Dynamics of charged particles

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1 The Lorentz force law

We began our study of the relativistic formulation of Maxwell's equations by writing the Lorentz force law in spacetime notation,

$$\frac{dp^{\alpha}}{d\tau}=\frac{q}{c}F^{\alpha\beta}u_{\beta}$$

If we break this into separate space and time components, the time component is

$$\begin{aligned}\frac{dp^{0}}{d\tau} &= \frac{q}{c}F^{0i}u_{i}\\ \frac{1}{c}\frac{dE}{d\tau} &= \frac{q}{c}\mathbf{E}\cdot\frac{d\mathbf{x}}{d\tau}\end{aligned}$$

and therefore,

$$dE=q\mathbf{E}\cdot d\mathbf{x}$$

This is exactly the energy change of a particle of charge q moving through a displacement $d\mathbf{x}$ in an electric field $\mathbf{E}$. Notice that E is the relativistic energy, $E=\gamma mc^{2}$.

For the spatial components,

$$\begin{aligned}\frac{dp^{i}}{d\tau} &= \frac{q}{c}F^{i\beta}u_{\beta}\\ \frac{dp^{i}}{d\tau} &= \frac{q}{c}F^{i0}u_{0}+\frac{q}{c}F^{ij}u_{j}\\ \gamma\frac{d\mathbf{p}}{dt} &= \frac{q}{c}\left(-\mathbf{E}\right)\left(-\gamma c\right)+\frac{q}{c}\varepsilon^{ijk}B_{k}\gamma v_{j}\\ \frac{d\mathbf{p}}{dt} &= q\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\end{aligned}$$

and once again, the momentum is relativistic, $\mathbf{p}=\gamma m\mathbf{v}$, while $\mathbf{v}=\frac{d\mathbf{x}}{dt}$.

2 Lagrangian

In classical mechanics, we may write the Lagrangian for a particle with kinetic energy $T=\frac{1}{2}m\mathbf{v}^{2}$ and potential $V=V\left(\mathbf{x}\right)$ as

$$L=T-V$$

Then the Euler-Lagrange equation of motion, equivalent to Newton's second law, is given by extrema of the action functional

$$S=\int Ldt$$

There are, in fact, infinitely many action functionals with the same extrema, so the Lagrangian is not unique. Rather, we usually seek the simplest action that gives the equation of motion we desire. When the problem turns relativistic the form $L = T - V$ no longer gives the correct equation, so we must start over.

One property we require of any relativistic action is that it be Lorentz invariant. With this in mind, consider the simplest Lorentz invariant quantity, the proper time. The proper time along an arbitrary spacetime path may be written as an integral,

$$\begin{aligned}
\tau &= \int_{\tau_1}^{\tau_2} d\tau \\
&= \int_{\tau_1}^{\tau_2} \sqrt{dt^2 - \frac{1}{c^2} d\mathbf{x}^2} \\
&= \int_{t_1}^{t_2} \sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2}\, dt
\end{aligned}$$

If we vary $\mathbf{x}(t)$ to find the extremal paths, we get

$$\begin{aligned}
0 &= \delta\tau \\
&= \int_{t_1}^{t_2} \delta\sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2}\, dt \\
&= \int_{t_1}^{t_2} \frac{1}{2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\left(-\frac{2}{c^2}\right)\frac{d\mathbf{x}}{dt}\cdot\frac{d\delta\mathbf{x}}{dt} dt \\
&= -\frac{1}{c^2}\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d\mathbf{x}}{dt}\cdot\delta\mathbf{x}\right) - \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d\mathbf{x}}{dt}\right)\cdot\delta\mathbf{x}\right] dt
\end{aligned}$$

The first integral vanishes because $\delta\mathbf{x}$ is required to vanish at the endpoints,

$$-\frac{1}{c^2}\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d\mathbf{x}}{dt}\cdot\delta\mathbf{x}\right) dt = -\left.\frac{1}{c^2}\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d\mathbf{x}}{dt}\cdot\delta\mathbf{x}\right|_{t_1}^{t_2} = 0$$

leaving us with

$$0 = \frac{1}{c^2}\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d\mathbf{x}}{dt}\right)\cdot\delta\mathbf{x} dt$$

Since $\delta\mathbf{x}$ is arbitrary over the range of integration, this vanishes if and only if

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d\mathbf{x}}{dt}\right) = 0$$

at every point. Recalling that $\gamma d\tau = dt$, this may be written as

$$\frac{d^2\mathbf{x}}{d\tau^2} = 0$$

Thus, we have the vanishing of the spatial components of the 4-acceleration, but not the time component.

