Additional Mathematical Tools: Summary

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Besides vector calculus, we will need some additional mathematical tools. Here I cite some essential results about cylindrical and spherical coordinates, rotations, the Dirac delta function, and the Helmholz theorem. I give a separate set of notes with more detailed derivations for those who wish to see them.

1 Cylindrical coordinates

Cylindrical coordinates are the radius, ρ, and angle, φ, from the x-axix in the xy plane, together with the usual Cartesian z coordinate. The transformation is given by

$$\begin{aligned} x &= \rho \cos\varphi \\ y &= \rho \sin\varphi \\ z &= z \end{aligned}$$

with inverse relations

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}$$

Unit vectors in the three (orthogonal) coordinate directions are

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\varphi \hat{\mathbf{i}} + \sin\varphi \hat{\mathbf{j}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi \hat{\mathbf{i}} + \cos\varphi \hat{\mathbf{j}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{k}} \end{aligned}$$

Inverting these relations we also have

$$\begin{aligned} \hat{\mathbf{i}} &= \cos\varphi \hat{\boldsymbol{\rho}} - \sin\varphi \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{j}} &= \sin\varphi \hat{\boldsymbol{\rho}} + \cos\varphi \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{k}} &= \hat{\mathbf{z}} \end{aligned}$$

We can find the length, dl, of infinitesimal displacements

$$dl^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

We also need derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\varphi \frac{\partial}{\partial \rho} - \sin\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial y} &= \sin\varphi \frac{\partial}{\partial \rho} + \cos\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} \end{aligned}$$

From these we find the gradient, divergence, curl and Laplacian in cylindrical coordinates to be:

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \\
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho v_\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}v_\varphi + \frac{\partial}{\partial z}v_z \\
\boldsymbol{\nabla}\times\mathbf{v} &= \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\partial_\varphi v_z - \partial_z v_\varphi\right) + \hat{\boldsymbol{\varphi}}\left(\partial_z v_\rho - \partial_\rho v_z\right) + \hat{\mathbf{k}}\left(\frac{1}{\rho}\partial_\rho\left(\rho v_\varphi\right) - \frac{1}{\rho}\partial_\varphi v_\rho\right) \\
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

2 Spherical coordinates

Spherical coordinates include the radial distance, r, the angle down from the z-axis to the vector, θ, and the angle, φ, in the xy-plane measured from the x-axis. The transformation and its inverse are given by

$$\begin{aligned}
x &= r\sin\theta\cos\varphi \\
y &= r\sin\theta\sin\varphi \\
z &= r\cos\theta
\end{aligned}$$

and

$$\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \\
&= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\
\varphi &= \tan^{-1}\frac{y}{x}
\end{aligned}$$

Unit vectors in the three (orthogonal) coordinate directions are

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \sin\theta\cos\varphi\hat{\mathbf{i}} + \sin\theta\sin\varphi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\hat{\mathbf{i}} + \cos\theta\sin\varphi\hat{\mathbf{j}} - \hat{\mathbf{k}}\sin\theta \\
\hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\mathbf{i}} + \cos\varphi\hat{\mathbf{j}}
\end{aligned}$$

Inverting these relations we also have

$$\begin{aligned}
\hat{\mathbf{i}} &= \hat{\boldsymbol{r}}\sin\theta\cos\varphi + \hat{\boldsymbol{\theta}}\cos\theta\cos\varphi - \hat{\boldsymbol{\varphi}}\sin\varphi \\
\hat{\mathbf{j}} &= \hat{\boldsymbol{r}}\sin\theta\sin\varphi + \hat{\boldsymbol{\theta}}\cos\theta\sin\varphi + \hat{\boldsymbol{\varphi}}\cos\varphi \\
\hat{\mathbf{k}} &= \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta
\end{aligned}$$

We can find the length, dl, of infinitesimal displacements

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$$

The gradient, divergence, curl and Laplacian in spherical coordinates are:

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \\
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}v_\varphi
\end{aligned}$$

$$\nabla \times \mathbf{v} = \left(\hat{\boldsymbol{r}} \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial\theta} (\sin\theta v_\varphi) - \frac{\partial}{\partial\varphi} v_\theta \right) + \hat{\boldsymbol{\theta}} \left(\frac{1}{r\sin\theta} \frac{\partial}{\partial\varphi} (v_r) - \frac{1}{r} \frac{\partial}{\partial r} (r v_\varphi) \right) + \hat{\boldsymbol{\varphi}} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) - \frac{1}{r} \frac{\partial}{\partial\theta} v_r \right) \right)$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial\varphi^2}$$

