Electromagnetic invariants

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1 Two Lorentz invariants of electromagnetic fields

In solving problems in electromagnetism, it is helpful to know of two quantities which are Lorentz invariant. Given the Faraday tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

and its dual

$$\begin{aligned} \mathcal{F}_{\alpha\beta} &= \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu} \\ &= \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & E^z & -E^y \\ -B^y & -E^z & 0 & E^x \\ -B^z & E^y & -E^x & 0 \end{pmatrix} \end{aligned}$$

we may construct two scalars. The first is $F^{\alpha\beta}F_{\beta\alpha}$

$$\begin{aligned} F^{\alpha\beta}F_{\beta\alpha} &= tr\begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^x \\ E^z & B^y & -B^x & 0 \end{pmatrix} \\ &= 2\left(\mathbf{E}^2 - \mathbf{B}^2\right) \end{aligned}$$

The second is $F^{\alpha\beta}\mathcal{F}_{\beta\alpha}$

$$\begin{aligned} F^{\alpha\beta}\mathcal{F}_{\beta\alpha} &= tr\begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}\begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & -E^z & E^y \\ B^y & E^z & 0 & -E^x \\ B^z & -E^y & E^x & 0 \end{pmatrix} \\ &= 4\mathbf{E}\cdot\mathbf{B} \end{aligned}$$

The first means that if $|\mathbf{E}| > |\mathbf{B}|$ in any one frame of reference, then $|\mathbf{E}| > |\mathbf{B}|$ in *every* frame of reference; if the magnitudes are equal in one frame, they are equal in all, and if $|\mathbf{B}| > |\mathbf{E}|$ in one frame, then $|\mathbf{B}| > |\mathbf{E}|$ in all frames. The second relation tells us the same about the angle between $\mathbf{E}$ and $\mathbf{B}$. If they are orthogonal in one frame, they are orthogonal in all.

2 Motion of charges using invariants

Now suppose we reconsider the motion of a particle in orthogonal $\mathbf{E}$ and $\mathbf{B}$ fields in an inertial frame O. Let $|\mathbf{E}| > |\mathbf{B}|$. Then, viewing the fields from an inertial frame of reference $\tilde{O}$ moving with 3-velocity $\mathbf{u}$, we have the Lorentz force law

$$\frac{d\tilde{p}^{\alpha}}{d\tau} = \frac{q}{c}\tilde{F}^{\alpha\beta}\tilde{u}_{\beta}$$

where

$$\begin{aligned}
\tilde{\mathbf{E}} &= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
\tilde{\mathbf{B}} &= \gamma\left(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\boldsymbol{\beta}
\end{aligned}$$

for $\boldsymbol{\beta} = \frac{\mathbf{u}}{c}$. If $\mathbf{u}$ is orthogonal to both $\mathbf{E}$ and $\mathbf{B}$, then,

$$\begin{aligned}
\tilde{\mathbf{E}} &= \gamma\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right) \\
\tilde{\mathbf{B}} &= \gamma\left(\mathbf{B} - \frac{1}{c}\mathbf{u}\times\mathbf{E}\right)
\end{aligned}$$

Choose coordinates such that

$$\begin{aligned}
\mathbf{E} &= E\hat{\mathbf{i}} \\
\mathbf{B} &= B\hat{\mathbf{j}} \\
\mathbf{u} &= u\hat{\mathbf{k}}
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{\mathbf{E}} &= \gamma\left(E\hat{\mathbf{i}} - \frac{u}{c}B\hat{\mathbf{i}}\right) \\
\tilde{\mathbf{B}} &= \gamma\left(B\hat{\mathbf{j}} - \frac{u}{c}E\hat{\mathbf{j}}\right)
\end{aligned}$$

and since $E > B$ we may choose the magnitude of u so that $\tilde{\mathbf{B}} = 0$. Writing

$$\begin{aligned}
\mathbf{u} &= \frac{cB}{E}\hat{\mathbf{k}} \\
&= \frac{c}{E^2}\mathbf{E}\times\mathbf{B}
\end{aligned}$$

so that

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - \frac{c^2B^2}{c^2E^2}}} \\
&= \frac{E}{\sqrt{E^2 - B^2}}
\end{aligned}$$

we find

$$\begin{aligned}
\tilde{\mathbf{E}} &= \frac{\gamma}{E}\left(E^2 - B^2\right)\hat{\mathbf{i}} \\
&= \frac{E}{\sqrt{E^2 - B^2}}\frac{1}{E}\left(E^2 - B^2\right)\hat{\mathbf{i}} \\
&= \frac{1}{E}\sqrt{E^2 - B^2}\mathbf{E} \\
&= \frac{1}{\gamma}\mathbf{E} \\
\tilde{\mathbf{B}} &= 0
\end{aligned}$$

The motion in $\tilde{O}$ is the same hyperbolic acceleration we found for a constant electric field,

$$\begin{aligned}
\tilde{u}^0 &= c\cosh\frac{q\tilde{E}\tau}{mc} \\
\tilde{u}^x &= c\sinh\frac{q\tilde{E}\tau}{mc} \\
\tilde{u}^y &= 0 \\
\tilde{u}^z &= 0
\end{aligned}$$

and transforming back with a boost by $-\mathbf{u} = -\frac{cB}{E}\hat{\mathbf{k}}$, the 4-velocity in the original frame O is

$$\begin{aligned}
u^\alpha &= \Lambda^\alpha_{\ \beta}\tilde{u}^\beta \\
&= \begin{pmatrix} \gamma & & & \gamma\beta \\ & 1 & & \\ & & 1 & \\ \gamma\beta & & & \gamma \end{pmatrix} \begin{pmatrix} c\cosh\frac{q\tilde{E}\tau}{mc} \\ c\sinh\frac{q\tilde{E}\tau}{mc} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \gamma c\cosh\frac{qE\tau}{\gamma mc} \\ c\sinh\frac{qE\tau}{\gamma mc} \\ 0 \\ \gamma u\cosh\frac{qE\tau}{\gamma mc} \end{pmatrix}
\end{aligned}$$

so the particle now accelerates in the z-direction as well.

A similar argument holds when $B > E$, but now we transform the electric field to zero, $\tilde{\mathbf{E}} = 0$, with a boost

$$\begin{aligned}
\mathbf{u} &= \frac{cE}{B}\hat{\mathbf{k}} \\
&= \frac{c}{B^2}\mathbf{E}\times\mathbf{B}
\end{aligned}$$

The motion in $\tilde{O}$ is that due to a pure magnetic field with the appropriate initial velocity, and boosting back gives a steady drift in the z-direction.