Electric force and electric field

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The basic problem of mechanics or field theory is to predict the motions of particles or the time evolution of fields. For particles, this depends on Newton's second law, and this requires a knowledge of forces. Electric charges, which can be either positive or negative, exert forces on one another, so we begin by characterizing the direction and magnitude of those forces.

1 Coulomb's law and the electric field: point charges

1.1 Coulomb's law: the force between two charges

For a single point charge Q in the neighborhood of a second charge, q, the force *on* Q *due to* q is measured (by determining the acceleration) to be

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of Q from q and r is the separation of the charges. This is Coulomb's law, and it works for any combination of signs of q and Q. The constant

$$\epsilon_0 = 8.85 \times 10^{-12}\frac{C^2}{N \cdot m^2}$$

is the *permittivity of free space*.

Notice that we may write this equation in any coordinates. For example, suppose Q lies at vector position $\mathbf{x}$ and q lies at $\mathbf{x}_0$. Then

$$\begin{aligned} r &= |\mathbf{x} - \mathbf{x}_0| \\ \hat{\mathbf{r}} &= \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \end{aligned}$$

where $\mathbf{x}$ and $\mathbf{x}_0$ may be expressed in any coordinate system:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3}$$

This is *Coulomb's law.*

1.2 Linear superposition

Perhaps the most important property of electromagnetism is that it is *linear* in the fields, and the first place we see this is in the additivity of the force due to multiple charges. We again consider the force on Q at $\mathbf{x}$, but this time in the neighborhood of n different charges, $q_1, q_2, \ldots, q_n$ at positions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively. Then the force on Q is just the *sum* of the forces from the individual particles,

$$\mathbf{F} = \sum_{i=1}^{n}\frac{q_i Q}{4\pi\epsilon_0}\frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|^3} = \sum_{i=1}^{n}\frac{q_i Q}{4\pi\epsilon_0 r_i^2}\hat{\mathbf{r}}_i$$

1.3 The electric field

A key point here is that the linearity applies to Q, so that if we have various charges Q moving in the presence of many fixed charges q_i, we can define a quantity that gives the force for any Q in a way that depends only of the fixed charges. We define the electric field, $\mathbf{E}$, produced by the charges q_i at any location $\mathbf{x}$, as the force per unit charge experienced by an additional charge Q at that location, as

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= \frac{\mathbf{F}(\mathbf{x})}{Q} \\ &= \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3}\end{aligned}$$

In order to avoid any argument that Q alters the configuration of "fixed"charges, we may take the limit of small test charges,

$$\mathbf{E}(\mathbf{x}) = \lim_{Q\to 0} \frac{\mathbf{F}(\mathbf{x})}{Q}$$

but it the charges are truly fixed this doesn't make a difference.

2 Continuous distributions

At the microscopic level, charges are discrete, in units of $\pm e$, the charge of the electron or positron. However, over macroscopic distances there will be extremely large numbers of electrons and we may treat the charge distribution as continuous. For example, in one cubic micron of hydrogen gas at standard temperature and pressure, there are about 2.7×10^8 electrons. A difference of one electron therefore alters the charge by only parts per billion. For macroscopic volumes, the effect is much smaller. A 1 amp current carries 1 Coulomb per second, and a single electron contributes only $1.6 \times 10^{-19}C$ more or less to this. The approximation of continuous charge distributions is a very good one!

To formulate the equivalent of Coulomb's law for a continuous distribution of charge, we introduce the *charge density*, $\rho(\mathbf{x})$. We can define this as the total charge per unit volume for a volume centered at the position $\mathbf{x}$, in the limit as the volume becomes "small". What we mean by small is any size much smaller than the size over which ρ is changing, but large enough that the volume still contains many charges.

$$\rho(\mathbf{x}) = \lim_{\Delta V \to small} \frac{Q\, in\, \Delta V\, about\, \mathbf{x}}{\Delta V}$$

