Born approximations and the blue sky

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1 Perturbation theory of scattering

1.1 General formalism

Scattering can occur in a medium with spatially varying or time varying properties. If these variations are small, they may be treated perturbatively.

The analysis starts by assuming that

$$\begin{aligned} \mathbf{D} &\neq \epsilon_0 \mathbf{E} \\ \mathbf{H} &\neq \frac{1}{\mu_0}\mathbf{B} \end{aligned}$$

so we treat all four fields as independent. Let ϵ_0, μ_0 be the *unperturbed values* of the dielectric constant and permeability (and *not* the vacuum values). Then we can rewrite Maxwell's equations in terms of the small differences

$$\begin{aligned} &\mathbf{D} - \epsilon_0 \mathbf{E} \\ &\mathbf{B} - \mu_0 \mathbf{H} \end{aligned}$$

Starting from

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{D} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= 0 \end{aligned}$$

we have

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \\ &= \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times \mathbf{D} - \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times (\mathbf{D} - \epsilon_0 \mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

and taking the curl,

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \times \left(\frac{1}{\epsilon_0}\boldsymbol{\nabla} \times \mathbf{D} - \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times (\mathbf{D} - \epsilon_0 \mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} \right) \\ &= \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{D}) - \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\mathbf{D} - \epsilon_0 \mathbf{E})] + \boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} \\ &= \frac{1}{\epsilon_0}\boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{D}) - \frac{1}{\epsilon_0}\nabla^2 \mathbf{D} - \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\mathbf{D} - \epsilon_0 \mathbf{E})) + \mu_0 \boldsymbol{\nabla} \times \frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\nabla} \times \frac{\partial}{\partial t}(\mathbf{B} - \mu_0 \mathbf{H}) \\ &= -\frac{1}{\epsilon_0}\nabla^2 \mathbf{D} + \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} - \frac{1}{\epsilon_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\mathbf{D} - \epsilon_0 \mathbf{E})) + \boldsymbol{\nabla} \times \frac{\partial}{\partial t}(\mathbf{B} - \mu_0 \mathbf{H}) \end{aligned}$$

so that

$$\nabla^2\mathbf{D}-\mu_0\epsilon_0\frac{\partial^2\mathbf{D}}{\partial t^2}=-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E}))+\epsilon_0\frac{\partial}{\partial t}\left[\boldsymbol{\nabla}\times(\mathbf{B}-\mu_0\mathbf{H})\right]$$

where the terms on the right are small. With harmonic time dependence, and setting $\mu_0\epsilon_0\omega^2=k^2$, this becomes

$$\left(\nabla^2+k^2\right)\mathbf{D}=-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E}))-i\omega\epsilon_0\left[\boldsymbol{\nabla}\times(\mathbf{B}-\mu_0\mathbf{H})\right]$$

1.2 Born approximation

We can now perform a systematic perturbation theory, allowing small corrections to the permittivity and permeability,

$$\begin{aligned}\epsilon &= \epsilon^{(0)}+\epsilon^{(1)}(\mathbf{x})+\epsilon^{(2)}(\mathbf{x})+\ldots\\ \mu &= \mu^{(0)}+\mu^{(1)}(\mathbf{x})+\mu^{(2)}(\mathbf{x})+\ldots\end{aligned}$$

where $\epsilon^{(0)}=\epsilon_0,\mu^{(0)}=\mu_0$, and setting the fields equal to

$$\begin{aligned}\mathbf{D} &= \mathbf{D}^{(0)}+\mathbf{D}^{(1)}+\ldots\\ \mathbf{E} &= \mathbf{E}^{(0)}+\mathbf{E}^{(1)}+\ldots\\ \mathbf{D} &= \epsilon\mathbf{E}\\ &= \left(\epsilon^{(0)}+\epsilon^{(1)}+\epsilon^{(2)}+\ldots\right)\left(\mathbf{E}^{(0)}+\mathbf{E}^{(1)}+\ldots\right)\\ &= \epsilon^{(0)}\mathbf{E}^{(0)}+\left[\epsilon^{(0)}\mathbf{E}^{(1)}+\epsilon^{(1)}\mathbf{E}^{(0)}\right]+\left[\epsilon^{(1)}\mathbf{E}^{(1)}+\epsilon^{(2)}\mathbf{E}^{(0)}+\epsilon^{(0)}\mathbf{E}^{(2)}\right]+\ldots\\ \mathbf{B} &= \mathbf{B}^{(0)}+\mathbf{B}^{(1)}+\ldots\\ \mathbf{H} &= \mathbf{H}^{(0)}+\mathbf{H}^{(1)}+\mathbf{H}^{(2)}+\ldots\\ \mathbf{B} &= \mu\mathbf{H}\\ &= \left(\mu^{(0)}+\mu^{(1)}+\mu^{(2)}+\ldots\right)\left(\mathbf{H}^{(0)}+\mathbf{H}^{(1)}+\mathbf{H}^{(2)}+\ldots\right)\\ &= \mu^{(0)}\mathbf{H}^{(0)}+\left[\mu^{(1)}\mathbf{H}^{(0)}+\mu^{(0)}\mathbf{H}^{(1)}\right]+\left[\mu^{(2)}\mathbf{H}^{(0)}+\mu^{(1)}\mathbf{H}^{(1)}+\mu^{(0)}\mathbf{H}^{(2)}\right]+\ldots\end{aligned}$$

