Motion of charge particles

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We consider examples of charged particle motion in various fields, with the equation of motion

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}u_\beta$$

1 Constant electric field

Suppose we have a constant, uniform electric field, so that

$$\begin{aligned} F^{0i} &= E^i \\ F^{ij} &= 0 \end{aligned}$$

Then the time component of the Lorentz force law becomes

$$\begin{aligned} \frac{dp^0}{d\tau} &= \frac{q}{c}F^{0i}u_i \\ \frac{\gamma}{c}\frac{dE}{dt} &= \frac{q}{c}\gamma E^i v_i \end{aligned}$$

and the relativistic energy changes as

$$\begin{aligned} \frac{dE}{dt} &= q\mathbf{E}\cdot\mathbf{v} \\ dE &= q\mathbf{E}\cdot\mathbf{v}dt \\ E &= q\int \mathbf{E}\cdot d\mathbf{x} \end{aligned}$$

which is the usual integral of the Newtonian force.

For the spatial components

$$\begin{aligned} \frac{dp^i}{d\tau} &= \frac{q}{c}F^{i\alpha}u_\alpha \\ \frac{dp^i}{d\tau} &= -\frac{q}{c}E^i u_0 \\ \gamma\frac{dp^i}{dt} &= \frac{q}{c}\gamma E^i c \end{aligned}$$

so the change in relativistic momentum is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E}$$

To solve the problem in detail, it is easier to work in the spacetime variables and proper time. Choose the x-axis in the direction of the electric field so the equations of motion become

$$\begin{aligned}\frac{du^0}{d\tau} &= \frac{q}{mc}Eu^x \\ \frac{du^x}{d\tau} &= \frac{q}{mc}Eu^0 \\ \frac{du^y}{d\tau} &= 0 \\ \frac{du^z}{d\tau} &= 0\end{aligned}$$

Differentiate the first and substitute the second,

$$\begin{aligned}\frac{d^2u^0}{d\tau^2} &= \frac{q}{mc}E\frac{du^x}{d\tau} \\ &= \frac{q^2E^2}{m^2c^2}u^0 \\ \frac{d^2u^0}{d\tau^2} - \frac{q^2E^2}{m^2c^2}u^0 &= 0\end{aligned}$$

with the immediate solution

$$u^x = C_1 \exp\frac{qE\tau}{mc} + C_2 \exp\left(-\frac{qE\tau}{mc}\right)$$

It is easier to fit the initial conditions if we set $C_1 = \frac{1}{2}(A_1 + A_2)$ and $C_2 = \frac{1}{2}(A_1 - A_2)$, giving the equivalent form,

$$u^0 = A_1 \cosh\frac{qE\tau}{mc} + A_2 \sinh\frac{qE\tau}{mc}$$

where evaluating at $\tau = 0$ shows that the initial value is $A_1 = u_0^0$.

Then the second equation becomes

$$\frac{du^x}{d\tau} = \frac{qE}{mc}\left(u_0^0 \cosh\frac{qE\tau}{mc} + A_2 \sinh\frac{qE\tau}{mc}\right)$$

with the immediate integral,

$$u^x = u_0^0 \sinh\frac{qE\tau}{mc} + A_2 \cosh\frac{qE\tau}{mc}$$

and the initial value determines $A_2 = u_0^x$

The remaining equations are trivial, so the 4-velocity is given by

$$\begin{aligned}u^0 &= u_0^0 \cosh\frac{qE\tau}{mc} + u_0^x \sinh\frac{qE\tau}{mc} \\ u^x &= u_0^0 \sinh\frac{qE\tau}{mc} + u_0^x \cosh\frac{qE\tau}{mc} \\ u^y &= u_0^y \\ u^z &= u_0^z\end{aligned}$$

Writing $u^\alpha = \frac{dx^\alpha}{d\tau}$, we integrate again and fit the integration constants to the initial conditions to find

$$\begin{aligned}ct &= ct_0 + \frac{mcu_0^0}{qE}\sinh\frac{qE\tau}{mc} + \frac{mcu_0^x}{qE}\left(\cosh\frac{qE\tau}{mc} - 1\right) \\ x &= x_0 + \frac{mcu_0^0}{qE}\left(\cosh\frac{qE\tau}{mc} - 1\right) + \frac{mcu_0^x}{qE}\sinh\frac{qE\tau}{mc} \\ y &= y_0 + u_0^y\tau \\ z &= z_0 + u_0^z\tau\end{aligned}$$

