

Gauss's Law

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1 Gauss's law

Suppose we have a point charge, Q , at the origin. Then, in spherical coordinates, the electric field it produces is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$$

We can visualize the field by constructing a set of curves with one curve through each point of space, such that at any point, the electric field is a tangent vector to the curve through that point. For the point charge, these curves are just the rays emanating from the origin. These are called *flux lines*. We can choose a representative (and finite!) subset of these rays. If we do this so that the density of lines is proportional to the magnitude of \mathbf{E} , then we have a reasonable visual way to think of the field.

These field lines have a remarkable property. At a distance r from the origin, the number of lines per unit area is

$$\frac{N}{A} = \lambda E = \frac{\lambda Q}{4\pi\epsilon_0 r^2}$$

for some convenient constant λ (depending on how many lines we choose to draw). But the area of a sphere at the same radius is $4\pi r^2$, so the number of lines passing through the sphere is

$$\frac{N}{A} 4\pi r^2 = \frac{\lambda Q}{\epsilon_0} = \text{const.}$$

a *constant*. The same number of flux lines pass through any sphere about the origin.

Now let's intersect half the flux lines with a hemisphere of radius r_1 and the other half with a second hemisphere of radius r_2 . Connect the two hemispheres by an annulus to form a closed surface. The normal to the annulus is perpendicular to $\hat{\mathbf{r}}$. The total number of flux lines through this new closed surface is still the same, $\frac{\lambda Q}{\epsilon_0}$. It's not hard to see that we can continue to modify the surface, building a closed surface out of pieces of spherical surfaces at varying radii, together with radial planes with normals perpendicular to $\hat{\mathbf{r}}$. The flux line count remains the same. We can even have parts of the surface loop back beneath other parts, as long as we make the total surface closed, because there will necessarily be a third spherical piece as we bring the surface back out. The count is always the same.

This is a visual description of the content of Gauss's law. Instead of counting flux lines, we compute the actual electric flux through a closed surface:

$$\begin{aligned} \Phi_E &= \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} d^2x \\ &= \frac{1}{4\pi\epsilon_0} \oint_S \frac{Q}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} d^2x \end{aligned}$$

For a sphere at radius r , this gives

$$\begin{aligned}\Phi_E &= \frac{Q}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \left(\frac{1}{r^2}\right) r^2 \sin\theta d\theta d\varphi \\ &= \frac{Q}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \sin\theta d\theta d\varphi \\ &= \frac{Q}{\epsilon_0}\end{aligned}$$

Again, we may modify the shape of the closed surface, S , by combining pieces of spherical surfaces at arbitrary radii with pieces with $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = 0$. In fact, we can approach arbitrarily close to *any* closed surface with enough pieces of these two types.

We can make a similar argument for the electric flux through a closed surface if the surface does *not* contain the charge Q . In this case, however, equal numbers of flux lines enter and leave the contained volume so the total flux is zero. No matter where we place the charge, we can always build up an arbitrary closed surface by combining perpendicular bits with parts of spheres and evaluate the flux. The result only depends on whether Q is inside the surface or not.

Because Coulomb's force law is additive, we can take any number of charges, q_1, \dots, q_n , and evaluate the flux from each through a given surface. The result always returns the total of all the charges *inside* the surface:

$$\Phi_E = \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} d^2x = \sum_{q_i \text{ inside}} \frac{q_i}{\epsilon_0}$$

or simply

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} d^2x = \frac{Q_{\text{total enclosed}}}{\epsilon_0}$$

2 Examples using the integral form of Gauss's law

The integral form of Gauss's law is useful when there is a lot of symmetry to a charge distribution, and it makes those problems much easier. We consider some examples. Any of the following may be easily found using the integral form:

