Math reminder: Taylor series

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It is possible to write an infinite series in place of many functions. If we know all the derivatives of such an *analytic function* at a point, say, x_0, then we have the following theorem,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

where we define

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$$

This is called the Taylor series for the function $f(x)$. The expression is very useful for making approximations in physics. The usefulness comes when we are interested in the behavior of $f(x)$ when x is very close to x_0.

Suppose we let

$$\varepsilon = x - x_0 \ll 1$$

Then we may rewrite

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)\,\varepsilon^n \\ &= f(x_0) + f'(x_0)\,\varepsilon + \frac{1}{2} f''(x_0)\,\varepsilon^2 + \cdots \end{aligned}$$

where $f', f'', \ldots$ are derivatives of f. If we are willing to neglect terms of order ε^2, then we may write

$$f(x) \approx f(x_0) + f'(x_0)\,\varepsilon$$

One of the most common uses of this is when we have an expression of the form

$$f(x) = (1 + x)^{\alpha}$$

where $x \ll 1$. Then we have

$$f' = \alpha (1 + x)^{\alpha - 1}$$

and we may let $x_0 = 0$ to get

$$\begin{aligned} f(x) &\approx f(x_0 = 0) + f'(x_0 = 0)\,x \\ &= f(0) + f'(0)\,x \\ &= \left[(1 + x)^{\alpha}\right]_{x=0} + \left[\alpha (1 + x)^{\alpha - 1}\right]_{x=0} x \\ &= 1 + \alpha x \end{aligned}$$

Notice that α may have either sign. For example, suppose we want to approximate

$$\frac{1}{(r^2 + d^2)^{3/2}}$$

at large distances when $r \gg d$. Rewrite this as

$$\begin{aligned}\frac{1}{\left(r^2+d^2\right)^{3/2}} &= \frac{1}{r^3\left(1+\left(\frac{d}{r}\right)^2\right)^{3/2}} \\ &= \frac{1}{r^3}\left(1+\left(\frac{d}{r}\right)^2\right)^{-3/2}\end{aligned}$$

where $\frac{d}{r} \ll 1$ and the exponent is $-\frac{3}{2}$. We immediately have

$$\frac{1}{\left(r^2+d^2\right)^{3/2}} \approx \frac{1}{r^3}\left(1-\frac{3d^2}{2r^2}\right)$$

This expression is accurate as long as we can neglect terms of order $\frac{d^4}{r^4}$. To see this, carry the Taylor series further by computing another derivative:

$$\begin{aligned}f &= (1+x)^\alpha \\ f' &= \alpha(1+x)^{\alpha-1} \\ f'' &= \alpha(\alpha-1)(1+x)^{\alpha-2}\end{aligned}$$

and write the first three terms,

$$\begin{aligned}f(x) &\approx f(x_0)+f'(x_0)\varepsilon+\frac{1}{2}f''(x_0)\varepsilon^2 \\ &= f(0)+f'(0)\varepsilon+\frac{1}{2}f''(0)\varepsilon^2\end{aligned}$$

where $\varepsilon = \frac{d^2}{r^2} \ll 1$. Then

$$\begin{aligned}\frac{1}{\left(r^2+d^2\right)^{3/2}} &\approx \frac{1}{r^3}\left(1-\frac{3d^2}{2r^2}+\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\left(\frac{d}{r}\right)^4\right) \\ &= \frac{1}{r^3}\left(1-\frac{3d^2}{2r^2}+\frac{15}{8}\frac{d^4}{r^4}\right)\end{aligned}$$

The method not only gives us a simpler way to write the express, but also tells us when the approximation is a good one.

A few more useful Taylor series are, for $\alpha x \ll 1$,

$$\begin{aligned}e^{\alpha x} &\approx 1+\alpha x+\frac{1}{2}\alpha^2x^2 \\ \sin\alpha x &\approx \alpha x-\frac{1}{3!}\alpha^3x^3 \\ \cos\alpha x &\approx 1-\frac{1}{2!}\alpha^2x^2\end{aligned}$$

As one final example, consider the expansion of

$$f(x) = \frac{1}{\sqrt{\rho^2+a^2+2a\rho\cos\alpha}}$$

in the limit $\rho \gg a$. Then we may pull out a factor of ρ,

$$f(x) = \frac{1}{\rho\sqrt{1+\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha}}$$

Notice that the entire quantity $\frac{a^2}{\rho^2} + \frac{2a}{\rho}\cos\alpha$ is much less than one. Think of this as a single small thing. Since we know that $(1+x)^\alpha \approx 1 + \alpha x$ for small x, $f(x)$ becomes

$$\begin{aligned} f(x) &= \frac{1}{\rho\sqrt{1+\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha}} \\ &= \frac{1}{\rho}\left(1+\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)\right)^{-1/2} \\ &\approx \frac{1}{\rho}\left(1-\frac{1}{2}\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)\right) \end{aligned}$$

Since we have only kept the first order term in our expansion, we may only keep up to first order in our small parameter, $\frac{a}{\rho}$, so to this order we write

$$f(x) \approx \frac{1}{\rho}\left(1-\frac{a}{\rho}\cos\alpha\right)$$

If we wish to keep the result to second order, we must expand our Taylor series to this order as well:

$$(1+x)^\alpha \approx 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2$$

Then we approximate $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1}{\rho}\left(1+\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)\right)^{-1/2} \\ &\approx \frac{1}{\rho}\left(1-\frac{1}{2}\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)^2\right) \\ &= \frac{1}{\rho}\left(1-\frac{1}{2}\frac{a^2}{\rho^2}-\frac{a}{\rho}\cos\alpha+\frac{3}{8}\left(\left(\frac{a^2}{\rho^2}\right)^2+\frac{4a^3}{\rho^3}\cos\alpha+\frac{4a^2}{\rho^2}\cos^2\alpha\right)\right) \end{aligned}$$

Now drop everything of order $\frac{a^3}{\rho^3}$ or smaller, so to second order the result is

$$\begin{aligned} f(x) &\approx \frac{1}{\rho}\left(1-\frac{1}{2}\frac{a^2}{\rho^2}-\frac{a}{\rho}\cos\alpha+\frac{3a^2}{2\rho^2}\cos^2\alpha\right) \\ &= \frac{1}{\rho}\left(1-\frac{a}{\rho}\cos\alpha-\frac{1}{2}\frac{a^2}{\rho^2}\left(1+3\cos^2\alpha\right)\right) \end{aligned}$$

At higher order it is important to keep both terms of $\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)$ because the cross terms in $\left(\frac{a^2}{\rho^2}+\frac{2a}{\rho}\cos\alpha\right)^n$ contribute at lower orders.