3 Rotations

We wish to describe a rotation through an angle θ around and axis in $\hat{\mathbf{n}}$ direction. Positive θ corresponds to a counterclockwise rotation when looking down at the tip of $\hat{\mathbf{n}}$. Let the rotation, O, act on an arbitrary vector $\mathbf{v}$ to give the rotated vector $\mathbf{v}'$. To describe what happens, we decompose $\mathbf{v}$ into parts parallel and perpendicular to $\hat{\mathbf{n}}$. The parallel part has magnitude $(\hat{\mathbf{n}} \cdot \mathbf{v})$ and direction $\hat{\mathbf{n}}$ so we define

$$\mathbf{v}_\parallel = (\hat{\mathbf{n}} \cdot \mathbf{v})\, \hat{\mathbf{n}}$$

The part of $\mathbf{v}$ perpendicular to $\hat{\mathbf{n}}$ has *magnitude* $|\hat{\mathbf{n}} \times \mathbf{v}| = v \sin\alpha$ where α is the angle between the two vectors. However, the vector $\hat{\mathbf{n}} \times \mathbf{v}$ is perpendicular to *both* $\hat{\mathbf{n}}$ and $\mathbf{v}$. To write the perpendicular part of $\mathbf{v}$ as a vector $\mathbf{v}_\perp$, we need a vector with this magnitude which is perpendicular to $\hat{\mathbf{n}}$ but lies in the $\hat{\mathbf{n}}\mathbf{v}$-plane. We can get it by taking another cross product with $\hat{\mathbf{n}} \times \mathbf{v}$. Using the BAC-CAB rule,

$$\begin{aligned} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v}) &= \hat{\mathbf{n}}\, (\hat{\mathbf{n}} \cdot \mathbf{v}) - \mathbf{v}\, (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) \\ &= -\left[\mathbf{v} - \hat{\mathbf{n}}\, (\hat{\mathbf{n}} \cdot \mathbf{v}) \right] \end{aligned}$$

so we can write

$$\mathbf{v} = \hat{\mathbf{n}}\, (\hat{\mathbf{n}} \cdot \mathbf{v}) - \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v})$$

The two vectors on the right are perpendicular to one another and lie in the $\hat{\mathbf{n}}\mathbf{v}$-plane. We define the perpendicular component of $\mathbf{v}$ to be

$$\mathbf{v}_\perp = -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v})$$

Now we can easily write the rotated vector, $\mathbf{v}'$, because the parallel part is unchanged while the part lying in the $\hat{\mathbf{n}}\mathbf{v}$-plane is a simple 2-dimensional rotation. We therefore have

$$\begin{aligned} \mathbf{v}'_\parallel &= \mathbf{v}_\parallel \\ \mathbf{v}'_\perp &= \mathbf{v}_\perp \cos\theta + (\hat{\mathbf{n}} \times \mathbf{v}) \sin\theta \end{aligned}$$

Adding these and simplifying,

$$\mathbf{v}' = (\hat{\mathbf{n}} \cdot \mathbf{v})\, \hat{\mathbf{n}}\, (1 - \cos\theta) + \mathbf{v} \cos\theta + (\hat{\mathbf{n}} \times \mathbf{v}) \sin\theta$$

This gives the form of any vector $\mathbf{v}$ when rotated by an angle θ around the direction $\hat{\mathbf{n}}$.

To find the rotation matrix, O, we can look at the effect of this rotation on each of the three Cartesian unit vectors, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. This results in O as a matrix, $[O]$:

$$[O]_{ij} = \begin{bmatrix} n_x^2 (1 - \cos\theta) + \cos\theta & n_x n_y (1 - \cos\theta) - n_z \sin\theta & n_x n_z (1 - \cos\theta) + n_y \sin\theta \\ n_y n_x (1 - \cos\theta) + n_z \sin\theta & n_y^2 (1 - \cos\theta) + \cos\theta & n_y n_z (1 - \cos\theta) - n_x \sin\theta \\ n_z n_x (1 - \cos\theta) - n_y \sin\theta & n_y n_z (1 - \cos\theta) + n_x \sin\theta & n_z^2 (1 - \cos\theta) + \cos\theta \end{bmatrix}$$

You may use either the vector formula or the matrix to find $\mathbf{v}$.

