# Formal derivation of the Lorentz transformations (optional)

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## 1 Lorentz transformations

Here we develop the form of Lorentz transformations using techniques from the study of Lie algebras. We find the general form of infinitesimal transformations, then apply those infinitely many times to develop the full set of finite transformations.

We define Lorentz transformations to be those linear transformations of the spacetime coordinates $x^\alpha = (ct, x, y, z)$ , $\alpha = 0, 1, 2, 3$ for which

$$\begin{aligned} ds^2 &= \eta_{\alpha\beta} x^\alpha x^\beta \\ &= -c^2 t^2 + x^2 + y^2 + z^2 \end{aligned}$$

or equivalently, those linear transformations that preserve the wave equation. Either of these specifications leads to the necessary and sufficient condition

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{1}$$

### 1.1 Infinitesimal transformations

We find all Lorentz transformations by first considering infinitesmal ones,

$$\Lambda^\mu{}_\alpha = \delta^\mu_\alpha + \varepsilon^\mu{}_\alpha$$

The defining condition becomes

$$\begin{aligned} \eta_{\mu\nu} \left(\delta^\mu_\alpha + \varepsilon^\mu{}_\alpha\right) \left(\delta^\nu_\beta + \varepsilon^\nu{}_\beta\right) &= \eta_{\alpha\beta} \\ \eta_{\mu\nu} \delta^\mu_\alpha \delta^\nu_\beta + \eta_{\mu\nu} \delta^\mu_\alpha \varepsilon^\nu{}_\beta + \eta_{\mu\nu} \varepsilon^\mu{}_\alpha \delta^\nu_\beta + \eta_{\mu\nu} \varepsilon^\mu{}_\alpha \varepsilon^\nu{}_\beta &= \eta_{\alpha\beta} \\ \eta_{\alpha\beta} + \eta_{\alpha\nu} \varepsilon^\nu{}_\beta + \eta_{\mu\beta} \varepsilon^\mu{}_\alpha + O\left(\varepsilon^2\right) &= \eta_{\alpha\beta} \end{aligned}$$

Dropping the higher order term, cancelling $\eta_{\alpha\beta}$, and defining

$$\varepsilon_{\alpha\beta} \equiv \eta_{\alpha\mu} \varepsilon^\mu{}_\beta$$

we have

$$\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = 0$$

so that $\varepsilon_{\alpha\beta}$ must be antisymmetric.

The most general antisymmetric $4 \times 4$ matrix may be written as

$$\begin{pmatrix} 0 & a_3 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix} = a_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{aligned}
&+\cdots+b_1\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
=\ & a^i K_i + b^i J_i
\end{aligned}
$$

where

$$
\begin{aligned}
a^i &= (a_1, a_2, a_3) \\
b^i &= (b_1, b_2, b_3)
\end{aligned}
$$

and

$$
\begin{aligned}
[K_1]_{\alpha\beta} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
[K_2]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
[K_3]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

and finally,

$$
\begin{aligned}
[J_1]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{J}_1 \end{pmatrix} \\
[J_2]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{J}_2 \end{pmatrix} \\
[J_3]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{J}_3 \end{pmatrix}
\end{aligned}
$$

The $3 \times 3$ matrices $\tilde{J}_i$ have components given by the Levi-Civita tensor,

$$
\left[\tilde{J}_i\right]_{jk} = \varepsilon_{ijk}
$$

To compute the transformations, we need to raise the first index of $K_i$ and $J_i$ using the inverse metric. The three $K_i$ change:

$$
\begin{aligned}
[K_1]^{\alpha}_{\ \beta} &= \eta^{\alpha\mu} [K_1]_{\mu\beta} \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
[K_2]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
[K_3]_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

while the $J_i$ stay the same, that is, $[J_i]^{\alpha}_{\ \beta}$ has identical components to $[J_i]_{\alpha\beta}$. The transformations generated by the $J_i$ with therefore be different from those generated by $K_i$.

# 2 Finite transformations

We build up finite transformations as the limit of an infinite number of infinitesmial transformations.