We may get the time component if we parameterize the curve by an arbitrary parameter λ instead of t. Then with $x^\alpha = (t(\lambda), \mathbf{x}(\lambda))$ we have

$$\begin{aligned}
d\tau^2 &= dt^2 - \frac{1}{c^2} d\mathbf{x}^2 \\
&= \left(\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2 \right) d\lambda^2 \\
d\tau &= \sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2} d\lambda
\end{aligned}$$

Carrying out the variation again,

$$\begin{aligned}
0 &= \delta\tau \\
&= \int_{t_1}^{t_2} \delta \sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2} d\lambda \\
&= \int_{t_1}^{t_2} \frac{1}{2\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \left(2 \left(\frac{dt}{d\lambda} \frac{d\delta t}{d\lambda} \right) - \frac{2}{c^2} \frac{d\mathbf{x}}{d\lambda} \cdot \frac{d\delta\mathbf{x}}{d\lambda} \right) d\lambda \\
&= \int_{t_1}^{t_2} \left[\frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{dt}{d\lambda} \delta t \right) - \frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{dt}{d\lambda} \right) \delta t \right] d\lambda \\
&\quad - \int_{t_1}^{t_2} \left[\frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{1}{c^2} \frac{d\mathbf{x}}{d\lambda} \cdot \delta\mathbf{x} \right) - \frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{1}{c^2} \frac{d\mathbf{x}}{d\lambda} \right) \cdot \delta\mathbf{x} \right] d\lambda
\end{aligned}$$

Again, the total derivatives integrate to the endpoints and vanish, leaving

$$0 = \int_{t_1}^{t_2} \left[-\frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{dt}{d\lambda} \right) \delta t + \frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{1}{c^2} \frac{d\mathbf{x}}{d\lambda} \right) \cdot \delta\mathbf{x} \right] d\lambda$$

The four variations, $\delta t, \delta\mathbf{x}$ are all independent, so we now have four equations:

$$\begin{aligned}
\frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{dt}{d\lambda} \right) &= 0 \\
\frac{d}{d\lambda} \left(\frac{1}{\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2}} \frac{1}{c^2} \frac{d\mathbf{x}}{d\lambda} \right) &= 0
\end{aligned}$$

The parameter λ is arbitrary, and it is simplest to choose it equal to the proper time. Then the square root term is just one,

$$\begin{aligned}
\sqrt{\left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\lambda} \right)^2} &= \sqrt{\left(\frac{dt}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{d\tau} \right)^2} \\
&= \sqrt{\left(\frac{u^0}{c^2} \right)^2 - \frac{1}{c^2} (\mathbf{u})^2}
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{c} \quad \sqrt{\left(u^0\right)^2 - \left(\mathbf{u}\right)^2} \\
&= \tfrac{1}{c} \quad \sqrt{c^2} \\
&= \quad 1
\end{aligned}
$$

and the equations of motion give vanishing 4-acceleration,

$$
\begin{aligned}
\frac{d^2 t}{d\tau^2} &= 0 \\
\frac{d^2 \mathbf{x}}{d\tau^2} &= 0
\end{aligned}
$$

