Integral Theorems

September 14, 2015

1 Integral of the gradient

We begin by recalling the Fundamental Theorem of Calculus, that the integral is the inverse of the derivative,

$$F(b) - F(a) = \int f(x)\, dx$$

provided

$$f(x) = \frac{dF}{dx}$$

We apply this to the integral of a gradient, but since the gradient is a vector, we need to choose a path of integration. Let $t_i = \frac{dx_i}{d\lambda}$ along a curve, $C(\lambda) = \{x_i(\lambda)\}$. Then we may integrate $\nabla\phi$ along the curve by integrating the directional derivative, $\mathbf{t} \cdot \nabla\phi$,

$$\begin{aligned}
\int_a^b \mathbf{t} \cdot \nabla\phi d\lambda &= \int_a^b \sum_{i=1}^{3} \frac{dx_i}{d\lambda} \cdot \frac{\partial\phi}{\partial x_i} d\lambda \\
&= \int_a^b \frac{d\phi}{d\lambda} d\lambda \\
&= \phi(b) - \phi(a)
\end{aligned}$$

Notice that the result does not depend on which path we followed to get from a to b. The gradient therefore lets us generalize the Fundamental Theorem to higher dimensions.

2 Integral of a divergence

2.1 The divergence theorem

Now consider the volume integral of the divergence, $\boldsymbol{\nabla} \cdot \mathbf{v}$, of a vector field, $\mathbf{v}(x, y, z)$:

$$\int\limits_V d^3x\, \boldsymbol{\nabla} \cdot \mathbf{v}$$

where $d^3x = dxdydz$. We begin by letting the volume be a cube with sides of length L. Writing out the divergence in coordinates and distributing,

$$\begin{aligned}
\int\limits_V d^3x\, \boldsymbol{\nabla} \cdot \mathbf{v} &= \int\limits_0^L dx \int\limits_0^L dy \int\limits_0^L dz \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \\
&= \int\limits_0^L dy \int\limits_0^L dz \left(\int\limits_0^L dx \frac{\partial v_x}{\partial x} \right) + \int\limits_0^L dx \int\limits_0^L dz \left(\int\limits_0^L dy \frac{\partial v_y}{\partial y} \right) + \int\limits_0^L dx \int\limits_0^L dy \left(\int\limits_0^L dz \frac{\partial v_z}{\partial z} \right)
\end{aligned}$$

$$= \int_0^L dy \int_0^L dz\ \left(v_x(L,y,z) - v_x(0,y,z)\right) + \int_0^L dx \int_0^L dz\ \left(v_y(x,L,z) - v_y(x,0,z)\right)$$
$$+ \int_0^L dx \int_0^L dy\ \left(v_z(x,y,L) - v_z(x,y,0)\right)$$

Let $\hat{\mathbf{n}}$ be the outward normal at any point on the surface of the cube. For example, for the face of the cube lying in the xy plane at $z = 0$, the outward normal is $-\hat{\mathbf{k}}$. The parallel face of the cube, lying at $z = L$, has outward normal $+\hat{\mathbf{k}}$. With this in mind, consider the first of the six terms above,

$$\int_0^L dy \int_0^L dz\, v_x(L,y,z)$$

This is a surface integral over the face of the cube lying at $x = L$, where the outward normal is $\hat{\mathbf{n}}(L,y,z) = +\hat{\mathbf{i}}$. We can therefore write this integrand as

$$v_x(L,y,z) = \hat{\mathbf{n}} \cdot \mathbf{v}(L,y,z)$$

The second term in our sum is

$$\int_0^L dy \int_0^L dz\ \left(-v_x(0,y,z)\right)$$

and the outward normal to the cube on the face with $x = 0$ is $\hat{\mathbf{n}}(0,y,z) = -\hat{\mathbf{i}}$. This integral is therefore

$$\int_0^L dy \int_0^L dz\ \left(-v_x(0,y,z)\right) = \int_0^L dy \int_0^L dz\, \hat{\mathbf{n}}(0,y,z) \cdot \mathbf{v}(0,y,z)$$