2.1 Nested charge and spheres

A charge q lies at the origin and a sphere of radius R carrying a total charge Q on its surface is centered on the origin. We may choose *any* closed surface as our "Gaussian surface". First, imagine a sphere of radius r with $0 < r < R$, centered at the origin. By the symmetry of the problem, we know the electric field must be in the radial $\hat{\mathbf{r}}$ direction, and its magnitude can only depend on r , not on the angular direction. We may therefore write $\mathbf{E} = E(r) \hat{\mathbf{r}}$. At the same time, the outward normal to the imaginary spherical surface is also in the $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ direction. Gauss's law therefore tells us

$$\begin{aligned}\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} d^2x &= \frac{Q_{\text{total enclosed}}}{\epsilon_0} \\ \oint_S (E(r) \hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} d^2x &= \frac{q}{\epsilon_0}\end{aligned}$$

where the charge enclosed is only q because $r < R$. The integral over S is just over θ and φ , and it just gives the area of the sphere:

$$\begin{aligned} \frac{q}{\epsilon_0} &= \oint_S E(r) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} r^2 \sin \theta d\theta d\varphi \\ &= r^2 E(r) \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi \\ &= 4\pi r^2 E(r) \end{aligned}$$

The electric field is therefore

$$\mathbf{E} = E(r) \hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

in the region inside the sphere.

Now let $r > R$. The symmetry is still spherical. The only thing that changes is that the total enclosed charge is now $q + Q$, so the electric field outside is

$$\mathbf{E} = \frac{q + Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

Notice that the effect of the sphere is not felt at all inside.

2.2 Infinite line charge

Consider the infinite line charge of the previous section, lying along the x -axis with charge per unit length λ . Because the line is infinite, symmetry tells us that the field can have no x -component, and because of the rotational symmetry around the line charge, there can be no dependence on the angle. The field must therefore be radially away from the wire, and we may write

$$\mathbf{E} = E(\rho) \hat{\boldsymbol{\rho}}$$

where $\hat{\boldsymbol{\rho}} = \frac{1}{\rho} (y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$. For our Gaussian surface, we imagine a cylindrical shape of length L and radius R , centered on the x -axis. This shape has three distinct surfaces, each with its own normal vector. The surface must be closed, so in addition to the curved surface at $\rho = R$, we have two endcaps. The outward normals are then:

$$\hat{\mathbf{n}} = \begin{cases} -\hat{\mathbf{i}} & \text{left end cap} \\ \hat{\boldsymbol{\rho}} & \text{curved side} \\ +\hat{\mathbf{i}} & \text{right end cap} \end{cases}$$

Noticing that the amount of charge contained within the cylinder is $Q_{\text{enclosed}} = \lambda L$, and dividing the integral of Gauss's law into three separate terms, we have

$$\begin{aligned} \frac{Q_{\text{total enclosed}}}{\epsilon_0} &= \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} d^2x \\ \frac{\lambda L}{\epsilon_0} &= \int_{\text{left cap}} (E(\rho) \hat{\boldsymbol{\rho}}) \cdot (-\hat{\mathbf{i}}) d^2x + \int_{\text{curved side}} (E(\rho) \hat{\boldsymbol{\rho}}) \cdot (\hat{\boldsymbol{\rho}}) d^2x + \int_{\text{right cap}} (E(\rho) \hat{\boldsymbol{\rho}}) \cdot (\hat{\mathbf{i}}) d^2x \\ \frac{\lambda L}{\epsilon_0} &= \int_{\text{curved side}} (E(\rho) \hat{\boldsymbol{\rho}}) \cdot (\hat{\boldsymbol{\rho}}) d^2x \end{aligned}$$

where the first and third integrals vanish because

$$\hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{i}} = \frac{1}{\rho} (y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}}) = 0$$

For the final integral lies at constant radius, $\rho = R$, so

$$\begin{aligned} \frac{\lambda L}{\epsilon_0} &= \int_0^{2\pi} E(\rho = R) (\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}}) R d\varphi \int_{x_0}^{x_0+L} dx \\ &= RE(R) \int_0^{2\pi} d\varphi \int_{x_0}^{x_0+L} dx \\ &= RE(R) 2\pi L \end{aligned}$$

where we take the cylinder to run from x_0 to $x_0 + L$. At a radius R , the electric field has magnitude $E(R) = \frac{\lambda}{2\pi R\epsilon_0}$, but we could have taken R to be any value and we may write $E(\rho) = \frac{\lambda}{2\pi\rho\epsilon_0}$. The electric field is therefore,

$$\begin{aligned} \mathbf{E} &= E(\rho) \hat{\boldsymbol{\rho}} \\ &= \frac{\lambda}{2\pi\epsilon_0\rho} \hat{\boldsymbol{\rho}} \end{aligned}$$

