1 Lorentz transformation of the Maxwell equations

1.1 The transformations of the fields

Now that we have written the Maxwell equations in covariant form, we know exactly how they transform under Lorentz transformations. Consider a boost in the x-direction, from $\mathcal{O}$ to $\tilde{\mathcal{O}}$ given by the transformation matrix

$$M^{\alpha}_{\ \beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, since the Faraday tensor is a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor, it transforms as

$$\begin{aligned} \tilde{F}^{\alpha\beta} &= M^{\alpha}_{\ \mu} M^{\alpha}_{\ \nu} F^{\mu\nu} \\ &= M^{\alpha}_{\ \mu} F^{\mu\nu} M^{\alpha}_{\ \nu} \\ &= M^{\alpha}_{\ \mu} F^{\mu\nu} \left[M^{t}\right]_{\nu}^{\ \alpha} \end{aligned}$$

where the rearrangement may now be written as the matrix product,

$$\tilde{F} = MFM^{t}$$

We find,

$$\begin{aligned} \tilde{F}^{\alpha\beta} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma\beta E^1 & \gamma E^1 & E^2 & E^3 \\ -\gamma E^1 & \gamma\beta E^1 & -B^3 & B^2 \\ -\gamma E^2 - \gamma\beta B^3 & \gamma\beta E^2 + \gamma B^3 & 0 & -B^1 \\ -\gamma E^3 + \gamma\beta B^2 & \gamma\beta E^3 - \gamma B^2 & B^1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \left(\gamma^2 - \gamma^2\beta^2\right) E^1 & \gamma E^2 + \gamma\beta B^3 & \gamma E^3 - \gamma\beta B^2 \\ \left(\gamma^2\beta^2 - \gamma^2\right) E^1 & 0 & -\gamma\beta E^2 - \gamma B^3 & -\gamma\beta E^3 + \gamma B^2 \\ -\gamma E^2 - \gamma\beta B^3 & \gamma\beta E^2 + \gamma B^3 & 0 & -B^1 \\ -\gamma E^3 + \gamma\beta B^2 & \gamma\beta E^3 - \gamma B^2 & B^1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E^1 & \gamma\left(E^2 + \beta B^3\right) & \gamma\left(E^3 - \beta B^2\right) \\ -E^1 & 0 & -\gamma\left(B^3 + \beta E^2\right) & \gamma\left(B^2 - \beta E^3\right) \\ -\gamma\left(E^2 + \beta B^3\right) & \gamma\left(B^3 + \beta E^2\right) & 0 & -B^1 \\ -\gamma\left(E^3 - \beta B^2\right) & -\gamma\left(B^2 - \beta E^3\right) & B^1 & 0 \end{pmatrix} \end{aligned}$$

Comparing components with

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & \tilde{E}^1 & \tilde{E}^2 & \tilde{E}^3 \\ -\tilde{E}^1 & 0 & -\tilde{B}^3 & \tilde{B}^2 \\ -\tilde{E}^2 & \tilde{B}^3 & 0 & -\tilde{B}^1 \\ -\tilde{E}^3 & -\tilde{B}^2 & \tilde{B}^1 & 0 \end{pmatrix}$$

we see that

$$\begin{aligned} \tilde{E}^1 &= E^1 \\ \tilde{E}^2 &= \gamma\left(E^2 - \beta B^3\right) \end{aligned}$$

$$
\begin{aligned}
\tilde{E}^3 &= \gamma\left(E^3 + \beta B^2\right) \\
\tilde{B}^1 &= B^1 \\
\tilde{B}^2 &= \gamma\left(B^2 + \beta E^3\right) \\
\tilde{B}^3 &= \gamma\left(B^3 - \beta E^2\right)
\end{aligned}
$$

which we can write vectorially in terms of the components of $\mathbf{E}$ and $\mathbf{B}$ parallel and perpenducular to $\boldsymbol{\beta}$,

$$
\begin{aligned}
\mathbf{E}_{\parallel} &= \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
\mathbf{E}_{\perp} &= \mathbf{E} - \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}
\end{aligned}
$$

and similarly for $\mathbf{B}$. Notice that with $\boldsymbol{\beta} = \beta\hat{\mathbf{i}}$ and $\mathbf{B}_{\perp}$ in the yz plane,