## 2.1 Rotations

Consider the transformations involving $J_i$ first. A general, infinitesimal $J_i$-type transformation is given by

$$\Lambda = 1 + \mathbf{b} \cdot \mathbf{J}$$

To find a *finite* transformation we take the limit of many infinitesimal ones. Write the infinitesimal vector $\mathbf{b}$ as $\mathbf{b} = \varepsilon \mathbf{n}$ where $\mathbf{n}$ is a unit vector. Then define the finite transformation,

$$\Lambda(\mathbf{n}, \theta) = \lim_{n \to \infty} (1 + \varepsilon \mathbf{n} \cdot \mathbf{J})^n$$

where we take the limit in such a way that $\varepsilon n \to \theta$. To evaluate this we use the binomial theorem,

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k$$

to write

$$
\begin{aligned}
\Lambda(\mathbf{n}, \theta) &= \lim_{n \to \infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} 1^{n-k} (\varepsilon \mathbf{n} \cdot \mathbf{J})^k \\
&= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \varepsilon^k (\mathbf{n} \cdot \mathbf{J})^k
\end{aligned}
$$

Now we need powers of $\mathbf{n} \cdot \mathbf{J}$. This is easiest if we focus on the nontrivial $3 \times 3$ part, since

$$(\mathbf{n} \cdot \mathbf{J})^n = \begin{pmatrix} 0 & 0 \\ 0 & \left(\mathbf{n} \cdot \tilde{\mathbf{J}}\right)^n \end{pmatrix}$$

Since every term in $\Lambda$ lies in the lower $3 \times 3$ corner, this transformation only affects $x, y, z$ and not $t$. Let $\tilde{\Lambda}(\mathbf{n}, \theta)$ be the 3-dim transformation,

$$\tilde{\Lambda}(\mathbf{n}, \theta) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \varepsilon^k \left(\mathbf{n} \cdot \tilde{\mathbf{J}}\right)^k$$

with $\tilde{\mathbf{J}}$ the $3\times 3$ form of the generators. We find

$$
\begin{aligned}
\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^2 &= \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} -n_2^2-n_3^2 & n_1n_2 & n_1n_3 \\ -n_1n_2 & -n_1^2-n_3^2 & n_2n_3 \\ n_1n_3 & n_2n_3 & -n_1^2-n_2^2 \end{pmatrix} \\
&= \begin{pmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_1n_2 & n_2n_2 & n_2n_3 \\ n_1n_3 & n_2n_3 & n_3n_3 \end{pmatrix} - \left(n_1^2+n_2^2+n_3^2\right)\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \\
\left[\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^2\right]^i_{\ j} &= -\left(\delta^i_{\ j} - n^i n_j\right)
\end{aligned}
$$

Taking one more power,

$$
\begin{aligned}
\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^3 &= \left[\begin{pmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_1n_2 & n_2n_2 & n_2n_3 \\ n_1n_3 & n_2n_3 & n_3n_3 \end{pmatrix} - \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right]\begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_1n_2 & n_2n_2 & n_2n_3 \\ n_1n_3 & n_2n_3 & n_3n_3 \end{pmatrix}\begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right) \\
&= -\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)
\end{aligned}
$$

so we have come back to the original matrix except for a sign. If we define $\tilde{\mathbf{M}} \equiv -\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^2$ then we may write all powers as

$$
\begin{aligned}
\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^{2m} &= (-1)^m\,\tilde{\mathbf{M}} \\
\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^{2m+1} &= (-1)^m\,\mathbf{n}\cdot\tilde{\mathbf{J}}
\end{aligned}
$$

and the series becomes

$$
\begin{aligned}
\tilde{\Lambda}(\mathbf{n},\theta) &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\varepsilon^k\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^k \\
&= \lim_{n\to\infty}\left[\tilde{\mathbf{1}} + \sum_{m=1}^{n}\frac{n!}{(2m)!\,(n-2m)!}\varepsilon^{2m}\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^{2m} + \sum_{m=0}^{n}\frac{n!}{(2m+1)!\,(n-2m-1)!}\varepsilon^{2m+1}\left(\mathbf{n}\cdot\tilde{\mathbf{J}}\right)^{2m+1}\right] \\
&= \tilde{\mathbf{1}} - \tilde{\mathbf{M}} + \tilde{\mathbf{M}}\lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m\,n!}{(2m)!\,(n-2m)!}\varepsilon^{2m} + \mathbf{n}\cdot\tilde{\mathbf{J}}\lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m\,n!}{(2m+1)!\,(n-2m-1)!}\varepsilon^{2m+1}
\end{aligned}
$$