If the particle starts from rest at the origin, then

$$
\begin{aligned}
ct &= \frac{mc^2}{qE}\sinh\frac{qE\tau}{mc} \\
x &= \frac{mc^2}{qE}\left(\cosh\frac{qE\tau}{mc} - 1\right) \\
y &= 0 \\
z &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
u^0 &= c\cosh\frac{qE\tau}{mc} \\
u^x &= c\sinh\frac{qE\tau}{mc} \\
u^y &= 0 \\
u^z &= 0
\end{aligned}
$$

After a proper time τ the 3-velocity is

$$
\begin{aligned}
\frac{v^x}{c} &= \frac{u^x}{u^0} \\
v^x &= c\tanh\frac{qE\tau}{m}
\end{aligned}
$$

2 Constant magnetic field

For a constant magnetic field, we have

$$
\begin{aligned}
F^{0i} &= 0 \\
F^{ij} &= \varepsilon^{ijk}B_k
\end{aligned}
$$

so the time component of the equation of motion is

$$
\frac{1}{c}\frac{dE}{d\tau} = \frac{q}{c}F^{0i}u_i = 0
$$

The energy of the particle remains constant.

For the spatial components,

$$
\begin{aligned}
\frac{dp^i}{d\tau} &= \frac{q}{c}F^{i0}u_0 + \frac{q}{c}F^{ij}u_j \\
\gamma\frac{dp^i}{dt} &= \frac{q}{c}\varepsilon^{ijk}B_k\gamma v_j
\end{aligned}
$$

or simply

$$
\frac{d\mathbf{p}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}
$$

Since E is constant, the magnitude of the velocity does not change so γ is constant. Then with $\mathbf{p} = \gamma m\mathbf{v}$ the equation of motion is

$$
\frac{d\mathbf{v}}{dt} = \mathbf{v}\times\frac{q\mathbf{B}}{mc\gamma}
$$

Define the cyclotron frequency,

$$
\boldsymbol{\omega}_B \equiv \frac{q\mathbf{B}}{mc\gamma}
$$

so that

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_B$$

To solve, choose the z-axis to lie along the magnetic field, and let the yz-plane contain the initial velocity,

$$\begin{aligned} \boldsymbol{\omega}_B &= \omega_B \hat{\mathbf{k}} \\ \mathbf{v}_0 &= v_{0x}\hat{\mathbf{j}} + v_{0z}\hat{\mathbf{k}} \end{aligned}$$

with initial position on the x-axis at x_0

$$\mathbf{x}_0 = x_0\hat{\mathbf{i}}$$

Then

$$\begin{aligned} \frac{dv_x}{dt} &= v_y\omega_B \\ \frac{dv_y}{dt} &= -v_x\omega_B \\ \frac{dv_z}{dt} &= 0 \end{aligned}$$

Differentiating the second,

$$\begin{aligned} \frac{d^2v_y}{dt^2} &= -\frac{dv_x}{dt}\omega_B \\ &= -v_y\omega_B^2 \end{aligned}$$

with the immediate solution (satisfying the initial condition),

$$v_y = v_{0y}\cos\omega_B t$$

The equation for v_x is then

$$\frac{dv_x}{dt} = v_{0y}\omega_B\cos\omega_B t$$

so that

$$v_x = v_{0y}\sin\omega_B t$$

The velocity in the z-direction remains at the initial value.