$$
\begin{aligned}
\boldsymbol{\beta}\times\mathbf{B}_{\perp} &= \hat{\mathbf{i}}\left(\beta^2 B^3 - \beta^3 B^2\right) + \hat{\mathbf{j}}\left(\beta^3 B^1 - \beta^1 B^3\right) + \hat{\mathbf{k}}\left(\beta^1 B^2 - \beta^2 B^1\right) \\
&= \hat{\mathbf{j}}\left(-\beta B^3\right) + \hat{\mathbf{k}}\left(\beta B^2\right)
\end{aligned}
$$

so that the transformation of the parallel and perpendicular parts of the fields may be written as

$$
\begin{aligned}
\tilde{\mathbf{E}}_{\parallel} &= \mathbf{E}_{\parallel} \\
\tilde{\mathbf{E}}_{\perp} &= \gamma\mathbf{E}_{\perp} + \gamma\boldsymbol{\beta}\times\mathbf{B}_{\perp} \\
\tilde{\mathbf{B}}_{\parallel} &= \mathbf{B}_{\parallel} \\
\tilde{\mathbf{B}}_{\perp} &= \gamma\mathbf{B}_{\perp} - \gamma\boldsymbol{\beta}\times\mathbf{E}_{\perp}
\end{aligned}
$$

and we may reconstruct the full vectors,

$$
\begin{aligned}
\tilde{\mathbf{E}} &= \tilde{\mathbf{E}}_{\parallel} + \tilde{\mathbf{E}}_{\perp} \\
&= \mathbf{E}_{\parallel} + \gamma\mathbf{E}_{\perp} + \gamma\boldsymbol{\beta}\times\mathbf{B}_{\perp} \\
&= \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} + \gamma\left(\mathbf{E} - \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}\right) + \gamma\boldsymbol{\beta}\times\left(\mathbf{B} - \frac{1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\boldsymbol{\beta}\right) \\
&= \gamma\mathbf{E} + \left(\frac{1-\gamma}{\beta^2}\right)\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} + \gamma\boldsymbol{\beta}\times\mathbf{B} \\
&= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) + \left(\frac{1-\gamma}{\beta^2}\right)\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}
\end{aligned}
$$

Now, with

$$
\begin{aligned}
\gamma^2 &= \frac{1}{1-\beta^2} \\
1-\beta^2 &= \frac{1}{\gamma^2} \\
\beta^2 &= 1 - \frac{1}{\gamma^2}
\end{aligned}
$$

this becomes

$$
\begin{aligned}
\tilde{\mathbf{E}} &= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) + \frac{1-\gamma}{1-\frac{1}{\gamma^2}}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
&= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) + \gamma^2\frac{1-\gamma}{\gamma^2-1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta} \\
&= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}
\end{aligned}
$$

Similarly, for the transformed magnetic field

$$\begin{aligned}
\tilde{\mathbf{B}} &= \mathbf{B}_{\parallel} + \gamma \mathbf{B}_{\perp} - \gamma \boldsymbol{\beta} \times \mathbf{E}_{\perp} \\
&= \frac{1}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta} + \gamma \left(\mathbf{B} - \frac{1}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta} \right) - \gamma \boldsymbol{\beta} \times \left(\mathbf{E} - \frac{1}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{E}) \boldsymbol{\beta} \right) \\
&= \gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) + \frac{1-\gamma}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta} \\
&= \gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta}
\end{aligned}$$

so the complete transformation laws are

$$\begin{aligned}
\tilde{\mathbf{E}} &= \gamma (\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{E}) \boldsymbol{\beta} \\
\tilde{\mathbf{B}} &= \gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta}
\end{aligned}$$

These now hold for any directions of the boost and fields.