2.3 Infinite line charge with radially varying charge density

Now suppose we have an infinitely long cylinder of radius R , carrying a total charge per unit length of λ , but which increases linearly with radius. We choose cylindrical coordinates with the z -axis along the center of the conductor. Then the charge density is

$$\rho(\mathbf{x}) = k\rho\Theta(R - \rho)$$

for some constant k . If we integrate this over all ρ and φ , and length L of the wire, we must get the total charge on that segment, λL :

$$\begin{aligned} \lambda L &= \int_0^L dz \int_0^{2\pi} d\varphi \int_0^\infty [k\rho\Theta(R - \rho)] \rho d\rho \\ &= 2\pi kL \int_0^R \rho^2 d\rho \\ &= 2\pi kL \frac{R^3}{3} \end{aligned}$$

so we find the constant

$$k = \frac{3\lambda}{2\pi R^3}$$

and we write the charge density as

$$\rho(\mathbf{x}) = \frac{3\lambda\rho'}{2\pi R^3} \Theta(R - \rho')$$

Notice that the units are correct, charge per unit volume.

Now, because the conductor is long, we have cylindrical symmetry. The electric field must be in the $\hat{\boldsymbol{\rho}}$ direction and be independent of z and φ , so $\mathbf{E} = E(\rho) \hat{\boldsymbol{\rho}}$. Imagine a closed (Gaussian) surface of length L , coaxial with the cylinder and of radius $\rho_0 > R$. The electric flux across this surface may be divided into three parts,

$$\oint_{\text{closed cylinder}} \mathbf{E} \cdot \hat{\mathbf{n}} da = \int_{\text{left end}} \mathbf{E} \cdot \hat{\mathbf{n}} da + \int_{\text{round side}} \mathbf{E} \cdot \hat{\mathbf{n}} da + \int_{\text{right end}} \mathbf{E} \cdot \hat{\mathbf{n}} da$$

The outward normal on the left end is $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, so the integrand of the first integral is $\mathbf{E} \cdot \hat{\mathbf{n}}|_{left\ end} = -E(\rho) \hat{\rho} \cdot \hat{\mathbf{k}} = 0$. Similarly, the outward normal on the right end is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, so the integrand vanishes, $\mathbf{E} \cdot \hat{\mathbf{n}}|_{right\ end} = 0$. The outward normal on the circular side of the gaussian surface is $\hat{\mathbf{n}}|_{side} = \hat{\rho}$, so we have

$$\begin{aligned} \oint_{closed\ cylinder} \mathbf{E} \cdot \hat{\mathbf{n}} da &= \int_0^L dz \int_0^{2\pi} \rho_0 d\varphi (\mathbf{E} \cdot \hat{\rho})|_{side} \\ &= \int_0^L dz \int_0^{2\pi} \rho_0 d\varphi E(\rho_0) \hat{\rho} \cdot \hat{\rho} \\ &= 2\pi \rho_0 L E(\rho_0) \end{aligned}$$

Equating this to the total charge enclosed by the gaussian surface gives

$$\begin{aligned} \oint_{closed\ cylinder} \mathbf{E} \cdot \hat{\mathbf{n}} da &= \frac{1}{\epsilon_0} \int_{enclosed\ volume} \rho(\mathbf{x}) d^3x \\ 2\pi \rho_0 L E(\rho_0) &= \frac{1}{\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\rho_0 > R} \rho' d\rho' \int_0^L dz' \rho(\mathbf{x}) \end{aligned}$$