Integrate again to find the position. Letting the initial value of x be $x_0 = -\frac{v_{0x}}{\omega_B}$:

$$\begin{aligned} x &= -\frac{v_{0x}}{\omega_B}\cos\omega_B t \\ y &= \frac{v_{0x}}{\omega_B}\sin\omega_B t \\ z &= v_{0z}t \end{aligned}$$

The initial position above was chosen so that the motion is a spiral around the z-axis. The projection of the motion into the xy-plane is a circle with radius $\frac{v_{0x}}{\omega_B}$, with velocity v_{0x}. The direction of the motion obeys a left-hand rule: looking down the z-axis, it is clockwise. As γ increases at high velocity, the radius of the circle grows arbitrarily large,

$$R = \frac{v_{0x}mc\gamma}{qB} = \frac{v_{0x}mc}{qB\sqrt{1-\frac{v_0^2}{c^2}}}$$

but does so faster than the nonrelativistic result, $R_{nonrel} = \frac{mv_{0x}}{q(B/c)}$. Unlike the nonrelativistic case, the frequency is velocity dependent,

$$\omega_B = \frac{qB}{mc}\sqrt{1-\frac{v_0^2}{c^2}}$$

decreasing to zero as the speed approaches the speed of light. The more rapidly increasing radius makes it harder to complete a circuit.

3 Orthogonal electric and magnetic fields

Next, consider motion with both electric and magnetic fields, orthogonal to one another,

$$\begin{aligned}\mathbf{E} &= E\hat{\mathbf{i}} \\ \mathbf{B} &= B\hat{\mathbf{j}}\end{aligned}$$

The equation of motion is

$$\begin{aligned}\frac{dp^\alpha}{d\tau} &= \frac{q}{c}F^{\alpha\beta}u_\beta \\ \frac{du^\alpha}{d\tau} &= \frac{q}{mc}\begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix}\begin{pmatrix} -u^0 \\ u^x \\ u^y \\ u^z \end{pmatrix}\end{aligned}$$

and therefore,

$$\begin{aligned}\frac{du^0}{d\tau} &= \frac{qE}{mc}u^x \\ \frac{du^x}{d\tau} &= \frac{q}{mc}\left(Eu^0 - Bu^z\right) \\ \frac{du^y}{d\tau} &= 0 \\ \frac{du^z}{d\tau} &= \frac{qB}{mc}u^x\end{aligned}$$

This time, the energy is changing so that γ is no longer constant. The y equation integrates immediately to give $y = y_0 + u_0^y\tau$.

3.1 Solving for u^x

Differentiate the equation of motion,

$$\begin{aligned}\frac{d^2p^\alpha}{d\tau^2} &= \frac{q}{c}F^{\alpha\beta}\frac{du_\beta}{d\tau} \\ &= \frac{q}{mc}F^{\alpha\beta}\frac{dp_\beta}{d\tau} \\ &= \frac{q}{mc}F^{\alpha\beta}\left(\frac{q}{c}F_{\beta\mu}u^\mu\right) \\ &= \frac{q^2}{m^2c^2}\left(F^{\alpha\beta}F_{\beta\mu}\right)p^\mu\end{aligned}$$

The product matrix is

$$\begin{aligned}F^{\alpha\beta}F_{\beta\mu} &= \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} E^2 & 0 & 0 & -EB \\ 0 & E^2 - B^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ BE & 0 & 0 & -B^2 \end{pmatrix}\end{aligned}$$

The equation in the x direction separates,

$$m\frac{d^2u^x}{d\tau^2} = \frac{q^2}{m^2c^2}\left(E^2 - B^2\right) mu^x$$

so that

$$u^x = A_1 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)$$

3.2 The 4-velocity

Now return to the original equations,

$$\begin{aligned}
\frac{du^0}{d\tau} &= \frac{qE}{mc}u^x \\
\frac{du^x}{d\tau} &= \frac{q}{mc}\left(Eu^0 - Bu^z\right) \\
\frac{du^y}{d\tau} &= 0 \\
\frac{du^z}{d\tau} &= \frac{qB}{mc}u^x
\end{aligned}$$

$$\begin{aligned}
\frac{du^0}{d\tau} &= \frac{qE}{mc}u^x \\
&= \frac{qE}{mc}\left(A_1 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{du^z}{d\tau} &= \frac{qB}{mc}u^x \\
&= \frac{qB}{mc}\left(A_1 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right)
\end{aligned}$$

Integrating to find u^0 and u^z,

$$\begin{aligned}
u^0 &= A_3 + \frac{qE}{mc\frac{q}{mc}\sqrt{E^2 - B^2}}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right) \\
&= A_3 + \frac{E}{\sqrt{E^2 - B^2}}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right) \\
u^z &= A_4 + \frac{B}{\sqrt{E^2 - B^2}}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right)
\end{aligned}$$