1.2 Example: Pure electric field from a rapidly moving frame

Suppose we have a pure electric field in the original frame, with $\mathbf{B} = 0$. Then in $\tilde{\mathcal{O}}$, as the speed approaches the speed of light, $\boldsymbol{\beta}$ is essentially a unit vector, $\boldsymbol{\beta} \to \hat{\boldsymbol{\beta}}$, $\frac{\gamma^2}{\gamma+1} \to \gamma$, and the electric field approaches

$$\begin{aligned}
\tilde{\mathbf{E}} &= \gamma \mathbf{E} - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{E}) \boldsymbol{\beta} \\
&\longrightarrow \gamma \left(\mathbf{E} - \left(\hat{\boldsymbol{\beta}} \cdot \mathbf{E} \right) \hat{\boldsymbol{\beta}} \right) \\
&= \gamma \mathbf{E}_{\perp}
\end{aligned}$$

so the electric field flattens into the plane orthogonal to the motion. At the same time, the magnetic field approaches

$$\tilde{\mathbf{B}} = -\gamma \hat{\boldsymbol{\beta}} \times \mathbf{E}$$

which also lies in the orthogonal plane, and is perpendicular to $\mathbf{E}$. The Poynting vector of the field in the rapidly moving frame is

$$\begin{aligned}
\tilde{\mathbf{S}} &= \frac{1}{\mu_0} \tilde{\mathbf{E}} \times \tilde{\mathbf{B}} \\
&= -\frac{\gamma^2}{\mu_0} \mathbf{E}_{\perp} \times \left(\hat{\boldsymbol{\beta}} \times \mathbf{E} \right) \\
&= -\frac{\gamma^2}{\mu_0} \left(E_{\perp}^2 \hat{\boldsymbol{\beta}} - \mathbf{E} \left(\hat{\boldsymbol{\beta}} \cdot \mathbf{E}_{\perp} \right) \right) \\
&= -\frac{\gamma^2}{\mu_0} E_{\perp}^2 \hat{\boldsymbol{\beta}}
\end{aligned}$$

where we have used the BAC-CAB rule on the triple cross product. Not surprisingly, we find a strong flux opposing the observer's motion.

2 Thomas precession and the BMT equation

First consider orbital angular momentum,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

We may immediately generalize this to spacetime by defining

$$L^{\alpha\beta} \equiv x^\alpha p^\beta - x^\beta p^\alpha$$

where x^α is the position 4-vector of a particle and p^α is its momentum. Consider the dual,

$$\begin{aligned} \mathcal{L}_{\mu\nu} &\equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}L^{\alpha\beta} \\ &= \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}x^\alpha p^\beta \end{aligned}$$

and contract with $\frac{1}{c}$ times the *observer's* 4-velocity,

$$\begin{aligned} L_\nu &\equiv \frac{1}{c}u^\mu \mathcal{L}_{\mu\nu} \\ &= \frac{1}{c}u^\mu \varepsilon_{\mu\nu\alpha\beta}x^\alpha p^\beta \end{aligned}$$

Then in the rest frame of the observer, where $u^\mu = (c, \mathbf{0})$, L_ν becomes

$$\begin{aligned} L_\nu &\equiv \frac{1}{c}u^\mu \mathcal{L}_{\mu\nu} \\ &= \varepsilon_{0\nu\alpha\beta}x^\alpha p^\beta \\ &= \left(0, \varepsilon_{ijk}x^j p^k\right) \\ &= (0, \mathbf{L}) \end{aligned}$$

This shows that we expect angular momentum to be described by a spacelike vector, and we'll treat the spin vector in the same way. This property, of reducing to a purely spacelike vector in the rest frame, may be characterized by orthogonality to the 4-velocity, $u^\alpha L_\alpha = 0$.

We are now in a position to consider the fully relativistic evolution of spin angular momentum. We know that the equation of motion of the 3-dimensional spin vector, $\mathbf{s}$, in the rest frame $\tilde{\mathcal{O}}$ of the particle is

$$\frac{d\mathbf{s}}{d\tilde{t}} = \frac{ge}{2mc}\mathbf{s} \times \tilde{\mathbf{B}}$$

and want to generalize this to a covariant expression. We begin by generalizing the spin 3-vector to a spin 4-vector, which we assume reduces to

$$\tilde{s}^\alpha = (0, \mathbf{s})$$

in the rest frame $\tilde{\mathcal{O}}$, or equivalently,

$$\tilde{u}_\alpha \tilde{s}^\alpha = 0$$

and this relation is invariant.

