,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$$

Consider the integral of $\mathbf{E} \cdot d\mathbf{l}$ along an arbitrary curve, C,

$$\begin{aligned}
\int_C \mathbf{E} \cdot d\mathbf{l} &= \frac{Q}{4\pi\epsilon_0} \int_C \frac{1}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{l} \\
&= \frac{Q}{4\pi\epsilon_0} \int_{r_i}^{r_f} \frac{1}{r^2} dr \\
&= \frac{Q}{4\pi\epsilon_0} \left(-\frac{1}{r_f} + \frac{1}{r_i} \right)
\end{aligned}$$

where r_i and r_f are the initial and final radii of the curve.

Now suppose the curve is a closed loop. Then $r_i = r_f$ and the integral vanishes, regardless of the closed curve, C,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{Q}{4\pi\epsilon_0} \oint_C \frac{1}{r^2} dr = 0$$

Notice that this result depends only on the relative position of the curve and the charge, not on the charge being at the origin.

Now, suppose there is more than one charge. Since the electric field $\mathbf{E}$ is the sum of the fields from each charge, $\mathbf{E}_i$, and the line integral for each vanishes, the sum vanishes,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \sum_{i=1}^{n} \oint_C \mathbf{E}_i \cdot d\mathbf{l} = 0$$

as long as the curve doesn't pass through any of the charges. The result holds equally well in the limit of a charge density, and we conclude that

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

for any curve C in empty space.

Now apply Stokes' theorem. We have

$$\begin{aligned}
0 &= \oint_C \mathbf{E} \cdot d\mathbf{l} \\
&= \iint_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{n} \, d^2x
\end{aligned}$$

where S is now an arbitrary surface with boundary C. This can only be the case for all surfaces if the integrand vanishes. Moreover, since $\mathbf{n}$ is arbitrary as well, we have

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

in free space for the electric field of any static charge distribution.

The vanishing of closed line integrals of the electric field, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$, (or equivalently, the vanishing curl of $\mathbf{E}$, since Stokes' theorem makes these statements equivalent), means that we may define a function from the integral of the electric field along curves,

$$V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E} \cdot d\mathbf{l}$$

This integral is independent of our choice of the path of integration, and therefore depends only on the endpoint $\mathbf{x}$. To see this, subtract the integral along any two curves between the same endpoints,

$$V_{C_1}(\mathbf{x}) - V_{C_2}(\mathbf{x}) = -\int_{\mathbf{x}_0,C_1}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} + \int_{\mathbf{x}_0,C_2}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l}$$

Since integrating along C_1 from $\mathbf{x}_0$ to $\mathbf{x}$ just gives the negative of the integral along $-C_1$ from $\mathbf{x}$ to $\mathbf{x}_0$, we may combine the two integrals on the right into a single closed line integral, which then vanishes:

$$-\int_{\mathbf{x}_0,C_1}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} + \int_{\mathbf{x}_0,C_2}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} = \oint_{C_2-C_1} \mathbf{E}\cdot d\mathbf{l} = 0$$

Therefore, $V_{C_1} = V_{C_2}$ for any two paths between the same endpoints, and $V(\mathbf{x})$ is a function.

$V(\mathbf{x})$ is called the *electric potential.* From it, we may find the electric field by taking the gradient,

$$\mathbf{E} = -\boldsymbol{\nabla} V(\mathbf{x}) \tag{1}$$

2.2 An alternative proof

Consider the gradient of $\frac{1}{|\mathbf{x}-\mathbf{x}_i|}$,

$$\begin{aligned}
\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{x}_i|} &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\frac{1}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}} \\
&= -\frac{1}{2}\frac{1}{\left[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\right]^{3/2}}\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\right] \\
&= -\frac{1}{2}\frac{1}{\left[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\right]^{3/2}}\left[2(x-x_i)\hat{\mathbf{i}} + 2(y-y_i)\hat{\mathbf{j}} + 2(z-z_i)\hat{\mathbf{k}}\right] \\
&= -\frac{1}{\left[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\right]^{3/2}}\left[(x-x_i)\hat{\mathbf{i}} + (y-y_i)\hat{\mathbf{j}} + (z-z_i)\hat{\mathbf{k}}\right] \\
&= -\frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3}
\end{aligned}$$

