Motion in nonuniform fields

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1 Gradient orthogonal to the field

We consider field lines which curve gently away from straight.

1.1 Lorentz force equations

Let the field lines be described by circles of large radius lying in the xy plane. The field is entirely in the φ-direction,

$$\mathbf{B} = B_\varphi \hat{\boldsymbol{\varphi}}$$

This needs to solve the field equations,

$$\begin{aligned}
0 &= \boldsymbol{\nabla}\cdot\mathbf{B} \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\varphi}{\partial\varphi} + \frac{\partial B_z}{\partial z} \\
0 &= \boldsymbol{\nabla}\times\mathbf{B} \\
&= \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial B_z}{\partial\rho} - \rho\frac{\partial B_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho}\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial\rho}(\rho B_\varphi) - \frac{\partial B_\rho}{\partial\varphi}\right)
\end{aligned}$$

so we must have

$$\begin{aligned}
\frac{\partial B_\varphi}{\partial\varphi} &= 0 \\
-\rho\frac{\partial B_\varphi}{\partial z} &= 0 \\
\frac{\partial}{\partial\rho}(\rho B_\varphi) &= 0
\end{aligned}$$

Therefore, we set

$$\mathbf{B} = \frac{B_0 R}{\rho}\hat{\boldsymbol{\varphi}}$$

where a neighborhood of the large constant radius R is where we consider the gently curving field.

The spatial part of the equation of motion is

$$\frac{d\mathbf{v}}{dt} = \frac{q}{mc}\mathbf{v}\times\mathbf{B}$$

where the cross product is given by

$$\mathbf{v}\times\mathbf{B} = -\hat{\boldsymbol{\rho}}\dot{z}B_\varphi + \hat{\mathbf{z}}\dot{\rho}B_\varphi$$

where the velocity and acceleration are given by

$$\begin{aligned}\frac{d}{dt}(\rho\hat{\boldsymbol{\rho}}+z\hat{\mathbf{z}}) &= \dot{\rho}\hat{\boldsymbol{\rho}}+\rho\dot{\varphi}\hat{\boldsymbol{\varphi}}+\dot{z}\hat{\mathbf{z}} \\ \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}(\dot{\rho}\hat{\boldsymbol{\rho}}+\rho\dot{\varphi}\hat{\boldsymbol{\varphi}}+\dot{z}\hat{\mathbf{z}}) \\ &= \ddot{\rho}\hat{\boldsymbol{\rho}}+2\dot{\rho}\dot{\varphi}\hat{\boldsymbol{\varphi}}+\rho\ddot{\varphi}\hat{\boldsymbol{\varphi}}-\rho\dot{\varphi}^2\hat{\boldsymbol{\rho}}+\ddot{z}\hat{\mathbf{z}}\end{aligned}$$

Therefore,

$$\ddot{\rho}\hat{\boldsymbol{\rho}}+2\dot{\rho}\dot{\varphi}\hat{\boldsymbol{\varphi}}+\rho\ddot{\varphi}\hat{\boldsymbol{\varphi}}-\rho\dot{\varphi}^2\hat{\boldsymbol{\rho}}+\ddot{z}\hat{\mathbf{z}}=\frac{q}{mc}\left(-\hat{\boldsymbol{\rho}}v_z\frac{B_0R}{\rho}+\hat{\mathbf{z}}v_\rho\frac{B_0R}{\rho}\right)$$

or, separating components and defining $\omega_B\equiv\frac{qB_0}{mc}$,

$$\begin{aligned}\ddot{\rho}\hat{\boldsymbol{\rho}}-\rho\dot{\varphi}^2\hat{\boldsymbol{\rho}} &= -\omega_B\dot{z}\frac{R}{\rho}\hat{\boldsymbol{\rho}} \\ (\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi})\hat{\boldsymbol{\varphi}} &= 0 \\ \ddot{z}\hat{\mathbf{z}} &= \omega_B\dot{\rho}\frac{R}{\rho}\hat{\mathbf{z}}\end{aligned}$$

1.2 Solving for the motion

Solving, the φ equation gives a conservation law,

$$\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi}=\frac{d}{dt}\left(\rho^2\dot{\varphi}\right)=0$$

so $\rho^2\dot{\varphi}=Rv_\parallel=\mathit{constant}$. The z equation also integrated immediately,

$$\begin{aligned}\ddot{z} &= \omega_BR\frac{\dot{\rho}}{\rho} \\ d\dot{z} &= \omega_BRd(\ln\rho) \\ \dot{z} &= \dot{z}_0+\omega_BR\left(\ln\frac{\rho}{R}\right)\end{aligned}$$

(Jackson is missing a factor of R here). We will return to integrate this equation again once we can approximate ρ.

