Vectors

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Our basic notion of a vector is as a displacement, directed from one point of Euclidean space to another, and therefore having direction and magnitude. We will write vectors in boldface here, but for problems and lectures it is faster to use an arrow, $\vec{v}$. Since we will write the magnitude of the vector $\vec{v}$ or $\mathbf{v}$ as simply v it is important to always use some distinguishing notation.

While we may always think of a vector as direction and magnitude, we can give a more abstract definition that doesn't depend on these intuitions. This has the advantage of allowing us easy algebraic manipulations. We give an algebraic definition, then describe two important ways of combining two vectors to get a third: the *dot product* and the *cross product*. Finally, we discuss rotations, which leave the important vector relationships invariant.

1 Definition:

A *vector space*, $\mathcal{V}$, is a collection of objects called vectors, $\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots \in \mathcal{V}$ together with an operation, addition, satisfying the following properties:

1. The sum of any two vectors, $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, denoted $\mathbf{w} = \mathbf{u} + \mathbf{v}$, is in the vector space as well, $\mathbf{w} \in \mathcal{V}$. (Closure)

2. There is an identity vector, $\mathbf{0}$, called the zero vector, such that for all vectors, $\mathbf{u} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

3. For every $\mathbf{u} \in \mathcal{V}$, there is an inverse vector in $\mathcal{V}$, called $(-\mathbf{u})$, such that

$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$$

4. Vector addition is associative. For any three vectors, $\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots \in \mathcal{V}$,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

and therefore we may unambiguously write $\mathbf{u} + \mathbf{v} + \mathbf{w}$ for the sum of more than two vectors.

5. Vector addition is commutative. For any two vectors, $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

There is also a second operation, scalar multiplication (for our purposes by real or complex numbers, $a, b, \ldots \in \mathcal{R} \text{ or } \mathcal{C}$), such that for any two scalars, a, b, and any two vectors, $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{w} = a\mathbf{u} + b\mathbf{v}$$

is also in $\mathcal{V}$. Finally, we require the distributive laws,

$$
\begin{aligned}
(a+b)\,\mathbf{u} &= a\mathbf{u} + b\mathbf{u} \\
a\,(\mathbf{u}+\mathbf{v}) &= a\mathbf{u} + a\mathbf{v}
\end{aligned}
$$

A vector space is n-dimensional if n is the smallest number such that any collection of $n+1$ distinct vectors, $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n+1}\}$ is linearly dependent, in the sense that there exist numbers $a_1, a_2, \ldots a_{n+1}$ such that

$$
a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_{n+1}\mathbf{u}_{n+1} = 0
$$

It is then always possible to choose a *basis*, that is, a set of n independent vectors, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ such that every vector in V may be written as a linear combination of the form,

$$
\mathbf{u} = a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n
$$

for some real (complex) numbers, $a_1, \ldots, a_n$. These numbers are the *components* of the vector $\mathbf{u}$ in that basis, and $\mathbf{u}$ may be conveniently written as

$$
\mathbf{u} = (a_1, a_2, \ldots, a_n)
$$

2 Dot product

2.1 Length

We require our vectors to have a real, positive length, u,

$$
u = |\mathbf{u}|
$$

which we will write using a dot:

$$
\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2
$$

We require this length to be linear in the sense that:

$$
|a\mathbf{u}| = a\,|\mathbf{u}|
$$

By extending linearity to sums of vectors, we can extend the dot product to a real-valued mapping of any pair of vectors. Noticing that linearity under vector addition would require

$$
\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u} \cdot (\mathbf{u}+\mathbf{v}) + \mathbf{v} \cdot (\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}
\end{aligned}
$$

we *define* the dot product of any two vectors to be

$$
\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}\left(|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2\right)
$$

From this it follows that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

Two vectors are said to be perpendicular or orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, and parallel if $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|$.

2.2 Orthonormal basis

Our intuitions about vectors are clearest if we choose an orthonormal basis. Given any vector $\mathbf{u}$, and its length u, the vector $\frac{1}{u}\mathbf{u}$ has unit length,

$$
\begin{aligned}
\left| \frac{1}{u}\mathbf{u} \right| &= \frac{1}{u}|\mathbf{u}| \\
&= \frac{1}{u}u \\
&= 1
\end{aligned}
$$

Suppose we have a basis, $\mathbf{e}_1, \ldots, \mathbf{e}_n$. We can then form a new basis which is orthonormal by using a procedure called Gram-Schmidt orthogonalization.