Using this, we may write the electric field of a charge q_i at an arbitrary point $\mathbf{x}^i$ as

$$\begin{aligned}
\mathbf{E}_i(\mathbf{x}) &= \frac{q_i}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3} \\
&= -\boldsymbol{\nabla}\frac{q_i}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_i|}
\end{aligned}$$

Now suppose we have a collection of charges, $q_i, i = 1,\ldots,N$. Then by superposition, the total electric field is

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \mathbf{E}_i(\mathbf{x}) \\
&= -\sum_{i=1}^{N}\boldsymbol{\nabla}\frac{q_i}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_i|} \\
&= -\boldsymbol{\nabla}\left(\sum_{i=1}^{N}\frac{q_i}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_i|}\right) \qquad (2)
\end{aligned}$$

Since the curl of the gradient of any function vanishes, we immediately have

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

for any electric field.

This approach gives us an expicit formula for finding the electric potential. Comparing eqs.(1) and (2) we see that for any distribution of charges,

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{x} - \mathbf{x}_i|}$$

In the continuum limit, this behaves just like the electric field and we have

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')\, d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

This scalar integration is generally easier than the vector integration for finding the electric field directy. Once we have the potential, we easily find the electric field using eq.(1).

The definition of the electric potential allows us to choose an arbitrary reference point for the zero of the potential. Starting again from the integral for the potential, we substitute the integral for the electric field,

$$\begin{aligned} V(\mathbf{x}) &= -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\mathbf{x}'') \cdot d\mathbf{x}'' \\ &= -\frac{1}{4\pi\epsilon_0} \int_{\mathbf{x}_0}^{\mathbf{x}} \int d^3x' \rho(\mathbf{x}') \frac{\mathbf{x}'' - \mathbf{x}'}{|\mathbf{x}'' - \mathbf{x}'|^3} \cdot d\mathbf{x}'' \\ &= -\frac{1}{4\pi\epsilon_0} \int d^3x' \rho(\mathbf{x}') \int_{\mathbf{x}_0}^{\mathbf{x}'} \frac{\mathbf{x}'' - \mathbf{x}'}{|\mathbf{x}'' - \mathbf{x}'|^3} \cdot d\mathbf{x}'' \end{aligned}$$

Now, interchanging the order of integration and using $\int_{\mathbf{x}_0}^{\mathbf{x}'} \frac{\mathbf{x}''-\mathbf{x}'}{|\mathbf{x}''-\mathbf{x}'|^3} \cdot d\mathbf{x}'' = -\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, the electric potential becomes

$$= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x}_0 - \mathbf{x}'|} \right) d^3x'$$

Frequently, we assume the reference point is at infinity. In this case we take $\mathbf{x}_0 \to \infty$ and the potential is simply

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')\, d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

3 Maxwell's equations for electrostatics

While the curl of $\mathbf{E}$ maybe be different from zero in the presence of a changing magnetic field, Maxwell's equations for *electrostatics* reduce to

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \rho(\mathbf{x}) \\ \boldsymbol{\nabla} \times \mathbf{E} &= 0 \end{aligned} \tag{3}$$

The Helmholz theorem tells us that knowing the divergence and curl of a vector field, together with boundary conditions, uniquely determines the field everywhere within the boundary. These equations therefore give a complete characterization of the electric field once we specify the charge density $\rho(\mathbf{x})$ in a volume V, and give boundary conditions on the boundary of V.

As we have seen above, the vanishing curl of $\mathbf{E}$ implies the existence of a potential. Furthermore, we may write the electrostatic equations in terms of the potential, eq.(1). Substituting this into the electrostatic equations, the curl of the gradient vanishes automatically, while Gauss's law becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho(\mathbf{x}) \tag{4}$$

This is the Poisson equation. Together with boundary conditions, this is gives a unique solution for the potential, which then determines the electric field. We will devote considerable attention to solving the Poisson equation. If there are no sources in a given region, $\rho(\mathbf{x}) = 0$, the Poisson equation reduces to the Laplace equation,

$$\nabla^2 V = 0 \tag{5}$$

Again, the Laplace equation together with boundary conditions determines the electric potential completely. The electric field is then found from

$$\mathbf{E} = -\boldsymbol{\nabla} V(\mathbf{x})$$

4 Boundary conditions

Frequently, we wish to solve for the electric field in some region which contains or is bounded by surfaces carrying charges. In such cases, we can solve for the fields on either side of the surface, then use boundary conditions to match the two solutions. For this, we need to know how the field and potential change when we cross a charge-carrying surface.