4 Dirac delta function

Let $f(x)$ be a *test function* – any smooth function which vanishes outside a certain compact region. Then the Dirac delta function $\delta(x - x_0)$ has the defining properties,

$$\begin{aligned} \delta(x - x_0) &= 0 \qquad for\ x \neq x_0 \\ \int_{-\infty}^{\infty} f(x)\,\delta(x - x_0)\,dx &= f(x_0) \quad for\ any\ test\ function\ f(x) \end{aligned}$$

It is useful to think of an analogy with the Kronecker delta,

$$\begin{aligned} v_i &= \sum_{i-1}^{3} \delta_{ij} v_j \\ f(x_0) &= \int_{-\infty}^{\infty} f(x)\,\delta(x - x_0)\,dx \end{aligned}$$

where the discrete sum in the first expression is replaced by a continuous "sum" over x in the second.

You may think of $\delta(x - x_0)$ as a limit of normalized Gaussians which get progressively taller and narrower,

$$\delta(x - x_0) = \lim_{n\to\infty} \frac{n}{\sqrt{2\varphi}} \exp\left[-\frac{1}{2} n^2 (x - x_0)^2\right]$$

Properly speaking, the delta function has meaning only when integrated.

We can evaluate integrals of expressions containing derivatives of a Dirac delta function as well, using integration by parts. For any test function $f(x)$, consider the integral

$$\int_{-\infty}^{\infty} dx f(x) \frac{d}{dx} \delta(x - x_0)$$

Using the product rule,

$$\begin{aligned} \frac{d}{dx}\left(f(x)\,\delta(x - x_0)\right) &= \frac{df}{dx}\delta(x - x_0) + f(x)\frac{d}{dx}\delta(x - x_0) \\ f(x)\frac{d}{dx}\delta(x - x_0) &= \frac{d}{dx}\left(f(x)\,\delta(x - x_0)\right) - \frac{df}{dx}\delta(x - x_0) \end{aligned}$$

so substituting, our integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} dx f(x) \frac{d}{dx}\delta(x - x_0) &= \int_{-\infty}^{\infty} dx \left[\frac{d}{dx}\left(f(x)\,\delta(x - x_0)\right) - \frac{df}{dx}\delta(x - x_0)\right] \\ &= \left(f(x)\,\delta(x - x_0)\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \frac{df}{dx}\delta(x - x_0) \end{aligned}$$

The integrated term vanishes both because $f(x)$ is a test function and because $\delta(x - x_0)$ vanishes away from x_0. In the second term, we use the basic property of the delta function to conclude

$$\int_{-\infty}^{\infty} dx f(x) \frac{d}{dx}\delta(x - x_0) = -\frac{df}{dx}(x_0)$$

In higher dimensions, we simply multiply Dirac delta functions

$$\delta^3(\mathbf{x}-\mathbf{x}_0)=\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

We also define the unit step function:

$$\Theta(x-a)\equiv\begin{cases}1 & x-a>0\\ 0 & x-a<0\end{cases}$$

We have the relation

$$\Theta(x-a)=\int_0^x \delta(x-a)\,dx$$

since the integral of the delta function vanishes unless the range of integration includes the singular point $x=a$.

The Dirac delta and step functions will allow us to write a complete expression for charge densities even when they are idealized.

5 Helmholtz theorem

We conclude by writing the Helmholtz theorem.

Suppose we have a vector field, $\mathbf{v}$, in some region or all space, where we know the values of $\mathbf{v}$ on the entire boundary or at infinity, and know its divergence and its curl, $\boldsymbol{\nabla}\cdot\mathbf{v}$ and $\boldsymbol{\nabla}\times\mathbf{v}$, everywhere. The Helmholz theorem then states that $\mathbf{v}$ is given by

$$\begin{aligned}\mathbf{v} &= \mathbf{u}+\mathbf{w}+\mathbf{s}\\ &= -\boldsymbol{\nabla}f+\boldsymbol{\nabla}\times\mathbf{A}+\mathbf{s}\end{aligned}$$

where:

$$\begin{aligned}\nabla^2\mathbf{s} &= 0\\ \mathbf{s}|_S &= \mathbf{v}|_S\end{aligned}$$

and

$$\begin{aligned}\nabla^2 f &= -\boldsymbol{\nabla}\cdot\mathbf{v}\\ \boldsymbol{\nabla}f|_S &= 0\end{aligned}$$

and

$$\begin{aligned}\nabla^2\mathbf{A} &= -\boldsymbol{\nabla}\times\mathbf{v}\\ \boldsymbol{\nabla}\cdot\mathbf{A} &= 0\\ \boldsymbol{\nabla}\times\mathbf{A}|_S &= 0\end{aligned}$$