To be a basis, no two basis vectors can be parallel (prove this!). Therefore, given any two, $\mathbf{e}_1, \mathbf{e}_2$ we make the first one unit length,

$$
\hat{\mathbf{e}}_1 = \frac{1}{e_1}\mathbf{e}_1
$$

Now construct a unit vector orthogonal to $\hat{\mathbf{e}}_1$ by writing

$$
\hat{\mathbf{e}}_2 = \frac{\mathbf{e}_2 - (\mathbf{e}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1}{|\mathbf{e}_2 - (\mathbf{e}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1|}
$$

We now have

$$
\begin{aligned}
\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 &= 1 \\
\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 &= 0 \\
\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 &= 1
\end{aligned}
$$

We continue in this way, adding one new vector at a time and constructing one perpendicular to all the previous ones, until we have done the full set. Then for all $i, j = 1, \ldots, n$,

$$
\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \tag{1}
$$

where the Kronecker delta, δ_{ij} is defined to be the matrix

$$
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
$$

This is just the identity matrix. Notice that eq.(1) is actually n^2 distinct equations, since each of i, j may take n different values.

2.3 Three dimensions

In Euclidean 3-space, we define a *Cartesian* orthonormal basis $\hat{\mathbf{e}}_i = \left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \right)$ for $i = 1, 2, 3$, where $\hat{\mathbf{i}}$ is a unit vector pointing in the $+x$ direction, $\hat{\mathbf{j}}$ a unit vector in the y-direction and $\hat{\mathbf{k}}$ is a unit vector point in the $+z$ directions. Then any vector may be written as

$$
\begin{aligned}
\vec{v} &= \mathbf{v} \\
&= v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}} \\
&= v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} = \sum_{i=1}^{3} v_i\hat{\mathbf{e}}_i \\
&= \sum_{i=1}^{3} v_i\hat{\mathbf{e}}_i
\end{aligned}
$$

We will use these various notations interchangeably. Notice that, while an arrow over a symbol or boldface indicates a vector, a carat over a vector, $\hat{\mathbf{k}}$, indicates that it is a *unit* vector. For any unit vector, we have $\hat{\mathbf{v}} \cdot \hat{\mathbf{v}} = 1$.

Using the distributive laws, we find the usual Pythagorean expression for the length of a vector:

$$
\begin{aligned}
v^2 &= \mathbf{v} \cdot \mathbf{v} \\
&= \left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \right) \cdot \left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \right) \\
&= \left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \right) \cdot v_x \hat{\mathbf{i}} + \left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \right) \cdot v_y \hat{\mathbf{j}} + \left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \right) \cdot v_z \hat{\mathbf{k}} \\
&= \left(v_x \hat{\mathbf{i}} \right) \cdot \left(v_x \hat{\mathbf{i}} \right) + \left(v_y \hat{\mathbf{j}} \right) \cdot \left(v_x \hat{\mathbf{i}} \right) + \left(v_z \hat{\mathbf{k}} \right) \cdot \left(v_x \hat{\mathbf{i}} \right) \\
&\quad + \left(v_x \hat{\mathbf{i}} \right) \cdot \left(v_y \hat{\mathbf{j}} \right) + \left(v_y \hat{\mathbf{j}} \right) \cdot \left(v_y \hat{\mathbf{j}} \right) + \left(v_z \hat{\mathbf{k}} \right) \cdot \left(v_y \hat{\mathbf{j}} \right) \\
&\quad + \left(v_x \hat{\mathbf{i}} \right) \cdot \left(v_z \hat{\mathbf{k}} \right) + \left(v_y \hat{\mathbf{j}} \right) \cdot \left(v_z \hat{\mathbf{k}} \right) + \left(v_z \hat{\mathbf{k}} \right) \cdot \left(v_z \hat{\mathbf{k}} \right) \\
&= v_x^2 \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + v_y v_x \hat{\mathbf{j}} \cdot \hat{\mathbf{i}} + v_z v_x \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} + v_x v_y \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + v_y^2 \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} + v_z v_y \hat{\mathbf{k}} \cdot \hat{\mathbf{j}} + v_x v_z \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} + v_y v_z \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} + v_z^2 \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \\
&= v_x^2 + v_y^2 + v_z^2
\end{aligned}
$$

where we have used the orthonormality.