Consider an arbitrary surface with charge density $\sigma(\mathbf{x})$. Choose a Gaussian surface in the shape of a tiny cylinder. Make it small enough that

1. The surface is essentially flat, and parallel to the flat faces of the Gaussian surface.
2. The radius of the cylinder is small enough that σ changes only negligibly across the surface contained.
3. The height of the cylinder is much shorter than the diameter, so the field across the side of the cylinder is negligible.

Then Gauss' law for the cylinder is just given by

$$\begin{aligned} \frac{\sigma A}{\epsilon_0} &= \int\limits_{top} \mathbf{E}\cdot\mathbf{n} d^2x + \int\limits_{bottom} \mathbf{E}\cdot(-\mathbf{n})\, d^2x \\ &\approx (\mathbf{E}_{top} - \mathbf{E}_{bottom})\cdot\mathbf{n}|\, A \end{aligned}$$

In the limit as we shrink the cylinder, we have the change in the normal component of the electric field across the surface.

$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\epsilon_0}$$

For the component of the electric field tangential to the surface, we use

$$\oint\limits_C \mathbf{E}\cdot d\mathbf{l} = 0$$

and choose the curve C to be a tiny rectangular loop with short sides of length d piercing the surface and long sides of length L parallel to it on opposite sides. With the loop small enough that $\mathbf{E}$ is essentially constant along L, we then have

$$\begin{aligned} 0 &= \oint_C \mathbf{E} \cdot d\mathbf{l} \\ &\approx \mathbf{E}_{\| top} L - \mathbf{E}_{\| bottom} L \end{aligned}$$

and therefore,

$$\mathbf{E}_{\| top} = \mathbf{E}_{\| bottom}$$

so the tangential component of the field is constant across the boundary surface.

In the case of a conductor, there is no electric field inside the conductor or the free charges would move. This means that $\mathbf{E}_{\| bottom} = 0$ and $\mathbf{E}_{\perp bottom} = 0$, so that the field above a conductor is normal to the surface and given by

$$\mathbf{E}_{above} = \frac{\sigma}{\epsilon_0} \mathbf{n}$$

Since $\mathbf{E} = -\boldsymbol{\nabla} V$, we can write this as

$$\begin{aligned} -\boldsymbol{\nabla} V &= \frac{\sigma}{\epsilon_0} \mathbf{n} \\ -\mathbf{n} \cdot \boldsymbol{\nabla} V &= \frac{\sigma}{\epsilon_0} \end{aligned}$$

and writing the directional derivative as $\mathbf{n} \cdot \boldsymbol{\nabla} V = \frac{\partial V}{\partial n}$ we have

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

so we can find the charge density from the potential.

5 Examples

We consider both discrete and continuous charge distributions.

5.1 Example: Point charge

The potential due to a point charge is immediate from our discussion of the curl. For a charge at the origin, and for some reference point $\mathbf{x}_i$,

$$\begin{aligned} V(\mathbf{x}; \mathbf{x}_i) &= -\int_{\mathbf{x}_i}^{\mathbf{x}} \mathbf{E} \cdot d\mathbf{l} \\ &= -\frac{Q}{4\pi\epsilon_0} \int_{r_i}^{r} \frac{1}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{l} \\ &= -\frac{Q}{4\pi\epsilon_0} \left(-\frac{1}{r} + \frac{1}{r_i} \right) \end{aligned}$$

It is often convenient to take the reference point, $\mathbf{x}_i$, to lie at an infinite distance so the potential is simply

$$V(\mathbf{x}; \mathbf{x}_i) = \frac{Q}{4\pi\epsilon_0 r}$$

More generally, for the source charge at the point $\mathbf{x}_0$, the potential is

$$V(\mathbf{x}; \mathbf{x}_0) = \frac{Q}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}_0|}$$