Notice that the volume integral extends only over the volume inside the surface. The integral becomes

$$\begin{aligned} 2\pi \rho_0 L E(\rho_0) &= \frac{1}{\epsilon_0} 2\pi L \int_0^{\rho_0 > R} \rho' d\rho' \frac{3\lambda \rho'}{2\pi R^3} \Theta(R - \rho') \\ &= \frac{3L\lambda}{\epsilon_0 R^3} \int_0^R (\rho')^2 d\rho' \\ &= \frac{3L\lambda}{\epsilon_0 R^3} \frac{R^3}{3} \end{aligned}$$

Therefore,

$$E(\rho_0) = \frac{\lambda}{2\pi \epsilon_0 \rho_0}$$

This holds for any $\rho_0 > R$, so

$$\mathbf{E}(\rho) = \frac{\lambda}{2\pi \epsilon_0 \rho} \hat{\rho}$$

Now let the radius of the gaussian cylinder be smaller than the radius of the cylinder, $\rho_0 < R$. Then the flux integral is exactly the same – we have all the same symmetries. However, the enclosed charge is reduced, giving

$$\begin{aligned} \oint_{closed\ cylinder} \mathbf{E} \cdot \hat{\mathbf{n}} da &= \frac{1}{\epsilon_0} \int_{enclosed\ volume} \rho(\mathbf{x}) d^3x \\ 2\pi \rho_0 L E(\rho_0) &= \frac{1}{\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\rho_0 < R} \rho' d\rho' \int_0^L dz' \rho(\mathbf{x}) \\ &= \frac{1}{\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\rho_0 < R} \rho' d\rho' \int_0^L dz' \frac{3\lambda \rho'}{2\pi R^3} \Theta(R - \rho') \end{aligned}$$

This time, the step function is 1 over the entire range of the integral, so it has no effect on the limits. We are left with

$$\begin{aligned}
2\pi\rho_0 L E(\rho_0) &= \frac{1}{\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\rho_0} \rho' d\rho' \int_0^L dz' \frac{3\lambda\rho'}{2\pi R^3} \\
&= \frac{2\pi L}{\epsilon_0} \frac{3\lambda}{2\pi R^3} \int_0^{\rho_0} (\rho')^2 d\rho' \\
&= \frac{2\pi L}{\epsilon_0} \frac{3\lambda}{2\pi R^3} \left(\frac{1}{3} \rho_0^3 \right) \\
&= \frac{L}{\epsilon_0} \frac{\lambda}{R^3} \rho_0^3
\end{aligned}$$

and therefore, since ρ_0 was arbitrary,

$$E(\rho) = \frac{\lambda\rho^2}{2\pi\epsilon_0 R^3}$$

The complete solution is then

$$\mathbf{E}(\rho) = \begin{cases} \frac{\lambda}{2\pi\epsilon_0 R^3} \hat{\rho} & \rho \geq R \\ \frac{\lambda\rho^2}{2\pi\epsilon_0 R^3} \hat{\rho} & \rho \leq R \end{cases}$$

2.4 Infinite plane

An infinite plane carries constant charge per unit area, σ . Let the plane lie at $z = 0$, and choose for our imaginary surface a cylinder that pierces the plane, with endcaps lying at distances $\pm d$ above and below the plane and parallel to it. Choose cylindrical coordinates so that $\hat{\rho} = \frac{1}{\sqrt{x^2+y^2}} (x\hat{i} + y\hat{j})$ points radially away from the center of our Gaussian cylinder. We again have three surfaces, this time with

$$\hat{\mathbf{n}} = \begin{cases} +\hat{\mathbf{k}} & \text{top end cap} \\ \hat{\rho} & \text{curved side} \\ -\hat{\mathbf{k}} & \text{bottom end cap} \end{cases}$$