In general, $\mathbf{s}$ satisfies both $\boldsymbol{\nabla}\cdot\mathbf{s}=0$ and $\boldsymbol{\nabla}\times\mathbf{s}=0$. For vanishing boundary conditions $\mathbf{s}|_S=0$, we have $\mathbf{s}(\mathbf{x})=0$.

6 Exercises (required)

6.1 Spherical coordinates

Show that the three vectors

$$\begin{aligned}\hat{\boldsymbol{r}} &= \sin\theta\cos\varphi\hat{\mathbf{i}}+\sin\theta\sin\varphi\hat{\mathbf{j}}+\cos\theta\hat{\mathbf{k}}\\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\hat{\mathbf{i}}+\cos\theta\sin\varphi\hat{\mathbf{j}}-\hat{\mathbf{k}}\sin\theta\\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\mathbf{i}}+\cos\varphi\hat{\mathbf{j}}\end{aligned}$$

are orthonormal.

6.2 Spherical and cylindrical basis vectors

Espress the cylindrical basis vectors $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}})$,

$$\begin{aligned}\hat{\boldsymbol{\rho}} &= \cos\varphi\hat{\mathbf{i}} + \sin\varphi\hat{\mathbf{j}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\mathbf{i}} + \cos\varphi\hat{\mathbf{j}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{k}}\end{aligned}$$

in terms of the spherical basis, $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\right)$.

6.3 Divergence and curl: cylindrical

Find the divergence and the curl of

$$\mathbf{v} = \rho\hat{\boldsymbol{\rho}} + \sin\varphi\hat{\boldsymbol{\varphi}} + z^2\hat{\mathbf{z}}$$

Find the divergence and the curl of

$$\mathbf{v} = \rho\hat{\boldsymbol{\varphi}}$$

6.4 Divergence and curl: spherical

Write each of the vectors of problem 6.3 in spherical coordinates and compute the divergence and curl of each.

$$\mathbf{v} = \rho\hat{\boldsymbol{\rho}} + \sin\varphi\hat{\boldsymbol{\varphi}} + z^2\hat{\mathbf{z}}$$

6.5 Dirac delta

Perform each of the following integrals:

$$\begin{aligned}A &= \int_{-\infty}^{\infty} \left(3x^3 - 2x^2 + x - 1\right)\delta(x)\,dx \\ B &= \int_{-\infty}^{\infty} \left(3x^3 - 2x^2 + x - 1\right)\delta(x-2)\,dx \\ C &= \int_{0}^{\infty} \left(3x^3 - 2x^2 + x - 1\right)\delta(x+2)\,dx \\ D &= \int_{2}^{4} e^{-x}\sin x\,\delta(x-3)\,dx\end{aligned}$$

6.6 Charge density

The Dirac delta has units of $\frac{1}{length}$ and can be used to write infinitesimally thin charge distributions. Thus, a charge density of σ (charge per unit area) may be written as a volume charge density as

$$\rho(\mathbf{x}) = \sigma\delta(z)$$

This exists for all $\mathbf{x}$ and represents a uniform charge per unit area on the xy-plane. If we want to cut off the distribution at a finite distance from the origin, we can use the unit step function. Thus

$$\rho(\mathbf{x}) = \sigma\delta(z)\,\Theta(R-\rho)$$

gives a charge density that vanishes when ρ gets bigger than R. The total charge may be found by integrating over all space:

$$\begin{aligned}
Q &= \int \rho(\mathbf{x})\, d^3x \\
&= \int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz \left[\sigma\delta(z)\,\Theta(R-\rho)\right] \\
&= \int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi \left[\sigma\Theta(R-\rho)\right] \\
&= \int_0^\infty \rho d\rho \left[2\pi\sigma\Theta(R-\rho)\right] \\
&= \int_0^R \rho d\rho \left[2\pi\sigma\right] \\
&= \sigma\pi R^2
\end{aligned}$$

Use these ideas to write the volume charge density in spherical coordinates of an infinitesimally thin hemisphere of charge of radius R and uniform charge density σ (charge per unit area).