Writing out all nine (or n^2) separate terms is tedious. However, if we use index notation, it's not so bad. Consider the following shorter version:

$$
\begin{aligned}
v^2 &= \mathbf{v} \cdot \mathbf{v} \\
&= \sum_{i=1}^{3} v_i \hat{\mathbf{e}}_i \cdot \sum_{j=1}^{3} v_j \hat{\mathbf{e}}_j \\
&= \sum_{i=1}^{3} v_i \sum_{j=1}^{3} v_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \\
&= \sum_{i=1}^{3} v_i \sum_{j=1}^{3} v_j \delta_{ij} \\
&= \sum_{i=1}^{3} v_i v_i \\
&= v_1^2 + v_2^2 + v_3^2
\end{aligned}
$$

where we use the fact that

$$
\sum_{j=1}^{3} v_j \delta_{ij} = v_i
$$

because the δ_{ij} vanishes unless $i = j$. Notice that in the first step is is crucial to use different summation indices in the two sums.

For a general dot product, we now have

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \frac{1}{2} \left(|\mathbf{u} + \mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2 \right) \\
&= \frac{1}{2} \left((u_x + v_x)^2 + (u_y + v_y)^2 + (u_z + v_z)^2 - \left(u_x^2 + u_y^2 + u_z^2 \right) - \left(v_x^2 + v_y^2 + v_z^2 \right) \right) \\
&= \frac{1}{2} \left(2 u_x v_x + 2 u_y v_y + 2 u_z v_z \right) \\
&= u_x v_x + u_y v_y + u_z v_z
\end{aligned}
$$

2.4 Angles

We can use the dot product to define the angle between two vectors, at least up to the sign of the angle. Suppose we have two vectors, $\mathbf{u}, \mathbf{v}$. If we choose the plane spanned by linear combinations of just these two to be the xy plane, and then choose the x-axis to lie along the direction of the first vector, then we have

$$\mathbf{u} = u\hat{\mathbf{i}}$$

while the second vector lies somewhere in the xy plane,

$$\begin{aligned} \mathbf{v} &= v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \\ &= v\cos\varphi\hat{\mathbf{i}} + v\sin\varphi\hat{\mathbf{j}} \end{aligned}$$

The dot product of $\mathbf{u}, \mathbf{v}$ is then

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u\hat{\mathbf{i}} \cdot \left(v\cos\varphi\hat{\mathbf{i}} + v\sin\varphi\hat{\mathbf{j}} \right) \\ &= uv\left(\cos\varphi\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + \sin\varphi\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} \right) \\ &= uv\cos\varphi \end{aligned}$$

and therefore, we find the angle from

$$\cos\varphi = \frac{\mathbf{u} \cdot \mathbf{v}}{uv}$$

The beauty of this construction is that a *rotation* is defined as a linear transformation of vectors that preserves their lengths. Since we have defined the dot product in terms of lengths, rotations also preserve the dot product, and the angle is now defined in terms of lengths and a dot product. Therefore, the expression above does not depend on our choosing the x and y axes the way we did. We may *always* find the cosine of the angle between two vectors by dividing their dot product by their norms, regardless of the directions of the coordinate axes.

This result gives us another convenient way to write the dot product, namely,

$$\mathbf{u} \cdot \mathbf{v} = uv\cos\varphi$$

where u, v are the lengths and φ the angle between the vectors.

3 Cross product

There is another way to define a product of two vectors, one which gives a third vector. It exists because given any two independent vectors in 3-space, there is a unique third vector orthogonal to them both. The first two define a plane, and the third is normal to that plane. Choosing the direction of the normal by a right-hand rule, we only need to specify the length of the product vector. We take it to be the area of the parallelogram defined by the first two vectors.