5.2 Example 2: Multiple point charges

Now consider multiple point charges at locations $q_1 = 2q, q_2 = -q$ and $q_3 = 3q$ located respectively at

$$\begin{aligned} \mathbf{x}_1 &= 3\hat{\mathbf{i}} \\ \mathbf{x}_2 &= \hat{\mathbf{i}} + 3\hat{\mathbf{j}} \\ \mathbf{x}_1 &= 2\hat{\mathbf{k}} \end{aligned}$$

The electric potential $V(\mathbf{x})$ at any position $\mathbf{x}$ is the linear superposition

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{x}-\mathbf{x}_i|} \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{\left|\mathbf{x}-3\hat{\mathbf{i}}\right|} - \frac{q}{\left|\mathbf{x}-\left(\hat{\mathbf{i}}+3\hat{\mathbf{j}}\right)\right|} + \frac{3q}{\left|\mathbf{x}-2\hat{\mathbf{k}}\right|}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{2q}{\sqrt{(x-3)^2+y^2+z^2}} - \frac{q}{\sqrt{(x-1)^2+(y-3)^2+z^2}} + \frac{3q}{\sqrt{x^2+y^2+(z-2)^2}}\right) \end{aligned}$$

Notice that we no longer have to keep track of directions.

5.3 Example 3: Infinite line charge

We know the electric field is

$$\mathbf{E} = E(\rho)\,\hat{\boldsymbol{\rho}} = \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}$$

so the potential only changes in the $\hat{\boldsymbol{\rho}}$ direction:

$$\begin{aligned} V(\rho) &= -\int_{\mathbf{x}_i}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} \\ &= -\int_{\rho_0}^{\rho}\frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}\cdot d\mathbf{l} \\ &= -\frac{\lambda}{2\pi\epsilon_0}\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} \\ &= -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{\rho}{\rho_0} \end{aligned}$$

Now try it directly from the charge density. In terms of the density per unit length, we have

$$V(\mathbf{x}) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}_0-\mathbf{x}'|}\right)dz'$$

where we keep the reference point arbitrary for the moment. The problem is symmetric about the z-axis, and is also invariant as we change z, so we may take $z = 0$. Then integrating, and *carefully* taking the limit as $z' \to \infty$

$$V(\mathbf{x}) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\rho^2+z'^2}} - \frac{1}{\sqrt{\rho_0^2+z'^2}}\right)dz'$$

$$
\begin{aligned}
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left(z' + \sqrt{\rho^2 + z'^2}\right) - \ln\left(z' + \sqrt{\rho_0^2 + z'^2}\right)\right)\Bigg|_{-\infty}^{\infty} \\
&= \frac{\lambda}{4\pi\epsilon_0}\lim_{z'\to\infty}\ln\left(\frac{\sqrt{\rho^2 + z'^2} + z'}{\sqrt{\rho_0^2 + z'^2} + z'}\right)\left(\frac{\sqrt{\rho_0^2 + z'^2} - z'}{\sqrt{\rho^2 + z'^2} - z'}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\lim_{z'\to\infty}\ln\left(\frac{\sqrt{\frac{\rho_0^2}{z'^2} + 1} - 1}{\sqrt{\frac{\rho^2}{z'^2} + 1} - 1}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\lim_{z'\to\infty}\ln\left(\frac{\frac{\rho_0^2}{2z'^2} + 1 - 1}{\frac{\rho^2}{2z'^2} + 1 - 1}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\lim_{z'\to\infty}\ln\left(\frac{\rho_0^2}{\rho^2}\right) \\
&= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\rho_0}{\rho}\right) \\
&= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\rho}{\rho_0}\right)
\end{aligned}
$$

5.4 Example: Midpoint of a finite line charge

For a finite line charge, we no longer can ignore z. Everything is still independent of φ and φ'. The integral becomes

$$
\begin{aligned}
V(\mathbf{x}) &= \frac{\lambda}{4\pi\epsilon_0}\int_a^b \frac{1}{|\mathbf{x} - \mathbf{x}'|}dz' \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_a^b \frac{1}{|\rho\hat{\boldsymbol{\rho}} + z\mathbf{k} - z'\mathbf{k}|}dz' \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_a^b \frac{1}{\sqrt{\rho^2 + (z - z')^2}}dz'
\end{aligned}
$$