To generalize the rest frame equation, we will clearly want the proper time derivative of the 4-vector spin

$$\frac{ds^\alpha}{d\tau}$$

expressed in terms of other 4-vectors. Rather than trying to transform the known expression, we try to write the most general covariant expression we can that reduces to the known non-relativistic expression. The only tensors relevant to the problem are

$$F^{\alpha\beta}, u^\alpha, s^\alpha, \frac{du^\alpha}{d\tau}$$

Notice that the 4-acceleration, $a^\alpha = \frac{du^\alpha}{d\tau}$, has a part expressible in terms of the first two but may also depend on non-electromagnetic forces, so we need to consider it separately. Our goal is to write $\frac{ds^\alpha}{d\tau}$ in terms of these. Looking at the classical limit in the rest frame, we expect that the correct expression is at most linear in the fields $F^{\alpha\beta}$, and each term is linear in the spin vector. Then there are only three possibile 4-vectors we can construct:

$$F^{\alpha\beta}s_\beta$$
$$\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha$$
$$\left(s_\beta \frac{du^\beta}{d\tau}\right)u^\alpha$$

(Notice that $\left(u^\beta s_\beta\right)u^\alpha = 0$, $F^{\alpha\beta}u_\beta$ is independent of s^α, and $\left(F^{\mu\nu}u_\mu u_\nu\right)u^\alpha = 0$. Since the acceleration is already linear in $F^{\alpha\beta}$ we do not consider terms involving $F^{\alpha\beta}a_\beta$).

We begin with an arbitrary linear combination of these, and ask what combination gives the right behavior in the particle rest frame,

$$\frac{ds^\alpha}{d\tau} = AF^{\alpha\beta}s_\beta + B\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha + C\left(s_\beta\frac{du^\beta}{d\tau}\right)u^\alpha \tag{1}$$

First, notice that since $u^\alpha s_\alpha = 0$, it follows that

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(u^\alpha s_\alpha\right) \\ &= \frac{du^\alpha}{d\tau}s_\alpha + u^\alpha\frac{ds_\alpha}{d\tau} \end{aligned}$$

so that

$$u^\alpha\frac{ds_\alpha}{d\tau} = -s_\alpha\frac{du^\alpha}{d\tau} \tag{2}$$

Rewrite this using the Lorentz force law, combined with other non-electromagnetic forces, F^α

$$m\frac{du^\alpha}{d\tau} = F^\alpha + \frac{e}{c}u_\beta F^{\alpha\beta}$$

Then the contraction of the rate of change of spin with the 4-velocity is

$$\begin{aligned} u^\alpha\frac{ds_\alpha}{d\tau} &= -s_\alpha\frac{du^\alpha}{d\tau} \\ &= -\frac{1}{m}s_\alpha\left(F^\alpha + \frac{e}{c}F^{\alpha\beta}u_\beta\right) \end{aligned}$$

Take this same contraction with our ansatz, eq.(1),

$$\begin{aligned} \frac{ds^\alpha}{d\tau} &= AF^{\alpha\beta}s_\beta + B\left(s_\mu F^{\mu\nu}u_\nu\right)u^\alpha + C\left(s_\beta\frac{du^\beta}{d\tau}\right)u^\alpha \\ u_\alpha\frac{ds^\alpha}{d\tau} &= Au_\alpha F^{\alpha\beta}s_\beta + B\left(s_\mu F^{\mu\nu}u_\nu\right)u_\alpha u^\alpha + C\left(s_\beta\frac{du^\beta}{d\tau}\right)u_\alpha u^\alpha \\ &= Au_\alpha F^{\alpha\beta}s_\beta - c^2B\left(s_\mu F^{\mu\nu}u_\nu\right) - c^2C\left(s_\beta\frac{du^\beta}{d\tau}\right) \end{aligned}$$

and substituting on the left and in the final term,

$$-\frac{1}{m}s_\alpha\left(F^\alpha + \frac{e}{mc}u_\beta F^{\beta\alpha}\right) = Au_\alpha F^{\alpha\beta}s_\beta - c^2B\left(s_\mu F^{\mu\nu}u_\nu\right) - \frac{c^2}{m}C\left(s_\beta\left(F^\beta + \frac{e}{mc}u_\alpha F^{\alpha\beta}\right)\right)$$