Checking this in the remaining equation,

$$\begin{aligned}
0 &= \frac{q}{mc}\left(Eu^0 - Bu^z\right) - \frac{du^x}{d\tau} \\
0 &= \frac{q}{mc}\left(Eu^0 - Bu^z\right) - \frac{q}{mc}\sqrt{E^2 - B^2}\left(A_1 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right) \\
0 &= \frac{q}{mc}\left(EA_3 - A_4B\right) + \frac{q}{mc}\frac{E^2 - B^2}{\sqrt{E^2 - B^2}}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right) \\
&\quad - \frac{q}{mc}\sqrt{E^2 - B^2}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2 - B^2}\right)\right) \\
&= \frac{q}{mc}\left(EA_3 - A_4B\right) \\
A_4 &= \frac{E}{B}A_3
\end{aligned}$$

This gives the 4-velocity as

$$\begin{aligned}
u^0 &= A_3 + \frac{E}{\sqrt{E^2-B^2}}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)\right) \\
u^x &= A_1 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + A_2 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
u^y &= u_0^y \\
u^z &= \frac{E}{B}A_3 + \frac{B}{\sqrt{E^2-B^2}}\left(A_1 \sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + A_2 \cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)\right)
\end{aligned}$$

3.3 Initial conditions of the 4-velocity

At $\tau = 0$,

$$\begin{aligned}
u_0^0 &= A_3 + \frac{E}{\sqrt{E^2-B^2}}A_2 \\
u_0^x &= A_1 \\
u_0^y &= u_0^y \\
u_0^z &= \frac{E}{B}A_3 + \frac{B}{\sqrt{E^2-B^2}}A_2
\end{aligned}$$

so we have $A_1 = u_0^x$ and

$$\begin{aligned}
Eu_0^0 &= EA_3 + \frac{E^2}{\sqrt{E^2-B^2}}A_2 \\
Bu_0^z &= EA_3 + \frac{B^2}{\sqrt{E^2-B^2}}A_2
\end{aligned}$$

Subtracting,

$$\begin{aligned}
Eu_0^0 - Bu_0^z &= \frac{E^2-B^2}{\sqrt{E^2-B^2}}A_2 \\
A_2 &= \frac{Eu_0^0 - Bu_0^z}{\sqrt{E^2-B^2}}
\end{aligned}$$

and therefore,

$$\begin{aligned}
u_0^0 &= A_3 + \frac{E}{\sqrt{E^2-B^2}}\frac{Eu_0^0 - Bu_0^z}{\sqrt{E^2-B^2}} \\
A_3 &= u_0^0 - \frac{E\left(Eu_0^0 - Bu_0^z\right)}{E^2-B^2} \\
&= \frac{-B^2u_0^0 + EBu_0^z}{E^2-B^2} \\
&= -\frac{B\left(Bu_0^0 - Eu_0^z\right)}{E^2-B^2}
\end{aligned}$$

Substituting this into the remaining equation as a check,

$$\begin{aligned}
Bu_0^z &= EA_3 + \frac{B^2}{\sqrt{E^2-B^2}}A_2 \\
&= -\frac{EB}{E^2-B^2}\left(Bu_0^0 - Eu_0^z\right) + \frac{B^2}{\sqrt{E^2-B^2}}\frac{Eu_0^0 - Bu_0^z}{\sqrt{E^2-B^2}}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{EB^2u_0^0 - E^2Bu_0^z}{E^2-B^2} + \frac{B^2Eu_0^0 - B^2Bu_0^z}{E^2-B^2} \\
&= \frac{B^2Eu_0^0 - B^2Bu_0^z - EB^2u_0^0 + E^2Bu_0^z}{E^2-B^2} \\
&= Bu_0^z
\end{aligned}
$$