Then, in a volume ΔV, the total charge is $\rho(\mathbf{x})\Delta V$. The continuous idealization is good enough that we may write this infinitesimally,

$$dQ = \rho(\mathbf{x})\, d^3x$$

With this preparation, we reconsider the electric field, replacing q_i with ρd^3x. Let q_i be replaced by $q_i = q(\mathbf{x}_i) = \rho(\mathbf{x}_i)\Delta V$ and take the infinitesimal limit

$$\mathbf{E}(\mathbf{x}) = \lim_{\Delta V\to 0} \sum_{i=1}^{n} \frac{\rho(\mathbf{x}_i)\Delta V_i}{4\pi\epsilon_0} \frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3}$$

In this limit, the sum becomes an integral, as the charge positions $\mathbf{x}_i \to \mathbf{x}'$ vary smoothly over all space,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

If the charges in question are restricted to a surface S or to a curve C, we can define the charge per unit area, $\sigma(\mathbf{x}')$, or the charge per unit length, $\lambda(\mathbf{x}')$, and the corresponding electric fields,

2.1 Example 1: Electric field of an infinite line charge

Suppose we have a infinitely long, straight wire with a constant charge density λ. What is the electric field at all points around the wire?

Because this system has cylindrical symmetry, we may simplify the problem a great deal. Choosing cylindrical coordinates to match the symmetry, we let the wire lie along the z-axis. Then there can be no φ- or z-dependence of the electric field, $\mathbf{E}(\mathbf{x}) = \mathbf{E}(\rho)$. Moreover, $\mathbf{E}(\rho)$ must be radial since there can be no directional preference for left or right along z or for clockwise or counterclockwise in the φ-direction, and we have $\mathbf{E}(\mathbf{x}) = E(\rho)\,\hat{\boldsymbol{\rho}}$. This reduces our problem to a 1-dimensional integral, and simplifies our general formula,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\limits_C \lambda(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}dx'$$

If we let the point $\mathbf{x}$ at which we wish to know the field lie at a the point $(\rho, 0, 0)$ then $\mathbf{E}(\mathbf{x})$ becomes

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int\limits_{-\infty}^{\infty}\lambda\frac{\rho\hat{\boldsymbol{\rho}}-\hat{\mathbf{k}}z'}{\left|\rho\hat{\boldsymbol{\rho}}-\hat{\mathbf{k}}z'\right|^3}dz' \\
&= \frac{\lambda}{4\pi\epsilon_0}\int\limits_{-\infty}^{\infty}\frac{\rho\hat{\boldsymbol{\rho}}-\hat{\mathbf{k}}z'}{\left(\rho^2+(z')^2\right)^{3/2}}dz'
\end{aligned}$$

Notice that the integral is over the source charge distribution along the z-axis.

By our initial reasoning, the $\hat{\mathbf{k}}$ term must give zero. This is easy to see. The $\hat{\mathbf{k}}$ term is proportional to the integral

$$I = \int\limits_{-\infty}^{\infty}\frac{z'}{\left(\rho^2+(z')^2\right)^{3/2}}dz'$$

Changing to a new coordinate, $z'' = -z'$, gives

$$\begin{aligned}
I &= \int\limits_{\infty}^{-\infty}\frac{-z''}{\left(\rho^2+(-z')^2\right)^{3/2}}(-dz'') \\
&= -\int\limits_{-\infty}^{\infty}\frac{z''}{\left(\rho^2+(z'')^2\right)^{3/2}}(dz') \\
&= -I
\end{aligned}$$

showing that the integral vanishes. This type of proof is called a parity argument.

We are left with

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{4\pi\epsilon_0}\rho\hat{\boldsymbol{\rho}}\int\limits_{-\infty}^{\infty}\frac{1}{\left(\rho^2+(z')^2\right)^{3/2}}dz'$$