While this is not hard to integrate directly with the substitutions $\zeta = z - z'$ and $\zeta = \rho\sinh\theta$, it is still easier to use an online integrator to find

$$
\begin{aligned}
V(\mathbf{x}) &= -\frac{\lambda}{4\pi\epsilon_0}\left(\ln\left(\sqrt{\rho^2 + (z-b)^2} + z - b\right) - \ln\left(\sqrt{\rho^2 + (z-a)^2} + z - a\right)\right) \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{\rho^2 + (z-b)^2} + z - b}{\sqrt{\rho^2 + (z-a)^2} + z - a}\right)
\end{aligned}
$$

Consider the potential at above the midpoint of a segment of length L. At the center, $z = 0$, so

$$
V = -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{\rho^2 + \left(\frac{L}{2}\right)^2} - \frac{L}{2}}{\sqrt{\rho^2 + \left(\frac{L}{2}\right)^2} + \frac{L}{2}}\right)
$$

For large ρ,

$$\begin{aligned}
V &= -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\rho\sqrt{1+\left(\frac{L}{2\rho}\right)^2}-\frac{L}{2}}{\rho\sqrt{+\left(\frac{L}{2\rho}\right)^2}+\frac{L}{2}}\right) \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{1+\left(\frac{L}{2\rho}\right)^2}-\frac{L}{2\rho}}{\sqrt{1+\left(\frac{L}{2\rho}\right)^2}+\frac{L}{2\rho}}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{1+\frac{1}{2}\left(\frac{L}{2\rho}\right)^2-\frac{L}{2\rho}}{1+\frac{1}{2}\left(\frac{L}{2\rho}\right)^2+\frac{L}{2\rho}}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{1-\frac{L}{2\rho}}{1+\frac{L}{2\rho}}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\left(1-\frac{L}{2\rho}\right)\left(1-\frac{L}{2\rho}\right)\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(1-\frac{L}{\rho}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\left(-\frac{L}{\rho}\right) \\
&= \frac{\lambda L}{4\pi\epsilon_0\rho}
\end{aligned}$$

Setting the total charge to $Q=\lambda L$ and taking the gradient, the electric field is

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla} V \\
&= \frac{Q}{4\pi\epsilon_0\rho^2}\hat{\boldsymbol{\rho}}
\end{aligned}$$

In the opposite limit, the wire looks extremely long. Letting $\frac{\rho}{L}\ll 1$, the potential becomes

$$\begin{aligned}
V &= -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\frac{L}{2}\sqrt{1+\frac{4\rho^2}{L^2}}-\frac{L}{2}}{\frac{L}{2}\sqrt{1+\frac{4\rho^2}{L^2}}+\frac{L}{2}}\right) \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{1+\frac{4\rho^2}{L^2}}-1}{\sqrt{1+\frac{4\rho^2}{L^2}}+1}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{1+\frac{2\rho^2}{L^2}-1}{1+\frac{2\rho^2}{L^2}+1}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\frac{\rho^2}{L^2}}{1+\frac{\rho^2}{L^2}}\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\rho^2}{L^2}\left(1-\frac{\rho^2}{L^2}\right)\right) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\rho^2}{L^2}\right)
\end{aligned}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\rho}{L}\right)$$

The electric field is then

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla} V \\
&= \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}
\end{aligned}$$

5.5 Circular disk: on axis

Consider a uniformly charged disk of radius R and total charge Q. Compare the direct integration to find the electric field on the z-axix (problem 2.4) to finding V first then taking the gradient.

5.5.1 Electric field by direct integration

To find the electric field directly by integration, we must compute

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$

where the charge density is given by

$$\rho(\mathbf{x}') = \frac{Q}{\pi R^2}\delta(z')\Theta(R-\rho')$$

Letting $\mathbf{x} = z\hat{\mathbf{k}}$ and $|\mathbf{x}-\mathbf{x}'| = \sqrt{(\rho')^2+(z-z')^2}$, the electric field becomes