Substituting into the 4-velocities and collecting terms gives their final form,

$$
\begin{aligned}
u^0 &= \frac{u_0^0}{E^2-B^2}\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - B^2\right) + \frac{E}{\sqrt{E^2-B^2}}u_0^x\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - \frac{EBu_0^z}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right.\right. \\
u^x &= u_0^x\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{Eu_0^0 - Bu_0^z}{\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
u^y &= u_0^y \\
u^z &= \frac{u_0^z}{E^2-B^2}\left(E^2 - B^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)\right) + \frac{B}{\sqrt{E^2-B^2}}u_0^x\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{BEu_0^0}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right.\right.
\end{aligned}
$$

3.4 The position

A second integration gives the positions,

$$
\begin{aligned}
ct &= a_0 + \frac{u_0^0}{E^2-B^2}\left(\frac{mc}{q}\frac{E^2}{\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - B^2\tau\right) + \frac{mc}{q}\frac{E}{E^2-B^2}u_0^x\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - \frac{EBu_0^z}{E^2-B^2}\left(\right. \\
&= a_0 + \frac{B\left(Eu_0^z - Bu_0^0\right)}{E^2-B^2}\tau + \frac{mc}{q}\frac{E}{E^2-B^2}u_0^x\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{mc}{q}\frac{E\left(Eu_0^0 - Bu_0^z\right)}{\left(E^2-B^2\right)^{3/2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
x &= a_1 + \frac{mcu_0^x}{q\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{mc}{q}\frac{Eu_0^0 - Bu_0^z}{E^2-B^2}\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
y &= y_0 + u_0^y\tau \\
z &= a_3 + \frac{E}{E^2-B^2}\left(Eu_0^z - Bu_0^0\right)\tau + \frac{mc}{q}\frac{Bu_0^x}{E^2-B^2}\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{mc}{q}\frac{B\left(Eu_0^0 - Bu_0^z\right)}{\left(E^2-B^2\right)^{3/2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)
\end{aligned}
$$

3.5 Initial position

For the initial positions,

$$
\begin{aligned}
a_0 &= ct_0 - \frac{mc}{q}\frac{E}{E^2-B^2}u_0^x \\
a_1 &= x_0 - \frac{mc}{q}\frac{Eu_0^0 - Bu_0^z}{E^2-B^2} \\
y_0 &= y_0 \\
a_3 &= z_0 - \frac{mc}{q}\frac{Bu_0^x}{E^2-B^2}
\end{aligned}
$$

so the general solution is

$$
\begin{aligned}
ct &= ct_0 + \frac{B\left(Eu_0^z - Bu_0^0\right)}{E^2-B^2}\tau + \frac{mc}{q}\frac{E}{E^2-B^2}u_0^x\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 1\right) + \frac{mc}{q}\frac{E\left(Eu_0^0 - Bu_0^z\right)}{\left(E^2-B^2\right)^{3/2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
x &= x_0 + \frac{mcu_0^x}{q\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{mc}{q}\frac{Eu_0^0 - Bu_0^z}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 1\right) \\
y &= y_0 + u_0^y\tau \\
z &= z_0 + \frac{E}{E^2-B^2}\left(Eu_0^z - Bu_0^0\right)\tau + \frac{mc}{q}\frac{Bu_0^x}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 1\right) + \frac{mc}{q}\frac{B\left(Eu_0^0 - Bu_0^z\right)}{\left(E^2-B^2\right)^{3/2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B\right.
\end{aligned}
$$

3.6 Example

Suppose the particle starts from rest at the origin at $t = \tau = 0$. Then $u_0^\alpha = (c, 0, 0, 0)$ and $x_0^\alpha = 0$ so

$$\begin{aligned}
ct &= \frac{c}{E^2 - B^2}\left(\frac{mc}{q}\frac{E^2}{\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - B^2\tau\right) \\
x &= \frac{mc^2}{q}\frac{E}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 1\right) \\
y &= 0 \\
z &= \frac{BE}{E^2-B^2}\left(\frac{mc^2}{q\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - c\tau\right)
\end{aligned}$$

The 4-velocity is

$$\begin{aligned}
u^0 &= \frac{c}{E^2-B^2}\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - B^2\right) \\
u^x &= \frac{Ec}{\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
u^y &= 0 \\
u^z &= \frac{BEc}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 1\right)
\end{aligned}$$