By symmetry, the electric field can only be in the direction normal to the plane and its magnitude can only depend on z . Therefore, $\mathbf{E} = E(z)\hat{\mathbf{k}}$. If the cross-sectional area of the cylinder is $A = \pi R^2$, we have

$$\begin{aligned}
\frac{Q_{\text{total enclosed}}}{\epsilon_0} &= \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} d^2x \\
\frac{\sigma A}{\epsilon_0} &= \int_{\text{top cap}} (E(z)\hat{\mathbf{k}}) \cdot (\hat{\mathbf{k}}) d^2x + \int_{\text{curved side}} (E(z)\hat{\mathbf{k}}) \cdot (\hat{\rho}) d^2x + \int_{\text{bottom cap}} (E(z)\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{k}}) d^2x \\
&= \int_{\text{top cap}} (E(d)\hat{\mathbf{k}}) \cdot (\hat{\mathbf{k}}) d^2x + \int_{\text{bottom cap}} (E(-d)\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{k}}) d^2x \\
&= E(d) \int_{\text{top cap}} d^2x - E(-d) \int_{\text{bottom cap}} d^2x
\end{aligned}$$

Each of the remaining integrals gives the area A of the caps, so we have

$$\begin{aligned}
\frac{\sigma A}{\epsilon_0} &= (E(d) - E(-d)) A \\
\frac{\sigma}{\epsilon_0} &= E(d) - E(-d)
\end{aligned}$$

Now we use one final symmetry. Flipping the whole system upside down, $z \leftrightarrow -z$ cannot change the fields. Therefore, if the electric field points in the positive z -direction above the plane, it must point in the $-z$ -direction below the plane. This gives

$$E(-z) = -E(z)$$

Therefore, the right hand side is just $E(d) - E(-d) = 2E(d)$. Since d is an arbitrary distance above the plane, we have

$$E(z) = \frac{\sigma}{2\epsilon_0}$$

for all z above the plane. The electric field everywhere is therefore,

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\mathbf{k}} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{k}} & z < 0 \end{cases}$$

2.5 Electric field of a solid conductor

A conductor is defined as a material which has free charges throughout. These charges will move if there is any electric field, so a conductor in equilibrium can have no electric field inside. All excess charge therefore is on the surface of the conductor, since this is the only place where the movement of the charges is restricted – they can't fly off the conductor. We therefore expect some surface charge density, σ . To find the magnitude of the field at the surface, we look very close to the surface, close enough that the conductor looks like an infinite plane. We now use the same cylindrical Gaussian surface we used for the infinite plane, with one difference: we know the field at the bottom cap is zero since that end lies inside the conductor. Therefore, we get

$$\begin{aligned} \frac{\sigma}{\epsilon_0} &= E(d) - E(-d) \\ &= E(d) \end{aligned}$$

for d extremely close to the conductor's surface. Notice that the field must be perpendicular to the surface, because any component of electric field parallel to the surface would cause the charges to move. We conclude that the electric field at the surface of a conductor is perpendicular to the surface with magnitude $E = \frac{\sigma}{\epsilon_0}$, where σ is the surface charge density. This density may vary from place to place on the surface.

2.6 Field inside a hollow conductor: Faraday cage

If we have a closed, conducting surface, it is possible to show that the field inside is zero. The proof requires more than Gauss's law, however. The charges on the surface rearrange until there is no electric field either within the closed surface, or tangential to the surface. Charges outside the surface can have no effect on the inside. This makes it possible to shield an experiment from stray electric fields by surrounding it with a conducting surface. In practice, it may be sufficient to make the cage of a wire grid instead of a solid sheet, depending on the wavelengths to be shielded and how complete the shielding must be.

3 Exercises (required)

It is important that you be able to apply the integral form of Gauss's law to configurations involving planes, line charges or spheres.

3.1 Gauss's law: electric field of a charged sphere

Use the integral form of Gauss's law to find the electric field for all r of a ball of radius R with charge density

$$\rho(\mathbf{x}) = \begin{cases} \lambda r^2 & r < R \\ 0 & r > R \end{cases}$$

and find λ in terms of the total charge Q .

3.2 Concentric cylinder and line charge (coaxial cable)

Consider an infinite cylindrical shell of radius R_2 and charge per unit length $-\lambda$ coaxial with a conducting wire of radius $R_1 < R_2$, carrying on its surface a total charge per unit length of $+\lambda$. Use the integral form of Gauss's law to find the electric field everywhere.

3.3 Infinite plane capacitor

Consider two parallel infinite planes carrying opposite, constant, charge σ per unit area. Use superposition.