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \frac{Q}{\pi R^2}\delta(z')\Theta(R-\rho')\frac{z\hat{\mathbf{k}}-\left(\rho'\hat{\boldsymbol{\rho}}'+z'\hat{\mathbf{k}}\right)}{\left((\rho')^2+(z-z')^2\right)^{3/2}}d^3x' \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi}d\varphi'\int_0^{\infty}\rho'd\rho'\int_{-\infty}^{\infty}dz'\delta(z')\Theta(R-\rho')\frac{z\hat{\mathbf{k}}-\left(\rho'\hat{\boldsymbol{\rho}}'+z'\hat{\mathbf{k}}\right)}{\left((\rho')^2+(z-z')^2\right)^{3/2}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi}d\varphi'\int_0^{R}\rho'd\rho'\frac{z\hat{\mathbf{k}}-\rho'\hat{\boldsymbol{\rho}}'}{\left((\rho')^2+z^2\right)^{3/2}}
\end{aligned}$$

Replacing $\hat{\boldsymbol{\rho}}' = \hat{\mathbf{i}}\cos\varphi' + \hat{\mathbf{j}}\sin\varphi'$, the integral over the second term gives

$$\begin{aligned}
\frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi}d\varphi'\int_0^{R}\rho'd\rho'\frac{z\hat{\mathbf{k}}-\rho'\hat{\boldsymbol{\rho}}'}{\left((\rho')^2+z^2\right)^{3/2}} &= -\frac{Q}{4\pi^2\epsilon_0 R^2}\int_0^{2\pi}d\varphi'\int_0^{R}\rho'd\rho'\frac{\rho'\left(\hat{\mathbf{i}}\cos\varphi'+\hat{\mathbf{j}}\sin\varphi'\right)}{\left((\rho')^2+z^2\right)^{3/2}} \\
&= -\frac{Q}{4\pi^2\epsilon_0 R^2}\left(\hat{\mathbf{i}}\int_0^{2\pi}\cos\varphi'd\varphi'+\hat{\mathbf{j}}\int_0^{2\pi}\sin\varphi'd\varphi'\right)\int_0^{R}\frac{(\rho')^2d\rho'}{\left((\rho')^2+z^2\right)^{3/2}}
\end{aligned}$$

and we see that the angular integrals vanish,

$$\int_0^{2\pi}\cos\varphi'd\varphi' = \int_0^{2\pi}\sin\varphi'd\varphi' = 0$$

so this term drops out.

For the remaining integral the angular integral is trivial„

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi} d\varphi' \int_0^R \rho' d\rho' \frac{z\hat{\mathbf{k}}}{\left((\rho')^2+z^2\right)^{3/2}} &= \frac{Qz}{4\pi^2\epsilon_0 R^2}\hat{\mathbf{k}}\int_0^{2\pi} d\varphi' \int_0^R \frac{\rho' d\rho'}{\left((\rho')^2+z^2\right)^{3/2}} \\ &= \frac{Qz}{4\pi^2\epsilon_0 R^2}2\pi\hat{\mathbf{k}}\int_0^R \frac{\rho' d\rho'}{\left((\rho')^2+z^2\right)^{3/2}}\end{aligned}$$

The final integral is

$$\begin{aligned}\int_0^R \frac{\rho' d\rho'}{\left((\rho')^2+z^2\right)^{3/2}} &= -\frac{1}{\sqrt{(\rho')^2+z^2}}\Bigg|_0^R \\ &= -\frac{1}{\sqrt{R^2+z^2}}+\frac{1}{z}\end{aligned}$$

and therefore,

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{2\pi\epsilon_0 R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{k}}$$

5.5.2 Using the potential

Now we have only a single integral,

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')\, d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

with $\rho(\mathbf{x}') = \frac{Q}{\pi R^2}\delta(z')\,\Theta(R-\rho')$. Since the gradient of V must lie along the z axis, we set $|\mathbf{x}-\mathbf{x}'| = \sqrt{(\rho')^2+(z-z')^2}$ as before

$$\begin{aligned}V(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2}\int \frac{\delta(z')\,\Theta(R-\rho')\, d^3x'}{\sqrt{(\rho')^2+(z-z')^2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi} d\varphi' \int_0^{\infty} \rho' d\rho' \int_{-\infty}^{\infty} dz'\delta(z')\,\Theta(R-\rho')\frac{1}{\sqrt{(\rho')^2+(z-z')^2}} \\ &= \frac{1}{2\pi\epsilon_0}\frac{Q}{R^2}\int_0^R \frac{\rho' d\rho'}{\sqrt{(\rho-\rho')^2+z^2}}\end{aligned}$$