Now expand ρ near R as

$$\rho=R+x$$

and expand in powers of $\frac{x}{R}\ll 1$,

$$\begin{aligned}\ddot{\rho}-\rho\dot{\varphi}^2 &= -\omega_BR\frac{\dot{z}}{\rho} \\ \ddot{x}-(R+x)\left(\frac{Rv_\parallel}{\rho^2}\right)^2 &= -\omega_BR\frac{\dot{z}_0+\omega_BR\ln\left(1+\frac{x}{R}\right)}{R+x} \\ \ddot{x}-(R+x)\frac{v_\parallel^2}{R^2}\left(1-\frac{4x}{R}\right) &= -\omega_B\left(\dot{z}_0+\omega_BR\ln\frac{x}{R}\right)\left(1-\frac{x}{R}\right) \\ \ddot{x}-\left(1-\frac{3x}{R}\right)\frac{v_\parallel^2}{R} &= -\omega_B\left(\dot{z}_0\left(1-\frac{x}{R}\right)+\omega_Bx\right)\end{aligned}$$

Treating $\dot{z}_0$ as small as well, this becomes

$$\ddot{x}+\left(\omega_B^2+\frac{3v_\parallel^2}{R^2}\right)x=\frac{v_\parallel^2}{R}-\omega_B\dot{z}_0$$

Write $x = \xi - a$ so that

$$\ddot{\xi} + \left(\omega_B^2 + \frac{3v_\parallel^2}{R^2}\right)\xi = \frac{v_\parallel^2}{R} - \omega_B \dot{z}_0 + a\left(\omega_B^2 + \frac{3v_\parallel^2}{R^2}\right)$$

and set

$$\begin{aligned} a &= -\frac{\frac{v_\parallel^2}{R} - \omega_B \dot{z}_0}{\omega_B^2 + \frac{3v_\parallel^2}{R^2}} \\ &= R\frac{\omega_B R\dot{z}_0 - v_\parallel^2}{R^2\omega_B^2 + 3v_\parallel^2} \end{aligned}$$

Then

$$\ddot{\xi} + \left(\omega_B^2 + \frac{3v_\parallel^2}{R^2}\right)\xi = 0$$

is a simple harmonic oscillation and the time-averaged value of $x$is

$$\begin{aligned} \langle x \rangle &= \langle \xi - a \rangle \\ &= -\langle a \rangle \\ &= -R\frac{\omega_B R\dot{z}_0 - v_\parallel^2}{R^2\omega_B^2 + 3v_\parallel^2} \end{aligned}$$

We need the radius of the spiral to be much less than R, where setting the centripetal acceleration equal to the force per unit mass shows that the radius must satisfy

$$\begin{aligned} \frac{v^2}{a} &= \frac{qvB}{mc} \\ \omega_B^2 a &= \omega_B v \\ a &= \frac{v}{\omega_B} \end{aligned}$$

Then large R implies

$$\begin{aligned} a &\ll R \\ v_\parallel &\ll R\omega_B \end{aligned}$$

Using this in the average value of x,

$$\begin{aligned} \langle x \rangle &= -R\frac{\omega_B R\dot{z}_0 - v_\parallel^2}{R^2\omega_B^2 + 3v_\parallel^2} \\ &= \frac{v_\parallel^2 - \omega_B R\dot{z}_0}{\omega_B^2 R} \\ &= \frac{v_\parallel^2}{\omega_B^2 R} - \frac{\dot{z}_0}{\omega_B} \end{aligned}$$

Finally, we return to the second integral of the z equation.

$$\begin{aligned} \dot{z} &= \dot{z}_0 + \omega_B R\left(\ln\frac{\rho}{R}\right) \\ &= \dot{z}_0 + \omega_B R\ln\left(1 + \frac{x}{R}\right) \\ &= \dot{z}_0 + \omega_B x \end{aligned}$$

Therefore, the average $z$velocity is given by,

$$\begin{aligned}\langle \dot{z} \rangle &= \dot{z}_0 + \omega_B \langle x \rangle \\ &= \dot{z}_0 + \omega_B \left(\frac{v_\parallel^2}{\omega_B^2 R} - \frac{\dot{z}_0}{\omega_B} \right) \\ &= \frac{v_\parallel^2}{\omega_B R}\end{aligned}$$

This is the curvature drift. It is orthogonal to both the dominant field lines and the direction in which they curve.