Finally, we may write the whole calculation in spacetime notation. The proper time is

$$
d\tau = \sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}} d\lambda
$$

and the variation is

$$
\begin{aligned}
0 &= \delta\tau \\
&= \int_{t_1}^{t_2} \delta\sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}} d\lambda \\
&= \int_{t_1}^{t_2} \frac{1}{2\sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}}} 2\left(\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{d\delta x^\beta}{d\lambda}\right) d\lambda \\
&= \int_{t_1}^{t_2} \frac{1}{2\sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}}} 2\left(\frac{dx^\alpha}{d\lambda}\frac{d\delta x_\alpha}{d\lambda}\right) d\lambda \\
&= \int_{t_1}^{t_2} \left(\frac{d}{d\lambda}\left(\frac{1}{\sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}}}\left(\frac{dx^\alpha}{d\lambda}\delta x_\alpha\right)\right) - \frac{d}{d\lambda}\left(\frac{1}{\sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}}}\frac{dx^\alpha}{d\lambda}\right)\delta x_\alpha\right) d\lambda \\
&= -\int_{t_1}^{t_2} \frac{d}{d\lambda}\left(\frac{1}{\sqrt{\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}}}\frac{dx^\alpha}{d\lambda}\right)\delta x_\alpha d\lambda
\end{aligned}
$$

where the integrated term still vanishes at the endpoints. The δx_α are arbitrary so, replacing $\lambda = \tau$ as before,

$$
\frac{d^2 x^\alpha}{d\tau^2} = 0
$$

This may also be writting using the momentum,

$$
S = \int_{t_1}^{t_2} \sqrt{p^\alpha p_\alpha} d\tau
$$

and since there is only one term to the final equation, we do not need the square root,

$$
S = \int_{t_1}^{t_2} p^\alpha p_\alpha d\tau
$$

or may mix the 4-velocity and 4-momentum,

$$S = \int_{t_1}^{t_2} u^\alpha p_\alpha d\tau$$

Any of these will reproduce

$$\frac{dp^\alpha}{d\tau} = 0$$

3 Electromagnetic interaction

The problem of writing an action functional becomes more difficult when we wish to include an interaction. Suppose we begin with

$$S = \int_{t_1}^{t_2} \sqrt{-p^\alpha p_\alpha} d\tau$$

and try to add a term proportional to the electric potential $\phi(x^\alpha)$,

$$S = \int_{t_1}^{t_2} \left(\sqrt{-p^\alpha p_\alpha} + a\phi\right) d\tau$$

for some constant a. Then varying, we find

$$\begin{aligned}
0 &= \delta S \\
&= \delta \int_{t_1}^{t_2} \left(\sqrt{-p^\alpha p_\alpha} + a\phi\right) d\tau \\
&= \int_{t_1}^{t_2} \left(-\frac{1}{2\sqrt{-p^\alpha p_\alpha}} 2p_\alpha \delta p^\alpha + a\delta\phi\right) d\tau \\
&= \int_{t_1}^{t_2} \left(-\frac{mp_\alpha}{\sqrt{-p^\alpha p_\alpha}} \frac{d}{d\tau}\delta x^\alpha + a\frac{\partial\phi}{\partial x^\alpha}\delta x^\alpha\right) d\tau \\
&= \int_{t_1}^{t_2} \left(-\frac{d}{d\tau}\left(\frac{mp_\alpha}{\sqrt{-p^\alpha p_\alpha}}\delta x^\alpha\right) + \frac{d}{d\tau}\left(\frac{mp_\alpha}{\sqrt{-p^\alpha p_\alpha}}\right)\delta x^\alpha + a\frac{\partial\phi}{\partial x^\alpha}\delta x^\alpha\right) d\tau
\end{aligned}$$

The first term vanishes, and setting $p_\alpha p^\alpha = -m^2c^2$, we are left with

$$\begin{aligned}
\frac{1}{c}\frac{dp_\alpha}{d\tau} + a\frac{\partial\phi}{\partial x^\alpha} &= 0 \\
\frac{dp_\alpha}{d\tau} &= -ac\frac{\partial\phi}{\partial x^\alpha}
\end{aligned}$$

If we choose $a = \frac{q}{c}$, the spatial terms are correct, but the time component is wrong:

$$\frac{d\mathbf{p}}{d\tau} = -q\nabla\phi$$

but contracting with u^α,

$$\begin{aligned} u^\alpha \frac{dp_\alpha}{d\tau} &= -qu^\alpha \frac{\partial\phi}{\partial x^\alpha} \\ -\frac{1}{2}\frac{d}{d\tau}\left(mc^2\right) &= -qc\frac{\partial\phi}{\partial\tau} \\ 0 &= \frac{\partial\phi}{\partial\tau} \end{aligned}$$

The action works only for electrostatics.

