Electromagnetic invariants

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1 Two Lorentz invariants of electromagnetic fields

In solving problems in electromagnetism, it is helpful to know of two quantities which are Lorentz invariant. Given the Faraday tensor

\[ F^{\alpha \beta} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^z \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \]

and its dual

\[ F_{\alpha \beta} = \frac{1}{2} \varepsilon_{\alpha \beta \mu \nu} F^{\mu \nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & E^z & -E^y \\ -B^y & -E^z & 0 & E^x \\ -B^z & E^y & -E^x & 0 \end{pmatrix} \]

we may construct two scalars. The first is \( F^{\alpha \beta} F_{\beta \alpha} \)

\[ F^{\alpha \beta} F_{\beta \alpha} = \text{tr} \left( \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^z \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^z \\ E^z & B^y & -B^x & 0 \end{pmatrix} = 2 (E^2 - B^2) \]

The second is \( F^{\alpha \beta} F_{\beta \alpha} \)

\[ F^{\alpha \beta} F_{\beta \alpha} = \text{tr} \left( \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^z \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & -E^z & E^y \\ B^y & E^z & 0 & -E^x \\ B^z & -E^y & E^x & 0 \end{pmatrix} = 4 E \cdot B \]

The first means that if \( |E| > |B| \) in any one frame of reference, then \( |E| > |B| \) in every frame of reference; if the magnitudes are equal in one frame, they are equal in all, and if \( |B| > |E| \) in one frame, then \( |B| > |E| \) in all frames. The second relation tells us the same about the angle between \( E \) and \( B \). If they are orthogonal in one frame, they are orthogonal in all.
2 Motion of charges using invariants

Now suppose we reconsider the motion of a particle in orthogonal $E$ and $B$ fields in an inertial frame $O$. Let $|E| > |B|$. Then, viewing the fields from an inertial frame of reference $\tilde{O}$ moving with 3-velocity $u$, we have the Lorentz force law

$$\frac{d\vec{p}}{dt} = q\frac{c}{\gamma} \tilde{F}_{\alpha\beta} \tilde{u}_{\beta}$$

where

$$\tilde{E} = \gamma (E + \beta \times B) - \frac{\gamma^2}{\gamma + 1} (\beta \cdot E) \beta$$

$$\tilde{B} = \gamma (B - \beta \times E) - \frac{\gamma^2}{\gamma + 1} (\beta \cdot B) \beta$$

for $\beta = \frac{u}{c}$. If $u$ is orthogonal to both $E$ and $B$, then,

$$\tilde{E} = \gamma \left(E + \frac{1}{c} u \times B\right)$$

$$\tilde{B} = \gamma \left(B - \frac{1}{c} u \times E\right)$$

Choose coordinates such that

$$E = E_i$$

$$B = B_j$$

$$u = u k$$

Then

$$\tilde{E} = \gamma \left(E_i - \frac{u}{c} B_i\right)$$

$$\tilde{B} = \gamma \left(B_j - \frac{u}{c} E_j\right)$$

and since $E > B$ we may choose the magnitude of $u$ so that $\tilde{B} = 0$. Writing

$$u = \frac{c B}{E} \hat{k}$$

$$= \frac{c}{E^2} E \times B$$

so that

$$\gamma = \frac{1}{\sqrt{1 - \frac{c^2 B^2}{E^2}}}$$

$$= \frac{E}{\sqrt{E^2 - B^2}}$$

we find

$$\tilde{E} = \frac{\gamma}{E} (E^2 - B^2) \hat{i}$$

$$= \frac{E}{\sqrt{E^2 - B^2}} E \left(E^2 - B^2\right) \hat{i}$$

$$= \frac{1}{E} \sqrt{E^2 - B^2} E$$

$$= \frac{1}{\gamma} E$$

$$\tilde{B} = 0$$
The motion in $\tilde{O}$ is the same hyperbolic acceleration we found for a constant electric field,

\[
\begin{align*}
\tilde{u}^0 &= c \cosh \frac{q\tilde{E} \tau}{mc} \\
\tilde{u}^x &= c \sinh \frac{q\tilde{E} \tau}{mc} \\
\tilde{u}^y &= 0 \\
\tilde{u}^z &= 0
\end{align*}
\]

and transforming back with a boost by $-u = -\frac{cB}{E}\hat{k}$, the 4-velocity in the original frame $O$ is

\[
\begin{align*}
u^\alpha &= \Lambda^\alpha_\beta \tilde{u}^\beta \\
&= \begin{pmatrix} \gamma & \gamma \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \cosh \frac{q\tilde{E} \tau}{mc} \\ c \sinh \frac{q\tilde{E} \tau}{mc} \end{pmatrix} \\
&= \begin{pmatrix} \gamma c \cosh \frac{q\tilde{E} \tau}{mc} \\ e \sinh \frac{q\tilde{E} \tau}{mc} \\ 0 \\ \gamma u \cosh \frac{q\tilde{E} \tau}{mc} \end{pmatrix}
\end{align*}
\]

so the particle now accelerates in the $z$-direction as well.

A similar argument holds when $B > E$, but now we transform the electric field to zero, $\tilde{E} = 0$, with a boost

\[
\begin{align*}
u &= \frac{cE}{B} \hat{k} \\
&= \frac{c}{B^2} \mathbf{E} \times \mathbf{B}
\end{align*}
\]

The motion in $\tilde{O}$ is that due to a pure magnetic field with the appropriate initial velocity, and boosting back gives a steady drift in the $z$-direction.