In the same way, we consider the remaining pairs. With the outward normals at various points of the cube given by

$$\begin{aligned}
\hat{\mathbf{n}}(L,y,z) &= +\hat{\mathbf{i}} \\
\hat{\mathbf{n}}(0,y,z) &= -\hat{\mathbf{i}} \\
\hat{\mathbf{n}}(x,L,z) &= +\hat{\mathbf{j}} \\
\hat{\mathbf{n}}(x,0,z) &= -\hat{\mathbf{j}} \\
\hat{\mathbf{n}}(x,y,L) &= +\hat{\mathbf{k}} \\
\hat{\mathbf{n}}(x,y,0) &= -\hat{\mathbf{k}}
\end{aligned}$$

the integrand of each of the six double integrals may be written as $\hat{\mathbf{n}} \cdot \mathbf{v}$ evaluated on the appropriate side. Then the full integral of the divergence becomes

$$\begin{aligned}
\int_V d^3x\, \boldsymbol{\nabla} \cdot \mathbf{v} &= \int_0^L dy \int_0^L dz\ \left(\hat{\mathbf{n}} \cdot \mathbf{v}(L,y,z) + \hat{\mathbf{n}} \cdot \mathbf{v}(0,y,z)\right) + \int_0^L dx \int_0^L dz\ \left(\hat{\mathbf{n}} \cdot \mathbf{v}(x,L,z) + \hat{\mathbf{n}} \cdot \mathbf{v}(x,0,z)\right) \\
&\quad + \int_0^L dx \int_0^L dy\ \left(\hat{\mathbf{n}} \cdot \mathbf{v}(x,y,L) + \hat{\mathbf{n}} \cdot \mathbf{v}(x,y,0)\right) \\
&= \sum_{i=1}^{6\,sides} \int_{S_i} d^2x\ \left(\hat{\mathbf{n}}(S_i) \cdot \mathbf{v}(S_i)\right)
\end{aligned}$$

$$= \oint_S d^2x\, \hat{\mathbf{n}} \cdot \mathbf{v}$$

where the final integral is over the entire surface of the cube, and the integrand is the outward component of $\mathbf{v}$ evaluated on that surface. The integral therefore gives us the total outward flux of $\mathbf{v}$ across the boundary of the cube. The circle over the integral sign indicates that the surface S is closed. This is always true of the boundary of a volume.

To indicate that S is the boundary surface of V we write $S = \delta V$.

Now consider two cubes lying side by side and sharing a common side. On the common side the values of the vector field $\mathbf{v}$ are the same for the two cubes, but if the outward normal is $\hat{\mathbf{n}}$ for the first, then it is $-\hat{\mathbf{n}}$ for the second. When we sum over all six sides of the two cubes, these two "inside" contributions exactly cancel so the integral of the divergence is still the integral of the outward normal component over the outer surface of the total volume.

Finally, consider an arbitrary volume, V. Fill the volume as completely as possible with cubes of side ε, and apply the following results. All boundaries of cubes in the interior of V will face another volume with the opposite sign for the outward normal. The only non-cancelling contributions will come from the surfaces near the outer boundary of V. As we make the cube size smaller and smaller, this outer boundary of our collection of cubes becomes a better and better approximation to the actual surface of the boundary of V. In the limit as $\varepsilon \to 0$, they become identical.

This establishes the divergence theorem: Let an arbitrary volume V have boundary S. Then

$$\int_V d^3x\, \boldsymbol{\nabla} \cdot \mathbf{v} = \oint_S d^2x\, \hat{\mathbf{n}} \cdot \mathbf{v}$$

where $\hat{\mathbf{n}}$ is the unit outward normal on the boundary S of the volume V.

The divergence theorem tells us about the flux of some vector quantity $\mathbf{v}$ out of a region V.