Once again, let's start with two vectors $\mathbf{u}, \mathbf{v}$, and choose coordinates so that they lie in the xy plane. Then the area of the parallelogram formed by the two is given by

$$A = uv\sin\varphi$$

Clearly this area is unchanged by rotations, though the normal is not. Since the z direction is given by $\hat{\mathbf{k}}$, we have

$$\mathbf{u} \times \mathbf{v} = uv\sin\varphi\hat{\mathbf{k}}$$

The only part of this that changes with rotations is the direction of $\hat{\mathbf{k}}$. Let $\hat{\mathbf{n}}$ be the direction normal to both $\mathbf{u}$ and $\mathbf{v}$ as given by the right-hand rule. Then the cross product may be written in a way independent of coordinates as

$$\mathbf{u} \times \mathbf{v} = uv\sin\varphi\hat{\mathbf{n}}$$

Notice that for any vector, $\mathbf{v} \times \mathbf{v} = 0$.

We may make this definition completely independent of rotations by defining the *triple product*. Let $\mathbf{w}$ be an arbitrary third vector. Then in general, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ give the sides of a parallelepiped. Since the cross product gives the area of the base, while $h = \hat{\mathbf{n}} \cdot \mathbf{w}$ gives the height, the volume of the parallelepiped is

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = uv \sin\varphi\, \hat{\mathbf{n}} \cdot \mathbf{w} = Ah$$

The triple product is purely geometrical and therefore independent of how we rotate the vectors. By choosing $\mathbf{w}$ perpendicular to the base, we may recover the cross product from the triple product.

To find the component form of the cross product, we use our orthonormal basis.

Clearly, we have:

$$\begin{aligned}
\hat{\mathbf{i}} \times \hat{\mathbf{i}} &= 0 \\
\hat{\mathbf{j}} \times \hat{\mathbf{j}} &= 0 \\
\hat{\mathbf{k}} \times \hat{\mathbf{k}} &= 0
\end{aligned}$$

For the remaining products, notice that any pair of the unit vectors give the sides of a unit square. The angle is $90°$ and the area is 1. The only thing remaining to keep track of is the orientation. Using the right hand rule we see that

$$\begin{aligned}
\hat{\mathbf{i}} \times \hat{\mathbf{j}} &= \hat{\mathbf{k}} \\
\hat{\mathbf{j}} \times \hat{\mathbf{k}} &= \hat{\mathbf{i}} \\
\hat{\mathbf{k}} \times \hat{\mathbf{i}} &= \hat{\mathbf{j}}
\end{aligned}$$

Reversing the sign allows us to tell which direction the normal points, so that

$$\begin{aligned}
\hat{\mathbf{j}} \times \hat{\mathbf{i}} &= -\hat{\mathbf{k}} \\
\hat{\mathbf{k}} \times \hat{\mathbf{j}} &= -\hat{\mathbf{i}} \\
\hat{\mathbf{i}} \times \hat{\mathbf{k}} &= -\hat{\mathbf{j}}
\end{aligned}$$

We can now find components of the cross product. Let

$$\begin{aligned}
\mathbf{u} &= u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}} \\
\mathbf{v} &= v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}
\end{aligned}$$

Remembering that everything to do with vectors must be linear, we find

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \left(u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}\right) \times \left(v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}\right) \\
&= u_1 v_1 \hat{\mathbf{i}} \times \hat{\mathbf{i}} + u_2 v_1 \hat{\mathbf{j}} \times \hat{\mathbf{i}} + u_3 v_1 \hat{\mathbf{k}} \times \hat{\mathbf{i}} \\
&\quad + u_1 v_2 \hat{\mathbf{i}} \times \hat{\mathbf{j}} + u_2 v_2 \hat{\mathbf{j}} \times \hat{\mathbf{j}} + u_3 v_2 \hat{\mathbf{k}} \times \hat{\mathbf{j}} \\
&\quad + u_1 v_3 \hat{\mathbf{i}} \times \hat{\mathbf{k}} + u_2 v_3 \hat{\mathbf{j}} \times \hat{\mathbf{k}} + u_3 v_3 \hat{\mathbf{k}} \times \hat{\mathbf{k}} \\
&= -u_2 v_1 \hat{\mathbf{k}} + u_3 v_1 \hat{\mathbf{j}} \\
&\quad + u_1 v_2 \hat{\mathbf{k}} - u_3 v_2 \hat{\mathbf{i}} \\
&\quad - u_1 v_3 \hat{\mathbf{j}} + u_2 v_3 \hat{\mathbf{i}}
\end{aligned}$$

resulting in

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\hat{\mathbf{i}} + (u_3 v_1 - u_1 v_3)\,\hat{\mathbf{j}} + (u_1 v_2 - v_1 u_2)\,\hat{\mathbf{k}}$$