2 Magnetic bottle

2.1 The form of the field

Consider a field in cylindrical coordinates again, but this time let the field lie predominantly in the z-direction with increasing strength. Assuming azimuthal symmetry, the field will satisfy

$$\begin{aligned}0 &= \boldsymbol{\nabla} \cdot \mathbf{B} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial^2 B_z}{\partial z^2} \\ 0 &= \boldsymbol{\nabla} \times \mathbf{B} \\ &= \hat{\boldsymbol{\rho}} \left(\frac{1}{\rho} \frac{\partial B_z}{\partial \rho} - \rho \frac{\partial B_\varphi}{\partial z} \right) + \hat{\boldsymbol{\varphi}} \left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right) + \hat{\mathbf{z}} \left(\frac{\partial}{\partial \rho} (\rho B_\varphi) - \frac{\partial B_\rho}{\partial \varphi} \right)\end{aligned}$$

This time we have $B_\rho (\rho, z)$ and $B_z (\rho, z)$ are nonvanishing, so we have four equations,

$$\begin{aligned}0 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{\partial B_z}{\partial z} \\ 0 &= \frac{1}{\rho} \frac{\partial B_z}{\partial \rho} \\ 0 &= \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \\ 0 &= \frac{\partial B_\rho}{\partial \varphi}\end{aligned}$$

The second equation shows that B_z is a function of z only, and the third then gives $B_\rho = B_\rho (\rho)$. Substituting these into the first, we require

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) = -\frac{\partial B_z}{\partial z} = constant$$

Therfore, for some constants α and β,

$$B_z = \alpha - \beta B_0 z$$

and

$$\begin{aligned}\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) &= \beta B_0 \\ \frac{\partial}{\partial \rho} (\rho B_\rho) &= \beta B_0 \rho \\ \rho B_\rho &= \mu + \frac{1}{2} \beta B_0 \rho^2 \\ B_\rho &= \frac{\mu}{\rho} + \frac{1}{2} \beta B_0 \rho\end{aligned}$$

Let the field at $\rho = z = 0$ be $B_0\hat{\mathbf{k}}$, so that $\mu = 0$ and $\alpha = B_0$. The field is then

$$\mathbf{B} = \frac{1}{2}\beta B_0 \rho \hat{\boldsymbol{\rho}} + B_0 \left(1 - \beta z\right) \hat{\mathbf{k}}$$

everywhere.

2.2 Equations of motion

Let ρ_0 be the radius of the orbit at $z = 0$. We will assume

$$\frac{\dot{\rho}}{\rho_0 \omega_B} \ll 1$$

The Lorentz force law is given in cylindrical coordinates by

$$\frac{d\mathbf{v}}{dt} = \frac{q}{\gamma m c} \mathbf{v} \times \mathbf{B}$$

The acceleration was found above to be

$$\frac{d\mathbf{v}}{dt} = \left(\ddot{\rho} - \rho \dot{\varphi}^2\right) \hat{\boldsymbol{\rho}} + \left(\rho \ddot{\varphi} + 2\dot{\rho}\dot{\varphi}\right) \hat{\boldsymbol{\varphi}} + \left(\ddot{z}\right) \hat{\mathbf{z}}$$

and the cross product is given by

$$\begin{aligned}
\mathbf{v} \times \mathbf{B} &= \left(\dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\varphi}\hat{\boldsymbol{\varphi}} + \dot{z}\hat{\mathbf{k}}\right) \times \left(\frac{1}{2}\beta B_0 \rho \hat{\boldsymbol{\rho}} + B_0 \left(1 - \beta z\right) \hat{\mathbf{k}}\right) \\
&= \frac{1}{2}\beta B_0 \rho \left(\rho\dot{\varphi}\hat{\boldsymbol{\varphi}} + \dot{z}\hat{\mathbf{z}}\right) \times \hat{\boldsymbol{\rho}} + B_0 \left(1 - \beta z\right) \left(\dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\varphi}\hat{\boldsymbol{\varphi}}\right) \times \hat{\mathbf{k}} \\
&= \frac{1}{2}\beta B_0 \rho \left(-\rho\dot{\varphi}\hat{\mathbf{z}} + \dot{z}\hat{\boldsymbol{\varphi}}\right) + B_0 \left(1 - \beta z\right) \left(-\dot{\rho}\hat{\boldsymbol{\varphi}} + \rho\dot{\varphi}\hat{\boldsymbol{\rho}}\right) \\
&= B_0 \left(1 - \beta z\right) \rho\dot{\varphi}\hat{\boldsymbol{\rho}} + B_0 \left(\frac{1}{2}\beta\rho\dot{z} - B_0\dot{\rho} + \beta z\dot{\rho}\right) \hat{\boldsymbol{\varphi}} - \frac{1}{2}\beta B_0 \rho^2 \dot{\varphi}\hat{\mathbf{z}}
\end{aligned}$$