2.2 Example

Let a vector field $\mathbf{v}$ be given by

$$\mathbf{v} = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}$$

Then the divergence is

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{v} &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}\left(x^2z\right) \\ &= y + z + x^2 \end{aligned}$$

Let the volume V be the unit cube. Then

$$\begin{aligned} \int_V d^3x\, \boldsymbol{\nabla} \cdot \mathbf{v} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \left(y + z + x^2\right) \\ &= \int_0^1 dx \int_0^1 dy \left(y + \frac{1}{2} + x^2\right) \\ &= \int_0^1 dx \left(\frac{1}{2} + \frac{1}{2} + x^2\right) \\ &= \frac{4}{3} \end{aligned}$$

The surface integral is a sum of six terms

$$\begin{aligned}
\oint_S d^2x\,\hat{\mathbf{n}}\cdot\mathbf{v} &= \sum_{i=1}^{6\,sides}\int_{S_i} d^2x\;\left(\hat{\mathbf{n}}\left(S_i\right)\cdot\mathbf{v}\left(S_i\right)\right) \\
&= \int_0^1 dy\int_0^1 dz\;\left(v_x\left(x=1\right)-v_x\left(x=0\right)\right)+\int_0^1 dz\int_0^1 dx\;\left(v_y\left(y=1\right)-v_y\left(y=0\right)\right)+\int_0^1 dx\int_0^1 dy\;\left(v_z\left(z=1\right)-v_z\left(z\right.\right. \\
&= \int_0^1 dy\int_0^1 dz\;\left(y-0\right)+\int_0^1 dz\int_0^1 dx\;\left(z-0\right)+\int_0^1 dx\int_0^1 dy\;\left(x^2-0\right) \\
&= \frac{1}{2}+\frac{1}{2}+\frac{1}{3} \\
&= \frac{4}{3}
\end{aligned}$$

3 Stokes' theorem: Integral of a curl

Next, we consider the integral of the curl of a vector. Once again this is a vector so we again require a second vector. This time, however, we choose the second vector to be the normal to a surface. For concreteness, let the surface be a square region of side L, lying in the xy plane. The unit normal to this square is the unit vector $\mathbf{k}$ in the z-direction Then the integral of the curl of a vector field $\mathbf{v}$ dotted with $\hat{\mathbf{k}}$ over the region is

$$\iint_0^L \left(\boldsymbol{\nabla}\times\mathbf{v}\right)\cdot\hat{\mathbf{k}}\,d^2x = \iint_0^L\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)dxdy$$

Once again we can integrate exactly,

$$\begin{aligned}
\iint_0^L \left(\boldsymbol{\nabla}\times\mathbf{v}\right)\cdot\hat{\mathbf{k}}\,d^2x &= \iint_0^L \frac{\partial v_y}{\partial x}\,dxdy-\iint_0^L \frac{\partial v_x}{\partial y}\,dxdy \\
&= \int_0^L \left(v_y\left(L,y,z\right)-v_y\left(0,y,z\right)\right)\,dy-\int_0^L\left(v_x\left(x,L,z\right)-v_x\left(x,0,z\right)\right)\,dx \\
&= \int_0^L v_y\left(L,y,z\right)\,dy-\int_0^L v_y\left(0,y,z\right)\,dy-\int_0^L v_x\left(x,L,z\right)\,dx+\int_0^L v_x\left(x,0,z\right)\,dx \\
&= \int_0^L v_x\left(x,0,z\right)\,dx+\int_0^L v_y\left(L,y,z\right)\,dy-\int_0^L v_x\left(x,L,z\right)\,dx-\int_0^L v_y\left(0,y,z\right)\,dy
\end{aligned}$$

where the last line is just a rearrangement of the four integrals. We can write this in a simple form. Let the border of the square have an tangent vector $\mathbf{t}=\frac{d\mathbf{x}}{d\lambda}$ so that an infinitesimal displacement along the boundary is given by

$$d\mathbf{l}=\frac{d\mathbf{x}}{d\lambda}d\lambda$$

where we take the counterclockwise direction around the square. Notice that doing this, the right-hand rule gives us the direction of the normal, $\hat{\mathbf{k}}$. Following the boundary of our particular square, we have

$$\begin{aligned}
d\mathbf{l} &= \hat{\mathbf{i}}dx \quad \textit{bottom side} \\
d\mathbf{l} &= \hat{\mathbf{j}}dy \quad \textit{right side} \\
d\mathbf{l} &= -\hat{\mathbf{i}}dx \quad \textit{top side} \\
d\mathbf{l} &= -\hat{\mathbf{j}}dy \quad \textit{left side}
\end{aligned}$$