We notice that this is just the determinant of the matrix

$$\begin{pmatrix}
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 \\
\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}
\end{pmatrix}$$

While this result may also be achieved more easily using index notation, it requires use of the Levi-Civita symbol. We will introduce this later as an optional tool.

4   Rotations

We defined rotations as linear transformations that preserve the lengths of all vectors. We may write the components of a linear transformation of an arbitrary vector $\mathbf{u}$ as

$$\tilde{u}_i = \sum_{j=1}^{3} R_{ij} u_j$$

and the condition that makes it a rotation is that the length of $\tilde{\mathbf{u}}$ equals the length of $\mathbf{u}$. In components,

$$
\begin{aligned}
\sum_{i=1}^{3} u_i u_i \;&=\; \sum_{i=1}^{3} \tilde{u}_i \tilde{u}_i \\
&=\; \sum_{i=1}^{3} \left(\sum_{j=1}^{3} R_{ij} u_j \right) \left(\sum_{k=1}^{3} R_{ik} u_k \right) \\
&=\; \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} R_{ij} u_j R_{ik} u_k \\
&=\; \sum_{j=1}^{3} \sum_{k=1}^{3} \left(\sum_{i=1}^{3} R_{ij} R_{ik} \right) u_j u_k
\end{aligned}
$$

where we use associativity of addition. Now notice that this holds if (and only if),

$$\sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk}$$

for then the triple sum becomes

$$
\begin{aligned}
\sum_{j=1}^{3} \sum_{k=1}^{3} \left(\sum_{i=1}^{3} R_{ij} R_{ik} \right) u_j u_k \;&=\; \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_{jk} u_j u_k \\
&=\; \sum_{j=1}^{3} u_j u_j
\end{aligned}
$$

The transpose of a matrix, R_{ij} is just R_{ji}, so the condition we need in order to have a rotation is

$$R^t R = 1$$

A matrix is a rotation if and only if its transpose is its inverse.

Let's check that this works in 2-dimensions. Let

$$
\begin{aligned}
R_{ij} \;&=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
R_{ij}^t \;&=\; \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\end{aligned}
$$

Then imposing the condition, we have

$$
\begin{aligned}
\sum_{i=1}^{2} R_{ji}^t R_{ik} \;&=\; \delta_{jk} \\
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;&=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \;&=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Therefore, we need:

$$a^2 + c^2 = 1$$
$$ab + cd = 0$$
$$b^2 + d^2 = 1$$

We can get the first condition by writing $a = \cos\theta, c = \sin\theta$ for any angle θ, and similarly $b = \cos\varphi, d = \sin\varphi$ for some φ. The middle condition then tells us that

$$0 = \cos\theta\cos\varphi + \sin\theta\sin\varphi$$
$$= \cos(\theta - \varphi)$$

so we need $\varphi = \theta + \frac{\pi}{2}$. Since $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$ and $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$, our rotation matrix becomes

$$R_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which is just right.

5 Exercises

For the three vectors

$$\mathbf{u} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$
$$\mathbf{v} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$$
$$\mathbf{w} = 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

find the following:

1. Find

$$\mathbf{u} \times \mathbf{v}$$
$$\mathbf{u} \times \mathbf{w}$$
$$\mathbf{w} \times \mathbf{v}$$

2. By explicitly taking the dot and cross products find:

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$$
$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$$
$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

3. Find

$$\mathbf{v} \times (\mathbf{w} \times \mathbf{w})$$
$$\mathbf{w} \times (\mathbf{v} \times \mathbf{w})$$

4. Prove that for any three vectors (the BAC-CAB rule),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

You can just substitute general expressions for the products, but there's an easier way to do this if you use the rotational invariance of the result.