where we add and subtract $\tilde{\mathbf{M}}$ so that $m$ starts at 0 in the first sum. Now look at the remaining sums. Multiplying and dividing the first by $n^{2m}$,

$$
\lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m\,n!}{(2m)!\,(n-2m)!}\varepsilon^{2m} = \lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m\,n!}{(2m)!n^{2m}\,(n-2m)!}(n\varepsilon)^{2m}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m n(n-1)(n-2)\cdots(n-2m+1)}{(2m)!n^{2m}}(n\varepsilon)^{2m}\\
&= \lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots(1-\frac{2m-1}{n})}{(2m)!}(n\varepsilon)^{2m}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m\theta^{2m}}{(2m)!}\\
&= \cos\theta
\end{aligned}
$$

and similarly for the second

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m n!}{(2m+1)!(n-2m-1)!}\varepsilon^{2m+1} &= \lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m\eta^{2m+1}}{(2m+1)!}\\
&= \sin\theta
\end{aligned}
$$

The full transformation is therefore

$$
\tilde{\Lambda}(\mathbf{n},\theta) = \tilde{\mathbf{1}}-\tilde{\mathbf{M}}+\tilde{\mathbf{M}}\cos\theta+\mathbf{n}\cdot\tilde{\mathbf{J}}\sin\theta
$$

where

$$
\tilde{M}_{ij}=\delta_{ij}-n_in_j
$$

This is a rotation through an angle $\theta$ about the $\mathbf{n}$ direction. To see this, consider the effect on an arbitrary position vector, $x^i$. In components, remembering that $\left[\tilde{J}_i\right]_{jk}=\varepsilon_{ijk}$,

$$
\begin{aligned}
\tilde{\Lambda}^i{}_j x^j &= \left[\delta^i_j-\left(\delta^i_j-n^in_j\right)\right]x^j+\left(\delta^i_j-n^in_j\right)x^j\cos\theta+n_k\varepsilon^{ki}{}_j\sin\theta\\
&= n^in_jx^j+\left(x^i-n^in_jx^j\right)\cos\theta+n_k\varepsilon^{ki}{}_jx^j\sin\theta\\
&= n^in_jx^j+\left(x^i-n^in_jx^j\right)\cos\theta-\varepsilon^i{}_{kj}n^kx^j\sin\theta\\
\tilde{\mathbf{\Lambda}}\mathbf{x} &= \mathbf{n}(\mathbf{n}\cdot\mathbf{x})+(\mathbf{x}-\mathbf{n}(\mathbf{n}\cdot\mathbf{x}))\cos\theta-(\mathbf{n}\times\mathbf{x})\sin\theta
\end{aligned}
$$

Now divide $\mathbf{x}$ into parts parallel and perpendicular to $\mathbf{n}$,

$$
\begin{aligned}
\mathbf{x}_\parallel &= \mathbf{n}(\mathbf{n}\cdot\mathbf{x})\\
\mathbf{x}_\perp &= \mathbf{x}-\mathbf{n}(\mathbf{n}\cdot\mathbf{x})
\end{aligned}
$$

and notice that $\mathbf{n}\times\mathbf{x}$ is perpendicular to both.

$$
\tilde{\mathbf{\Lambda}}\mathbf{x}=\mathbf{x}_\parallel+\mathbf{x}_\perp(\cos\theta-1)-(\mathbf{n}\times\mathbf{x}_\perp)\sin\theta
$$

The part of $\mathbf{x}$ parallel to $\mathbf{n}$ is unaffected by the transformation, while the perpendicular part undergoes a rotation by $\theta$ in the plane perpendicular to $\mathbf{n}$