Let $z' = \rho\sinh\zeta, dz' = \rho\cosh\zeta d\zeta$. Then

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{4\pi\epsilon_0}\rho\hat{\boldsymbol{\rho}}\int\limits_{-\infty}^{\infty}\frac{\rho\cosh\zeta d\zeta}{\left(\rho^2+\rho^2\sinh^2\zeta\right)^{3/2}}dz'$$

$$
\begin{aligned}
&= \frac{\lambda}{4\pi\epsilon_0}\frac{\rho^2}{\rho^3}\hat{\boldsymbol{\rho}}\int_{-\infty}^{\infty}\frac{\cosh\zeta d\zeta}{\left(1+\sinh^2\zeta\right)^{3/2}}dz' \\
&= \frac{\lambda}{4\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}\int_{-\infty}^{\infty}\frac{d\zeta}{\cosh^2\zeta}dz' \\
&= \frac{\lambda}{4\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}\tanh\zeta|_{-\infty}^{\infty} \\
&= \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}
\end{aligned}
$$

$\mathbf{E}(\mathbf{x})$ is directed radially away from the wire and falls off as $\frac{1}{\rho}$.

2.2 Example 2: Electric field of a finite line charge

If instead we have a wire of finite length, $2L$, the symmetry argument only still holds at the center of the wire. Taking the center at $z=0$, the electric field is still independent of φ, but now depends on both ρ and z. Our integral must therefore be written with $\mathbf{x}=\rho\hat{\boldsymbol{\rho}}+\hat{\mathbf{k}}z$, though we may still take $\varphi=0$:

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\lambda\frac{\left(\rho\hat{\boldsymbol{\rho}}+\hat{\mathbf{k}}z\right)-\hat{\mathbf{k}}z'}{\left|\rho\hat{\boldsymbol{\rho}}+\hat{\mathbf{k}}z-\hat{\mathbf{k}}z'\right|^3}dz' \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\rho\hat{\boldsymbol{\rho}}-\hat{\mathbf{k}}\left(z'-z\right)}{\left(\rho^2+\left(z'-z\right)^2\right)^{3/2}}dz'
\end{aligned}
$$

This time we take $(z'-z)=\rho\sinh\zeta$, and the resulting integral is the same:

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{\lambda}{4\pi\epsilon_0}\int_{-L+z}^{L-z}\frac{\rho\hat{\boldsymbol{\rho}}-\hat{\mathbf{k}}\rho\sinh\zeta}{\left(\rho^2+\rho^2\sinh^2\zeta\right)^{3/2}}\rho\cosh\zeta d\zeta \\
&= \frac{\lambda}{4\pi\epsilon_0\rho}\int_{-L+z}^{L-z}\frac{\hat{\boldsymbol{\rho}}-\hat{\mathbf{k}}\sinh\zeta}{\cosh^3\zeta}\cosh\zeta d\zeta
\end{aligned}
$$

Notice that because the limits are asymmetrical in z, the parity argument no longer makes the $\hat{\mathbf{k}}$ integral vanish. Instead, we have

$$
\begin{aligned}
\int_{-L+z}^{L-z}\frac{\sinh\zeta}{\cosh^2\zeta}d\zeta &= -\left.\frac{1}{\cosh\zeta}\right|_{z'=-L}^{z'=L} \\
&= -\left.\frac{1}{\sqrt{1+\left(\frac{z'-z}{\rho}\right)^2}}\right|_{z'=-L}^{z'=L} \\
&= \frac{\rho}{\sqrt{\rho^2+\left(z+L\right)^2}}-\frac{\rho}{\sqrt{\rho^2+\left(z-L\right)^2}}
\end{aligned}
$$

while only the limits of the $\hat{\boldsymbol{\rho}}$ integral change,

$$\begin{aligned}
\int_{-L+z}^{L-z} \frac{1}{\cosh^2 \zeta} d\zeta &= \tanh \zeta \Big|_{z'=-L}^{z'=L} \\
&= \left. \frac{\left(\frac{z'-z}{\rho}\right)}{\sqrt{1+\left(\frac{z'-z}{\rho}\right)^2}} \right|_{z'=-L}^{z'=L} \\
&= \frac{L+z}{\sqrt{\rho^2+(L+z)^2}} - \frac{z-L}{\sqrt{\rho^2+(L-z)^2}}
\end{aligned}$$

Combining these, the electric field is:

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{\lambda}{4\pi\epsilon_0\rho} \int_{-L+z}^{L-z} \frac{\hat{\boldsymbol{\rho}} - \hat{\mathbf{k}} \sinh\zeta}{\cosh^3\zeta} \cosh\zeta d\zeta \\
&= \frac{\lambda}{4\pi\epsilon_0\rho} \left[\hat{\boldsymbol{\rho}} \left(\frac{L+z}{\sqrt{\rho^2+(L+z)^2}} - \frac{z-L}{\sqrt{\rho^2+(L-z)^2}} \right) - \hat{\mathbf{k}} \left(\frac{\rho}{\sqrt{\rho^2+(z+L)^2}} - \frac{\rho}{\sqrt{\rho^2+(z-L)^2}} \right) \right]
\end{aligned}$$

In the limit as $L \to \infty$ we recover the previous result:

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \lim_{L\to\infty} \frac{\lambda}{4\pi\epsilon_0\rho} \left[\hat{\boldsymbol{\rho}} \left(\frac{L+z}{\sqrt{\rho^2+(L+z)^2}} - \frac{z-L}{\sqrt{\rho^2+(L-z)^2}} \right) - \hat{\mathbf{k}} \left(\frac{\rho}{\sqrt{\rho^2+(z+L)^2}} - \frac{\rho}{\sqrt{\rho^2+(z-L)^2}} \right) \right] \\
&= \lim_{L\to\infty} \frac{\lambda}{4\pi\epsilon_0\rho} \left[\hat{\boldsymbol{\rho}} \left(\frac{L+z}{L} - \frac{z-L}{L} \right) - \hat{\mathbf{k}} \left(\frac{\rho}{L} - \frac{\rho}{L} \right) \right] \\
&= \frac{\lambda}{4\pi\epsilon_0\rho} \left[2\hat{\boldsymbol{\rho}} \right]
\end{aligned}$$

There is another interesting limit. If the point of interest lies at a distance $\sqrt{\rho^2+z^2} \gg L$ then the wire will start to look like a point charge with $Q = 2\lambda L$. To check that this holds, we need to expand each denominator in a Taylor series,

$$\begin{aligned}
\frac{1}{\sqrt{\rho^2+(L+z)^2}} &= \frac{1}{\sqrt{\rho^2+z^2}} \left(1 + \left(\frac{2zL}{\rho^2+z^2} + \frac{L^2}{\rho^2+z^2} \right) \right)^{-1/2} \\
&\approx \frac{1}{\sqrt{\rho^2+z^2}} \left(1 - \frac{zL}{\rho^2+z^2} - \frac{1}{2}\frac{L^2}{\rho^2+z^2} \right) \\
\frac{1}{\sqrt{\rho^2+(L-z)^2}} &\approx \frac{1}{\sqrt{\rho^2+z^2}} \left(1 + \frac{zL}{\rho^2+z^2} - \frac{1}{2}\frac{L^2}{\rho^2+z^2} \right)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{\lambda}{4\pi\epsilon_0\rho} \left[\hat{\boldsymbol{\rho}} \left(\frac{L+z}{\sqrt{\rho^2+(L+z)^2}} - \frac{z-L}{\sqrt{\rho^2+(L-z)^2}} \right) - \hat{\mathbf{k}} \left(\frac{\rho}{\sqrt{\rho^2+(z+L)^2}} - \frac{\rho}{\sqrt{\rho^2+(z-L)^2}} \right) \right] \\
&= \frac{\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}} \hat{\boldsymbol{\rho}} \left((L+z) \left(1 - \frac{zL}{\rho^2+z^2} - \frac{1}{2}\frac{L^2}{\rho^2+z^2} \right) - (z-L) \left(1 + \frac{zL}{\rho^2+z^2} - \frac{1}{2}\frac{L^2}{\rho^2+z^2} \right) \right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}}\hat{\mathbf{k}}\left(\rho\left(1-\frac{zL}{\rho^2+z^2}-\frac{1}{2}\frac{L^2}{\rho^2+z^2}\right)-\rho\left(1+\frac{zL}{\rho^2+z^2}-\frac{1}{2}\frac{L^2}{\rho^2+z^2}\right)\right) \\
= & \frac{\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}}\hat{\boldsymbol{\rho}}\left(L+z-\frac{zL(L+z)}{\rho^2+z^2}-\frac{1}{2}\frac{(L+z)L^2}{\rho^2+z^2}-(z-L)-\frac{zL(z-L)}{\rho^2+z^2}+\frac{1}{2}\frac{(z-L)L^2}{\rho^2+z^2}\right) \\
& -\frac{\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}}\hat{\mathbf{k}}\left(\rho-\frac{\rho zL}{\rho^2+z^2}-\frac{1}{2}\frac{\rho L^2}{\rho^2+z^2}-\rho-\frac{\rho zL}{\rho^2+z^2}+\frac{1}{2}\frac{\rho L^2}{\rho^2+z^2}\right) \\
= & \frac{\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}}\hat{\boldsymbol{\rho}}\left(2L-\frac{2zL(L+z)}{\rho^2+z^2}-L\frac{L^2}{\rho^2+z^2}\right)+\frac{\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}}\hat{\mathbf{k}}\left(\frac{2\rho zL}{\rho^2+z^2}\right) \\
= & \frac{2L\lambda}{4\pi\epsilon_0\rho\sqrt{\rho^2+z^2}}\left(\hat{\boldsymbol{\rho}}\left(1-\frac{z^2}{\rho^2+z^2}\right)+\hat{\mathbf{k}}\left(\frac{\rho z}{\rho^2+z^2}\right)\right) \\
= & \frac{2L\lambda}{4\pi\epsilon_0\rho\left(\rho^2+z^2\right)^{3/2}}\left(\hat{\boldsymbol{\rho}}\rho^2+\rho z\hat{\mathbf{k}}\right) \\
= & \frac{2L\lambda}{4\pi\epsilon_0\left(\rho^2+z^2\right)^{3/2}}\left(\rho\hat{\boldsymbol{\rho}}+z\hat{\mathbf{k}}\right)
\end{aligned}
$$