The integral is

$$\begin{aligned}\int_0^R \frac{\rho' d\rho'}{\sqrt{(\rho-\rho')^2+z^2}} &= \sqrt{(\rho')^2+z^2}\Big|_0^R \\ &= \sqrt{R^2+z^2}-z\end{aligned}$$

so the potential is

$$V(\mathbf{x}) = \frac{Q}{2\pi\epsilon_0 R^2}\left(\sqrt{R^2+z^2}-z\right)$$

The electric field is the gradient,

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= -\boldsymbol{\nabla}\left[\frac{Q}{2\pi\epsilon_0 R^2}\left(\sqrt{R^2+z^2}-z\right)\right] \\
&= -\frac{Q}{2\pi\epsilon_0 R^2}\hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\sqrt{R^2+z^2}-z\right) \\
&= -\frac{Q}{2\pi\epsilon_0 R^2}\hat{\mathbf{k}}\left(\frac{1}{2}\frac{2z}{\sqrt{R^2+z^2}}-1\right) \\
&= \frac{Q}{2\pi\epsilon_0 R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{k}}
\end{aligned}$$

In general, it is easier to do a single integral and take derivatives than it is to do three integrals.

6 Exercises

6.1 Electric potential by integrating the electric field

We have found that the electric field due to an infinitely long cylinder of radius R, carrying a total charge per unit length of λ and having charge density,

$$\rho(\mathbf{x}) = \frac{3\lambda\rho'}{2\pi R^3}\Theta(R-\rho')$$

is given by

$$\mathbf{E}(\rho) = \begin{cases} \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}} & \rho \geq R \\ \frac{\lambda\rho^2}{2\pi\epsilon_0 R^3}\hat{\boldsymbol{\rho}} & \rho \leq R \end{cases}$$

Find the potential both inside and outside the cylinder by integrating

$$V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l}$$

Take the zero of the potential to be at $\rho_0 = R$.

6.2 Electric potential directly

Consider a straight wire of length $2L$ with charge per unit length λ. Take the charge density to be

$$\rho(\mathbf{x}) = \frac{\lambda}{2\pi\rho'}\delta(z')\Theta(z'+L)\Theta(L-z')$$

By integrating

$$V(\rho) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

show that the potential may be written as

$$V(\rho) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{\rho^2+L^2}+L}{\sqrt{\rho^2+L^2}-L}\right)$$

This potential does not vanish at $\rho = R$ like the one in problem 6.1, but we can make it vanish there by subtracting the constant $V(R)$ to get

$$
\begin{aligned}
V(\rho) &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{\sqrt{\rho^2+L^2}+L}{\sqrt{\rho^2+L^2}-L}\right) - \ln\left(\frac{\sqrt{R^2+L^2}+L}{\sqrt{R^2+L^2}-L}\right)\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\left(\sqrt{\rho^2+L^2}+L\right)\left(\sqrt{R^2+L^2}-L\right)}{\left(\sqrt{\rho^2+L^2}-L\right)\left(\sqrt{R^2+L^2}+L\right)}
\end{aligned}
$$

Show that the limit of $V(\rho)$ as $L \to \infty$ agrees with the $\rho > R$ potential of problem 6.1. (You will need the Taylor series, $\sqrt{\rho^2+L^2} = L\sqrt{1+\frac{\rho^2}{L^2}} \approx L\left(1+\frac{1}{2}\frac{\rho^2}{L^2}\right)$ for $\rho \ll L$)

6.3 Electric potential from point particles

Find the electric potential produced by a system of three charges: $2q$ at the origin, $3q$ at position $\mathbf{x} = 2\hat{\mathbf{i}}+3\hat{\mathbf{j}}$, and $-q$ at $\mathbf{x} = 5\hat{\mathbf{k}}$, then take the gradient of $V(\mathbf{x})$ to find the electric field everywhere.

6.4 Electric field of a charged sphere

By integrating

$$
V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')\, d^3x'}{|\mathbf{x}-\mathbf{x}'|}
$$

find the electric potential for all r of a ball of radius R with charge density

$$
\rho(\mathbf{x}') = \begin{cases} \lambda r'^2 & r' < R \\ 0 & r' > R \end{cases}
$$

and total charge Q. You may use the charge density in terms of Q that you found for problem 3.1 of the Gauss's law notes. Use the potential to find the electric field.