In each of the four integrals above, the integrand may be written as $\mathbf{v} \cdot d\mathbf{l}$, including the signs. For example, the first integral, $\int_0^L v_x(x,0,z)\, dx$ is the integral of the x-component of $\mathbf{v}$, evaluated along the bottom of the square and integrated along that side. The four integrals together are just the integral of $\mathbf{v} \cdot d\mathbf{l}$ all the way around the square back to the starting point. The full path of integration is therefore a closed loop, indicated by a circle on the integral sign,

$$\oint_{square} \mathbf{v} \cdot d\mathbf{l}$$

and we have established

$$\iint_{square\,region} (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \hat{\mathbf{k}}\, d^2x = \oint_{square} \mathbf{v} \cdot d\mathbf{l}$$

As in the case of the divergence theorem, we can add regions together to get a more general law. Suppose we have an arbitrary curved surface embedded in 3-space. We can approximate the surface by small squares of side ε. For each such element of the surface at position $\mathbf{x}$, we evaluate the dot product $\mathbf{n}(\mathbf{x}) \cdot (\boldsymbol{\nabla} \times \mathbf{v}(\mathbf{x}))$ of the curl of our vector field, $\boldsymbol{\nabla} \times \mathbf{v}(\mathbf{x})$, with the normal to the surface element at that point, $\mathbf{n}(\mathbf{x})$. Adding these up all contributions on the shared sides of squares cancel because of our counterclockwise orientation, and the total integral gives the integral over the boundary of the full region. The relationship is exact in the limit as $\varepsilon \to 0$. This results in *Stokes' theorem*:

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, d^2x = \oint_{C=\delta S} \mathbf{v} \cdot d\mathbf{l}$$

where S is any 2-dimensional region, $C = \delta S$ is its boundary, and $d\mathbf{l}$ is an infinitesimal displacement in the clockwise direction (relative to the normal, $\hat{\mathbf{n}}$) along the boundary.

3.1 Example

Let

$$\mathbf{v} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$$

so it has a nice tidy curl,

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \left(\frac{\partial z^2}{\partial y} - \frac{\partial(-x)}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial y}{\partial z} - \frac{\partial z^2}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right)\hat{\mathbf{k}} \\
&= -2\hat{\mathbf{k}}
\end{aligned}$$

Integrate this over a hemisphere of radius R, with its boundary a circle of radius R in the xy plane. Then the surface integral is

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, d^2x = -2 \iint_S \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, d^2x$$

A surface element of a sphere may be written in spherical coordinates as $d^2x = R^2 \sin\theta d\theta d\phi$ and the dot product is $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = \cos\theta$. Therefore,

$$\begin{aligned}
\iint_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, d^2x &= -2 \iint_S \cos\theta\, R^2 \sin\theta d\theta d\phi \\
&= -2R^2 \iint_S \cos\theta \sin\theta d\theta d\phi
\end{aligned}$$

$$
\begin{aligned}
&= -4\pi R^2 \iint\limits_S \cos\theta \sin\theta d\theta \\
&= -2\pi R^2 \sin^2\theta\Big|_0^{\frac{\pi}{2}} \\
&= -2\pi R^2
\end{aligned}
$$

For the boundary integral, $d\mathbf{l} = \left(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right) d\varphi$

$$
\begin{aligned}
\oint\limits_{C=\delta S} \mathbf{v}\cdot d\mathbf{l} &= \oint\limits_{C=\delta S} \left(y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}\right)\cdot\left(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right) d\varphi \\
&= -\oint\limits_{C=\delta S} \left(y^2 + x^2\right) d\varphi \\
&= -\oint\limits_{C=\delta S} R^2 d\varphi \\
&= -2\pi R^2
\end{aligned}
$$

just as we got before.

