Pretest

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Potentially useful equations:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \qquad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enclosed}$$
$$\int_{-1}^{1} P_l(x)\,P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{ll'} \qquad \oint \mathbf{B}\cdot\hat{\mathbf{n}}d^2x = 0$$
$$\int_0^{\pi} P_l(\cos\theta)\,P_{l'}(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'} \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

Legendre polynomials:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2-1\right) \\ P_3(x) &= \frac{1}{2}\left(5x^3-3x\right) \end{aligned}$$

1 Two planes at a right angle and one charge

Consider two conducting half planes at right angles, connected along the z-axis. One has $y=0$ for all x,z, the other $x=0$ for all y,z. Let a single charge q be placed midway between the planes, at the point $(x,y,z) = (1,1,0)$. Find the electric potential everywhere in the positive xy-quadrant (i.e. all $(x>0, y>0, z)$)using the method of images.

2 Separation of variables in spherical coordinates

In spherical coordinates, (r,θ,φ), the Laplace equation for the potential is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

Apply the method of separation of variables to separate this into three ordinary differential equations.

3 A sphere at fixed potential

A hollow sphere of radius R has the potential

$$V(R,\theta) = V_0\left(\cos^2\theta - \frac{1}{3}\right)$$

on its surface. Using a Legendre series, find the potential, $V(r,\theta)$, everywhere inside the sphere. Start from the φ-independent solution to the Laplace equation in spherical coordinates,

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

and fit to the boundary conditions to find a unique solution.

4 Capacitor with dielectric

A dielectric slab fills the half-space given by $x > 0$ and all y, z. The remaining half-space is vacuum. There is an electric field in the vacuum region ($x < 0$) making an angle θ with the interface (the yz-plane). We may choose coordinates so that the electric field in the vacuum region is

$$\mathbf{E}_v = E_0 \sin\theta \hat{\mathbf{i}} + E_0 \cos\theta \hat{\mathbf{j}}$$

Use the boundary conditions for dielectrics to find the electric field in the dielectric, $\mathbf{E}_d$. Express your answer in terms of the tangent of the angle, $\tan\theta'$, that $\mathbf{E}_d$ makes with the yz-plane.

5 Biot-Savart applied to a circular current

When the currents are constant and restricted to wires, the Biot-Savart law is

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi} \int \frac{1}{|\mathbf{r}-\mathbf{r}'|^2} d\mathbf{l}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \end{aligned}$$

Use the law to find the magnetic field at points on the z-axis over a circular loop of wire of radius R lying in the xy-plane and centered at the origin, which carries a constant current I.

6 Applications of the field equations

Consider an infinite cylindrical wire of radius R carrying a current which increases from the center as r^2. If the total current in the wire is I, (a) write the current density, $\mathbf{J}$, then (b) use

$$I_{through\ A} = \iint_A \mathbf{J} \cdot \hat{\mathbf{n}} d^2x$$

together with Ampére's law to find the magnetic field at arbitrary distances r from the wire. Consider both $r < R$ and $r > R$.