Fortunately, this problem is corrected if we go to the full Lorentz potential instead of just ϕ. Since the 4-potential is a vector, not a scalar, we must contract with another 4-vector to have a scalar. The only other 4-vector in the problem is the 4-velocity, so we try a term proportional to $A_\alpha u^\alpha = -\gamma c\phi + \gamma \mathbf{A}\cdot\mathbf{v}$. To agree with the static result, we the proportionality constant to be $-\frac{q}{c^2}$

This turns out to be the right approach. Writing

$$S = \int_{t_1}^{t_2} \left(\sqrt{-p^\alpha p_\alpha} - \frac{q}{c^2}A_\alpha u^\alpha\right) d\tau$$

we vary

$$\begin{aligned} 0 &= \delta S \\ &= \delta \int_{t_1}^{t_2} \left(\sqrt{-p^\alpha p_\alpha} - \frac{q}{c^2}A_\alpha u^\alpha\right) d\tau \\ &= \int_{t_1}^{t_2} \left(-\frac{2p_\alpha}{2\sqrt{-p^\beta p_\beta}}\delta p^\alpha - \frac{q}{c^2}A_\alpha \delta u^\alpha - \frac{q}{c^2}\delta A_\alpha u^\alpha\right) d\tau \\ &= \int_{t_1}^{t_2} \left(-\frac{mp_\alpha}{\sqrt{-p^\beta p_\beta}}\frac{d}{d\tau}\delta x^\alpha - \frac{q}{c^2}A_\alpha \frac{d}{d\tau}\delta x^\alpha - \frac{q}{c^2}\frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta u^\alpha\right) d\tau \\ &= \int_{t_1}^{t_2} \frac{d}{d\tau}\left(-\frac{mp_\alpha}{\sqrt{-p^\beta p_\beta}}\delta x^\alpha - \frac{q}{c^2}A_\alpha \delta x^\alpha\right) d\tau \\ &\quad + \int_{t_1}^{t_2} \left(\frac{d}{d\tau}\left(\frac{mp_\alpha}{\sqrt{-p^\beta p_\beta}}\right)\delta x^\alpha + \frac{q}{c^2}\frac{dA_\alpha}{d\tau}\delta x^\alpha - \frac{q}{c^2}\frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta u^\alpha\right) d\tau \end{aligned}$$

We discard the two total derivative terms, leaving

$$0 = \int_{t_1}^{t_2} \left(\frac{d}{d\tau}\left(\frac{mp_\alpha}{\sqrt{-p^\mu p_\mu}}\right) + \frac{q}{c^2}\frac{dA_\alpha}{d\tau} - \frac{q}{c^2}\frac{\partial A_\beta}{\partial x^\alpha}u^\beta\right)\delta x^\alpha d\tau$$

Since the variation is arbitrary, the term in parentheses must vanish. Setting $\sqrt{-p^\mu p_\mu} = mc$, the equation of motion is

$$\frac{1}{c}\frac{dp_\alpha}{d\tau} = -\frac{q}{c^2}\left(\frac{dA_\alpha}{d\tau} - \frac{\partial A_\beta}{\partial x^\alpha}u^\beta\right)$$

Expanding with the chain rule,

$$\frac{dA_\alpha}{d\tau} = \frac{\partial A_\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau} = \frac{\partial A_\alpha}{\partial x^\beta}u^\beta$$

we have

$$
\begin{aligned}
\frac{1}{c}\frac{dp_\alpha}{d\tau} &= -\frac{q}{c^2}\left(\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha}\right) u^\beta \\
\frac{dp_\alpha}{d\tau} &= \frac{q}{c} F_{\alpha\beta} u^\beta
\end{aligned}
$$

Adjusting index positions we have the Lorentz force law,

$$
\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} u_\beta
$$

We therefore have a satisfactory action,

$$
S = \int\limits_{t_1}^{t_2} \left(\sqrt{-p^\alpha p_\alpha} - \frac{q}{c^2} A_\alpha u^\alpha\right) d\tau
$$