Defining $\omega_B \equiv \frac{qB_0}{\gamma m c}$ and separating components,

$$\begin{aligned}
\ddot{\rho} - \rho\dot{\varphi}^2 &= \left(1 - \beta z\right) \rho \omega_B \dot{\varphi} \\
\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\dot{\varphi}\right) = \rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} &= \frac{1}{2}\omega_B \beta \rho \dot{z} - \omega_B \dot{\rho} + \omega_B \beta z \dot{\rho} \\
\ddot{z} &= -\frac{1}{2}\beta \omega_B \rho^2 \dot{\varphi}
\end{aligned}$$

We will assume

$$\begin{aligned}
\frac{\rho - \rho_0}{\rho_0} &\ll 1 \\
\frac{\dot{\varphi} - |\omega_B|}{|\omega_B|} &\ll 1 \\
\beta &\ll 1
\end{aligned}$$

2.3 Perturbative solution

Expand about the initial conditions. For a positive charge, the orbits about the z-axis will be predominantly in the negative φ direction, so let

$$\begin{aligned} \rho &= \rho_0 + x \\ \dot{\varphi} &= -\omega_B + \omega_1 \end{aligned}$$

Expand the equations of motion, treating β, x and ω_1 as small. For the ρ equation,

$$\begin{aligned} \ddot{x} - (\rho_0 + x)(-\omega_B + \omega_1)^2 &= (1 - \beta z)(\rho_0 + x)\,\omega_B(-\omega_B + \omega_1) \\ \ddot{x} - \rho_0\omega_B^2 - \omega_B^2 x + 2\rho_0\omega_B\omega_1 &= -\rho_0\omega_B^2 + \rho_0\omega_B^2\beta z + \rho_0\omega_B\omega_1 - x\omega_B^2 \\ \ddot{x} &= \rho_0\omega_B^2\beta z - \rho_0\omega_B\omega_1 \end{aligned}$$

For the φ equation,

$$\begin{aligned} \frac{1}{\rho_0 + x}\frac{d}{dt}\left(\left(\rho_0^2 + 2\rho_0 x\right)(-\omega_B + \omega_1)\right) &= \frac{1}{2}\omega_B\beta\rho_0\dot{z} - \omega_B\dot{x} \\ \rho_0\dot{\omega}_1 - 2\dot{x}\omega_B &= \frac{1}{2}\omega_B\beta\rho_0\dot{z} - \omega_B\dot{x} \end{aligned}$$

and finally the z equation is simply

$$\ddot{z} = -\frac{1}{2}\beta\rho_0^2\omega_B^2$$

Collecting these, we now must solve

$$\begin{aligned} \ddot{x} &= \rho_0\omega_B^2\beta z - \rho_0\omega_B\omega_1 \\ \rho_0\dot{\omega}_1 - 2\dot{x}\omega_B &= \frac{1}{2}\omega_B\beta\rho_0\dot{z} - \omega_B\dot{x} \\ \ddot{z} &= -\frac{1}{2}\beta\rho_0^2\omega_B^2 \end{aligned}$$

We integrate the z equation immediately,

$$\begin{aligned} \dot{z} &= \dot{z}_0 - \frac{1}{2}\beta\rho_0^2\omega_B^2 t \\ z &= \dot{z}_0 t - \frac{1}{4}\beta\rho_0^2\omega_B^2 t^2 \end{aligned}$$

This already shows the effect of the magnetic bottle. As long as the approximations are valid, the velocity in the z direction decreases linearly to a turning point where the particle reverses direction back toward the weaker field.

The second equation shows that

$$\begin{aligned} V &\equiv \rho_0\omega_1 - x\omega_B - \frac{1}{2}\omega_B\beta\rho_0 z \\ &= \rho_0\omega_1 - x\omega_B - \frac{1}{2}\omega_B\beta\rho_0\left(\dot{z}_0 t - \frac{1}{4}\beta\rho_0^2\omega_B^2 t^2\right) \end{aligned}$$

remains constant. The initial conditions show that $V = 0$ so we have a solution for ω_1,

$$\omega_1 = \frac{\omega_B}{\rho_0}x + \frac{1}{2}\omega_B\beta\left(\dot{z}_0 t - \frac{1}{4}\beta\rho_0^2\omega_B^2 t^2\right)$$

in terms of x and t.