Since the non-electromagnetic force F^α is independent of the electromagnetic force, we may split this into two independent equations,

$$
\begin{aligned}
-\frac{1}{m} s_\alpha F^\alpha &= -\frac{c^2}{m} C s_\beta F^\beta \\
-\frac{e}{m^2 c} u_\beta F^{\beta\alpha} s_\alpha &= A u_\alpha F^{\alpha\beta} s_\beta - c^2 B \left(s_\mu F^{\mu\nu} u_\nu\right) - \frac{c^2}{m} C \left(\frac{e}{mc} u_\alpha F^{\alpha\beta} s_\beta\right)
\end{aligned}
$$

The first of these immediately gives $C = \frac{1}{c^2}$, while the second becomes

$$
\begin{aligned}
0 &= \left(A + c^2 B - \frac{c^2}{m} C \frac{e}{mc} + \frac{e}{m^2 c}\right) u_\beta F^{\beta\alpha} s_\alpha \\
&= \left(A + c^2 B\right) u_\beta F^{\beta\alpha} s_\alpha
\end{aligned}
$$

With these restrictions, the equation of motion reduces to

$$
\frac{ds^\alpha}{d\tau} = A F^{\alpha\beta} s_\beta - \frac{A}{c^2} \left(s_\mu F^{\mu\nu} u_\nu\right) u^\alpha + \frac{1}{c^2} \left(s_\beta \frac{du^\beta}{d\tau}\right) u^\alpha
$$

Now consider the limit to the rest frame, in the case of a pure magnetic field, so $\mathbf{E} = \mathbf{0}$ in $F^{\alpha\beta}$, and the 4-velocity is given by

$$
u^\alpha = (c, \mathbf{0})
$$

Then $s_\mu F^{\mu\nu} u_\nu = -c s_i F^{i0} = c s_i E^i = 0$ so the middle term drops out. Then the time component simply reproduces eq.(2), while the spatial components give

$$
\begin{aligned}
\frac{ds^i}{dt} &= A F^{ij} s_j \\
&= A \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B^3 & -B^2 \\ 0 & -B^3 & 0 & B^1 \\ 0 & B^2 & -B^1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ s^1 \\ s^2 \\ s^3 \end{pmatrix} \\
&= A \begin{pmatrix} 0 \\ B^3 s^2 - B^2 s^3 \\ B^1 s^3 - B^3 s^1 \\ B^2 s^1 - B^1 s^2 \end{pmatrix}
\end{aligned}
$$

or simply

$$
\frac{d\mathbf{s}}{dt} = A \left(\mathbf{s} \times \mathbf{B}\right)
$$

and comparing this to

$$
\frac{d\mathbf{s}}{dt} = \frac{ge}{2mc} \mathbf{s} \times \mathbf{B}
$$

we see that

$$
A = \frac{ge}{2mc}
$$

and using the Lorentz force law $\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^{\alpha\beta} u_\beta$ to substitute the electromagnetic force for the acceleration, the full equation of motion for $\mathbf{s}$ becomes

$$
\begin{aligned}
\frac{ds^\alpha}{d\tau} &= \frac{ge}{2mc} F^{\alpha\beta} s_\beta - \frac{ge}{2mc^3} \left(s_\mu F^{\mu\nu} u_\nu\right) u^\alpha + \frac{1}{c^2} \left(s_\beta \frac{du^\beta}{d\tau}\right) u^\alpha \\
&= \frac{ge}{2mc} F^{\alpha\beta} s_\beta - \frac{ge}{2mc^3} \left(s_\mu F^{\mu\nu} u_\nu\right) u^\alpha + \frac{1}{c^2} \left(s_\beta \left(\frac{e}{mc} F^{\alpha\beta} u_\beta\right)\right) u^\alpha \\
&= \frac{ge}{2mc} F^{\alpha\beta} s_\beta - \frac{(g-2)\, e}{2mc^3} \left(s_\mu F^{\mu\nu} u_\nu\right) u^\alpha
\end{aligned}
$$

This is the BMT equation. The signs differ from Jackson because we use the opposite convention for the sign of the metric, $\eta_{\alpha\beta}$.