Since we may write $\mathbf{r}=\rho\hat{\boldsymbol{\rho}}+z\hat{\mathbf{k}}$ and $Q=2L\lambda$, this is exactly

$$
\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}
$$

as we predicted.

2.3 Example 3: Infinite, parallel wires

As a third example, we compute the electric field due to a pair of infinite, charged, parallel wires. One, with charge per unit length λ, lies parallel to the z-axis passing through $(x,y)=\left(\frac{d}{2},0\right)$, while the second with charge per unit length $-\lambda$, is parallel but passing through $(x,y)=\left(-\frac{d}{2},0\right)$. The electric field from the first is found above to be

$$
\mathbf{E}_1(\mathbf{x}) = \frac{\lambda}{2\pi\epsilon_0\rho_1}\hat{\boldsymbol{\rho}}_1
$$

but we must offset the radial distance and direction. Changing to Cartesian coordinates, we may write

$$
\mathbf{E}_1(\mathbf{x}) = \frac{\lambda\left(\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}\right)}{2\pi\epsilon_0\left(\left(x-\frac{d}{2}\right)^2+y^2\right)}
$$

where the radial distance from the wire is $\rho_1=\sqrt{\left(x-\frac{d}{2}\right)^2+y^2}$, and the direction is $\boldsymbol{\rho}_1=+(x-d)\hat{\mathbf{i}}+y\hat{\mathbf{j}}$. The electric field from the second wire is similar, $\lambda\to-\lambda$ and replacing $x-\frac{d}{2}$ with $x+\frac{d}{2}$,

$$
\mathbf{E}_2(\mathbf{x}) = -\frac{\lambda\left(\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}\right)}{2\pi\epsilon_0\left(\left(x+\frac{d}{2}\right)^2+y^2\right)}
$$

and, using *superposition*, the total electric field is simply the sum of these

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \mathbf{E}_1(\mathbf{x})+\mathbf{E}_2(\mathbf{x}) \\
&= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\left(x-\frac{d}{2}\right)^2+y^2}-\frac{\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\left(x+\frac{d}{2}\right)^2+y^2}\right)
\end{aligned}
$$

Superposition makes this complicated field easy to write.