## 2.2 Boosts

Now consider the transformations generated by $K_i$. The basic approach is identical, with only the generators differing. Identical steps, taking the limit of many infinitesimal transformations, lead to

$$
\Lambda(\mathbf{n},\theta) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\varepsilon^k(\mathbf{n}\cdot\mathbf{K})^k
$$

where the limit is taken with $n\varepsilon \to \zeta$ where $\zeta$ is finite. The powers of $\mathbf{n}\cdot\mathbf{K}$ again split into even and odd. Starting with

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{K} &= \begin{pmatrix} 0 & -n_1 & -n_2 & -n_3 \\ -n_1 & 0 & 0 & 0 \\ -n_2 & 0 & 0 & 0 \\ -n_3 & 0 & 0 & 0 \end{pmatrix} \\
(\mathbf{n}\cdot\mathbf{K})^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1n_1 & n_2n_1 & n_3n_1 \\ 0 & n_1n_2 & n_2n_2 & n_3n_2 \\ 0 & n_1n_3 & n_2n_3 & n_3n_3 \end{pmatrix} \\
(\mathbf{n}\cdot\mathbf{K})^3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1n_1 & n_2n_1 & n_3n_1 \\ 0 & n_1n_2 & n_2n_2 & n_3n_2 \\ 0 & n_1n_3 & n_2n_3 & n_3n_3 \end{pmatrix} \begin{pmatrix} 0 & -n_1 & -n_2 & -n_3 \\ -n_1 & 0 & 0 & 0 \\ -n_2 & 0 & 0 & 0 \\ -n_3 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -n_1 & -n_2 & -n_3 \\ -n_1 & 0 & 0 & 0 \\ -n_2 & 0 & 0 & 0 \\ -n_3 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

so in general,

$$\begin{aligned}
(\mathbf{n}\cdot\mathbf{K})^{2m+1} &= \mathbf{n}\cdot\mathbf{K} \\
(\mathbf{n}\cdot\mathbf{K})^{2m} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1n_1 & n_2n_1 & n_3n_1 \\ 0 & n_1n_2 & n_2n_2 & n_3n_2 \\ 0 & n_1n_3 & n_2n_3 & n_3n_3 \end{pmatrix} \equiv \mathbf{N}
\end{aligned}$$

Notice that there is no alternating sign now. The power series rearranges as before to give

$$\begin{aligned}
\Lambda(\mathbf{n},\theta) &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\varepsilon^k(\mathbf{n}\cdot\mathbf{K})^k \\
&= \mathbf{1}-\mathbf{N}+\mathbf{N}\sum_{m=0}^{\infty}\frac{\zeta^{2m}}{(2m)!}+\mathbf{n}\cdot\mathbf{K}\sum_{m=0}^{\infty}\frac{\zeta^{2m+1}}{(2m+1)!} \\
&= \mathbf{1}-\mathbf{N}+\mathbf{N}\cosh\zeta+\mathbf{n}\cdot\mathbf{K}\sinh\zeta
\end{aligned}$$

If we define $n^\alpha = \left(0, n^i\right)$ and $m^\alpha = (1, \mathbf{0})$ then we may write

$$\begin{aligned}
[\mathbf{N}]^\alpha_{\ \beta} &= n^\alpha n_\beta + m^\alpha m_\beta \\
[\mathbf{n}\cdot\mathbf{K}]^\alpha_{\ \beta} &= -m^\alpha n_\beta - n^\alpha m_\beta
\end{aligned}$$

and the Lorentz boost becomes

$$\Lambda^\alpha_{\ \beta} = \delta^\alpha_\beta + (n^\alpha n_\beta + m^\alpha m_\beta)(\cosh\zeta - 1) - (m^\alpha n_\beta + n^\alpha m_\beta)\sinh\zeta$$

To see that this is a boost, let $\mathbf{n}$ lie in the $x$-direction. Then

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{K} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{N} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

and therefore

$$
\begin{aligned}
\Lambda(\mathbf{n},\theta) &= \mathbf{1}-\mathbf{N}+\mathbf{N}\cosh\zeta+\mathbf{n}\cdot\mathbf{K}\sinh\zeta \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cosh\zeta & 0 & 0 & 0 \\ 0 & \cosh\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

just as we found previously as on of the transformations preserving the wave equation.