Check that

$$\begin{aligned}
u^\alpha u_\alpha &= -\left(u^0\right)^2 + \left(u^x\right)^2 + \left(u^z\right)^2 \\
&= -\frac{c^2}{\left(E^2-B^2\right)^2}\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - B^2\right)^2 + \frac{E^2c^2}{E^2-B^2}\sinh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
&\quad + \frac{B^2E^2c^2}{\left(E^2-B^2\right)^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 1\right)^2 \\
&= -\frac{c^2}{\left(E^2-B^2\right)^2}\left(E^4\cosh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 2B^2E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + B^4\right) + \frac{E^2c^2}{E^2-B^2}\sinh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
&\quad + \frac{B^2E^2c^2}{\left(E^2-B^2\right)^2}\left(\cosh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - 2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + 1\right) \\
&= \left(\frac{B^2E^2c^2}{\left(E^2-B^2\right)^2} - \frac{c^2E^4}{\left(E^2-B^2\right)^2}\right)\cosh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{E^2c^2}{E^2-B^2}\sinh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) \\
&\quad + \left(\frac{2B^2E^2c^2}{\left(E^2-B^2\right)^2} - \frac{2B^2E^2c^2}{\left(E^2-B^2\right)^2}\right)\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) + \frac{B^2E^2c^2}{\left(E^2-B^2\right)^2} - \frac{B^4c^2}{\left(E^2-B^2\right)^2} \\
&= \frac{B^2c^2}{E^2-B^2} - \frac{c^2E^2}{E^2-B^2}\left(\cosh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right) - \sinh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)\right) \\
&= -c^2
\end{aligned}$$

as required.

The velocity in the $z$direction is

$$\begin{aligned}
\frac{v_z}{c} &= \frac{dz}{dt} \\
&= \frac{dz/d\tau}{dt/d\tau} \\
&= \frac{u^z}{u^0}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{BEc}{E^2-B^2}\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-1\right)}{\frac{c}{E^2-B^2}\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2\right)} \\
&= \frac{EB\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-1\right)}{E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2}
\end{aligned}
$$

so that

$$
\lim_{\tau\to\infty}\frac{v^z}{c}=\frac{B}{E}
$$

Similarly,

$$
\frac{v^x}{c}=\frac{\frac{Ec}{\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)}{\frac{c}{E^2-B^2}\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2\right)}
$$

so

$$
\lim_{\tau\to\infty}\frac{v^x}{c}=\frac{\sqrt{E^2-B^2}}{E}
$$

and the total asymptotic velocity is

$$
\begin{aligned}
\frac{v}{c} &= \sqrt{(v^x)^2+(v^z)^2} \\
&= \sqrt{\frac{E^2-B^2}{E^2}+\frac{B^2}{E^2}} \\
&= 1
\end{aligned}
$$

so the speed asymptotically approaches the speed of light.

Keeping the next order term,

$$
\begin{aligned}
\left(\frac{v}{c}\right)^2 &= \left(\frac{\frac{Ec}{\sqrt{E^2-B^2}}\sinh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)}{\frac{c}{E^2-B^2}\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2\right)}\right)^2+\left(\frac{EB\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-1\right)}{E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2}\right)^2 \\
&= \frac{E^2\left(E^2-B^2\right)\sinh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)}{\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2\right)^2}+\frac{E^2B^2\left(\cosh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)+1\right)}{\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2\right)^2} \\
&= \frac{1}{\left(E^2\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-B^2\right)^2}\left(E^4\sinh^2\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-2E^2B^2\left(\cosh\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)-1\right)\right)
\end{aligned}
$$

For large τ,

$$
\begin{aligned}
\left(\frac{v}{c}\right)^2 &\approx \frac{1}{E^4\exp\left(\frac{2q\tau}{mc}\sqrt{E^2-B^2}\right)}\left(E^4\exp\left(\frac{2q\tau}{mc}\sqrt{E^2-B^2}\right)-2E^2B^2\exp\left(\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)\right) \\
&= 1-\frac{2B^2}{E^2}\exp\left(-\frac{q\tau}{mc}\sqrt{E^2-B^2}\right)
\end{aligned}
$$

so the velocity becomes exponentially close to the speed of light.