This takes a simpler form if we approximate it for large distances from the wire. Then with $\rho_1, \rho_2 \gg d$, we set $\rho = \sqrt{x^2+y^2} \gg d$, and expand

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\left(x-\frac{d}{2}\right)^2+y^2} - \frac{\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\left(x+\frac{d}{2}\right)^2+y^2}\right) \\
&= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\rho^2-xd} - \frac{\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\rho^2+xd}\right) \\
&= \frac{\lambda}{2\pi\epsilon_0\rho^2}\left(\frac{\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{1-\frac{xd}{\rho^2}} - \frac{\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{1+\frac{xd}{\rho^2}}\right)
\end{aligned}$$

Then, expanding the denominator in a Taylor series we have

$$\left(1 \pm \frac{xd}{\rho^2}\right)^{-1} \approx \left(1 \mp \frac{xd}{\rho^2}\right)$$

so

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{\lambda}{2\pi\epsilon_0\rho^2}\left(\frac{\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{1-\frac{xd}{\rho^2}} - \frac{\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{1+\frac{xd}{\rho^2}}\right) \\
&\approx \frac{\lambda}{2\pi\epsilon_0\rho^2}\left(\left(\left(x-\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}\right)\left(1+\frac{xd}{\rho^2}\right) - \left(\left(x+\frac{d}{2}\right)\hat{\mathbf{i}}+y\hat{\mathbf{j}}\right)\left(1-\frac{xd}{\rho^2}\right)\right) \\
&= \frac{\lambda}{2\pi\epsilon_0\rho^2}\left(\left(\left(x-\frac{d}{2}+\frac{x^2d}{\rho^2}-\frac{xd^2}{2\rho^2}\right)-\left(x+\frac{d}{2}-\frac{x^2d}{\rho^2}-\frac{xd^2}{2\rho^2}\right)\right)\hat{\mathbf{i}} + \left(y\left(1+\frac{xd}{\rho^2}\right)-y\left(1-\frac{xd}{\rho^2}\right)\right)\hat{\mathbf{j}}\right) \\
&= \frac{\lambda}{2\pi\epsilon_0\rho^2}\left(\left(-d+\frac{2x^2d}{\rho^2}\right)\hat{\mathbf{i}}+\frac{2xyd}{\rho^2}\hat{\mathbf{j}}\right)
\end{aligned}$$

Combining terms

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{\lambda d}{2\pi\epsilon_0\rho^2}\left(\left(-1+\frac{2x^2}{\rho^2}\right)\hat{\mathbf{i}}+\frac{2xy}{\rho^2}\hat{\mathbf{j}}\right) \\
&= \frac{\lambda d}{2\pi\epsilon_0\rho^2}\left(\left(-1+2\cos^2\varphi\right)\hat{\mathbf{i}}+2\sin\varphi\cos\varphi\hat{\mathbf{j}}\right) \\
&= \frac{\lambda d}{2\pi\epsilon_0\rho^2}\left(\hat{\mathbf{i}}\cos 2\varphi+\hat{\mathbf{j}}\sin 2\varphi\right)
\end{aligned}$$

Sketch some of the flux lines and you will see loops from one wire to the other.

3 Exercises (required)

3.1 Exercise 2.1: Force between two electrons

An electron has a charge of $1.6 \times 10^{-19}\, Coulomb$. Find the force between two electrons at a distance of $10^{-10}\, meter$. Find the initial acceleration resulting from the electrostatic repulsion.

3.2 Exercise 2.2: Electric field from point particles

Find the electric field produced by a system of three charges: $2q$ at the origin, $3q$ at position $\mathbf{x} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, and $-q$ at $\mathbf{x} = 5\hat{\mathbf{k}}$.

3.3 Exercise 2.3: Infinite plane

Find, by direct integration, the electric field at a point a distance z above and infinite plane of constant charge density σ, located at $z = 0$.

3.4 Exercise 2.4: Disk

A thin circular disk lies in the xy plane at the origin. It has radius R and carries a total charge Q.

1. Find the electric field at any distance z along the z-axis.
2. Show that your answer gives the expected result for $z \gg R$.