For the first equation, we substitute

$$\begin{aligned}\ddot{x} &= \rho_0\omega_B^2\beta\left(\dot{z}_0 t-\frac{1}{4}\beta\rho_0^2\omega_B^2t^2\right)-x\omega_B^2-\frac{1}{2}\omega_B^2\beta\rho_0\left(\dot{z}_0 t-\frac{1}{4}\beta\rho_0^2\omega_B^2t^2\right)\\ \ddot{x}+\omega_B^2 x &= \frac{1}{2}\rho_0\omega_B^2\beta\left(\dot{z}_0 t-\frac{1}{4}\beta\rho_0^2\omega_B^2t^2\right)\end{aligned}$$

Therefore, x has an oscillatory part together with a polynomial term. Letting

$$x = A\sin\omega_B t + Bt$$

we substite and drop higher order terms,

$$\begin{aligned}-A\omega_B^2\sin\omega_B t+\omega_B^2\left(A\sin\omega_B t+Bt\right) &= \frac{1}{2}\rho_0\omega_B^2\beta\left(\dot{z}_0 t-\frac{1}{4}\beta\rho_0^2\omega_B^2t^2\right)\\ B\omega_B^2 t &= \frac{1}{2}\rho_0\omega_B^2\beta\dot{z}_0 t\end{aligned}$$

so we require

$$B = \frac{1}{2}\beta\rho_0\dot{z}_0$$

The initial values $x_0=0$ and $\dot{x}_0=0$ the give the constant A,

$$\begin{aligned}\dot{x}_0 &= A\omega_B+\frac{1}{2}\beta\rho_0\dot{z}_0\\ A &= -\frac{1}{2\omega_B}\beta\rho_0\dot{z}_0\end{aligned}$$

so that

$$x=-\frac{\beta\rho_0\dot{z}_0}{2\omega_B}\left(\sin\omega_B t+\omega_B t\right)$$

To first order, we therefore have

$$\begin{aligned}\rho &= \rho_0-\frac{1}{2\omega_B}\beta\rho_0\dot{z}_0\left(\sin\omega_B t+\omega_B t\right)\\ \dot{\varphi} &= -\omega_B-\frac{1}{2}\beta\dot{z}_0\sin\omega_B t+\beta\dot{z}_0\omega_B t\\ z &= \dot{z}_0 t-\frac{1}{4}\beta\rho_0^2\omega_B^2t^2\end{aligned}$$

As noted above, z initially increases as $\dot{z}_0 t$, but slows until it stops at time

$$t_{turn}=\frac{2\dot{z}_0}{\beta\rho_0^2\omega_B^2}$$

and moves back the way it came. At this turning point, the average radius of the orbits has decreased to

$$\begin{aligned}\langle\rho\rangle_{turn} &= \rho_0-\frac{1}{2\omega_B}\beta\rho_0\dot{z}_0\omega_B\frac{2\dot{z}_0}{\beta\rho_0^2\omega_B^2}\\ &= \rho_0-\frac{\dot{z}_0^2}{\rho_0\omega_B^2}\end{aligned}$$

where we have dropped the oscillating term and put t_{turn} for the time.

3 Adiabatic invariants

For periodic classical motions, the action integrals defined by

$$J_{(i)} \equiv \oint \pi_{(i)} dq_{(i)}$$

where $\pi_{(i)}$ is the momentum conjugate to the generalized coordinate $q_{(i)}$, are constants of the motion. Here the integral involves only changes in $q_{(i)}$.

$$\begin{aligned}
\frac{dJ_{(i)}}{dt} &= \oint \frac{d\pi_{(i)}}{dt} dq_{(i)} \\
&= \oint \left[H, \pi_{(i)}\right] dq_{(i)} \\
&= \oint \left(\frac{\partial H}{\partial q^k} \frac{\partial \pi_{(i)}}{\partial \pi_k} - \frac{\partial H}{\partial \pi_k} \frac{\partial \pi_{(i)}}{\partial q^k} \right) dq_{(i)} \\
&= \oint \frac{\partial H}{\partial q^k} \delta_{ik} dq_{(i)} \\
&= \oint \frac{\partial H}{\partial q^i} dq_{(i)} \\
&= \oint dH \\
&= 0
\end{aligned}$$

Even if the Hamiltonian is changing for general motions of the system, the periodicity forces the integral to vanish.

Now suppose the *system* is slowly changed, perhaps by changing the masses, charges or other parameters of the system. If the change is slow compared to the period, then $J_{(i)}$ is *still* constant. This can be used directly to find results for nonuniform fields.