Notice that the result is unchanged if we change S to *any* surface with the same boundary. For example, let S' be a flat disk of radius R lying in the xy plane, centered at the origin. Then the boundary is the same, but the surface integral becomes

$$
\begin{aligned}
\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{v})\cdot\hat{\mathbf{n}}\, d^2x &= \iint\limits_S \left(-2\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{k}}\, r dr d\phi \\
&= -2\int\limits_0^{2\pi} d\phi \int\limits_0^R r dr \\
&= -2\cdot 2\pi\cdot\frac{R^2}{2} \\
&= -2\pi R^2
\end{aligned}
$$

3.2 Example: Faraday's law

Faraday's law tells us that the time rate of change of magnetic flux through a surface, S, is the negative of the emf around the boundary, C, of that surface,

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

The emf is defined as the integral of the electric field along a curve, so we may write

$$\mathcal{E} = \oint\limits_{C=\delta S} \mathbf{E}\cdot d\mathbf{l}$$

while the magnetic flux is given by

$$\Phi = \iint\limits_S \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2x$$

where $\hat{\mathbf{n}}$ is normal to S.

Now apply Stokes' theorem to the emf,

$$\begin{aligned}\mathcal{E} &= \oint\limits_{C=\delta S} \mathbf{E}\cdot d\mathbf{l} \\ &= \iint\limits_{S} (\boldsymbol{\nabla}\times\mathbf{E})\cdot\hat{\mathbf{n}}\,d^2x\end{aligned}$$

Faraday's law now becomes

$$\iint\limits_{S} (\boldsymbol{\nabla}\times\mathbf{E})\cdot\hat{\mathbf{n}}\,d^2x = -\frac{d}{dt}\iint\limits_{S} \mathbf{B}\cdot\hat{\mathbf{n}}\,d^2x$$

Since t is independent of the x, y, z variables, we may interchange the order of integration and differentiation on the right, as long as we remember to change $\frac{d}{dt}$ to a partial derivative,

$$\frac{d}{dt}\iint\limits_{S} \mathbf{B}\cdot\hat{\mathbf{n}}\,d^2x = \iint\limits_{S} \frac{\partial\mathbf{B}}{\partial t}\cdot\hat{\mathbf{n}}\,d^2x$$

Now, bringing the magnetic field term to the left, we have

$$\iint\limits_{S} \left(\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}\right)\cdot\hat{\mathbf{n}}\,d^2x = 0$$

Since S may be any surface with the boundary C, the integrand must vanish everywhere (if it didn't, then putting a little bump in the surface to include some of the nonzero expression would change the value of the integral away from zero). We conclude

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$$

This gives us Faraday's law written as a differential equation.

4 Exercises (required)

Work the following problems:

1. Compute the curl of the vector field $\mathbf{v} = y^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$, then calculate the line integral along the following two paths from the origin to $(2,1,1)$:

   (a) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (2,1,1)$

   (b) $(0,0,0) \to (2,0,0) \to (2,1,0) \to (2,1,1)$

2. Integrate $f(\mathbf{x}) = xy^2$ over the volume with corners at the origin, $(0,0,0)$ and the three unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

3. Compute both sides of the divergence theorem for the vector field $\mathbf{v} = xy\hat{\mathbf{i}} + 2xz^3\hat{\mathbf{j}} + yx^2\hat{\mathbf{k}}$ where the volume is the unit cube (i.e. with corners at $\mathbf{0}, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}, \hat{\mathbf{i}}+\hat{\mathbf{j}}, \hat{\mathbf{i}}+\hat{\mathbf{k}}, \hat{\mathbf{j}}+\hat{\mathbf{k}}, \hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$).

4. Compute both sides of Stokes' theorem for the vector field $\mathbf{v} = -y^2\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ where the surface spans the triangle in the xy-plane with corners at $(0,0,0), (1,2,0), (-1,2,0)$.