This means that the differences on the right side of the wave equation are of higher order than the field on the left.

$$\begin{aligned}\mathbf{D}-\epsilon_0\mathbf{E} &= \mathbf{D}-\epsilon_0\left[\mathbf{E}^{(0)}+\mathbf{E}^{(1)}+\mathbf{E}^{(2)}+\ldots\right]\\ &= \left[\epsilon^{(1)}\mathbf{E}^{(0)}\right]+\left[\epsilon^{(1)}\mathbf{E}^{(1)}+\epsilon^{(2)}\mathbf{E}^{(0)}\right]+\ldots\\ \mathbf{B}-\mu_0\mathbf{H} &= \mathbf{B}-\mu_0\left[\mathbf{H}^{(0)}+\mathbf{H}^{(1)}+\mathbf{H}^{(2)}+\ldots\right]\\ &= \mu^{(0)}\mathbf{H}^{(0)}+\left[\mu^{(1)}\mathbf{H}^{(0)}+\mu^{(0)}\mathbf{H}^{(1)}\right]+\left[\mu^{(2)}\mathbf{H}^{(0)}+\mu^{(1)}\mathbf{H}^{(1)}+\mu^{(0)}\mathbf{H}^{(2)}\right]+\ldots\\ &\quad -\mu_0\left[\mathbf{H}^{(0)}+\mathbf{H}^{(1)}+\mathbf{H}^{(2)}+\ldots\right]\\ &= \left[\mu^{(1)}\mathbf{H}^{(0)}\right]+\left[\mu^{(2)}\mathbf{H}^{(0)}+\mu^{(1)}\mathbf{H}^{(1)}\right]+\ldots\end{aligned}$$

Notice that the small source terms only depend on lower order fields.

As a first approximation, we put in the zeroth order approximations for $\mathbf{D}$ and $\mathbf{B}$,

$$\begin{aligned}\left(\nabla^2+k^2\right)\mathbf{D}^{(0)} &= -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times(\mathbf{D}-\epsilon_0\mathbf{E})^{(0)}\right)-i\omega\epsilon_0\left[\boldsymbol{\nabla}\times(\mathbf{B}-\mu_0\mathbf{H})^{(0)}\right]\\ &= 0\end{aligned}$$

and we see that $\mathbf{D}^{(0)}$ is the plane wave solution. At the next order, we have

$$\left(\nabla^2+k^2\right)\mathbf{D}^{(1)} = -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left(\epsilon^{(1)}\mathbf{E}^{(0)}\right)\right)-i\omega\epsilon_0\left[\boldsymbol{\nabla}\times\left(\mu^{(1)}\mathbf{H}^{(0)}\right)\right]$$

The solution to this equation is the first Born approximation, and as we show below it gives the first approximation to the scattered wave. Notice again that the wave equation for $\mathbf{D}^{(1)}$ has sources that only depend on the lowest order solutions for the fields, $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$.

We may continue to arbitrary order, to find corrections. Thus, at second order, we have

$$\left(\nabla^2+k^2\right)\mathbf{D}^{(2)} = -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left[\epsilon^{(1)}\mathbf{E}^{(1)}+\epsilon^{(2)}\mathbf{E}^{(0)}\right]\right)-i\omega\epsilon_0\left(\boldsymbol{\nabla}\times\left[\mu^{(2)}\mathbf{H}^{(0)}+\mu^{(1)}\mathbf{H}^{(1)}\right]\right)$$

As long as we have a sensible perturbative series for the permittivity and permeability, we may continue this process to arbitrary order.

1.3 First Born approximation for light scattering

For light scattering in gasses, higher order Born approximations are not generally appropriate because we do not really know ϵ or μ. Instead, we can make use of the first Born approximation by supposing

$$\begin{aligned}\epsilon &= \epsilon_0+\delta\epsilon\left(\mathbf{x}\right)\\ \mu &= \mu_0+\delta\mu\left(\mathbf{x}\right)\end{aligned}$$

where ϵ_0 and μ_0 are constant (but not necessarily the vacuum values) and that there are small, position and/or time dependent fluctuations in addition. First Born approximation immediately gives

$$\left(\nabla^2+k^2\right)\mathbf{D}^{(0)}=0$$

for the incoming plane wave, and the first order correction

$$\begin{aligned}\left(\nabla^2+k^2\right)\mathbf{D}^{(1)} &= -\mu_0\epsilon_0\boldsymbol{J}\\ \mu_0\epsilon_0\boldsymbol{J} &= \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)+i\omega\epsilon_0\left[\boldsymbol{\nabla}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right]\end{aligned}$$

This has solution

$$\mathbf{D}^{(1)}=\frac{1}{4\pi}\int \mathrm{d}^3x\frac{e^{ik\left|\mathbf{x}-\mathbf{x}'\right|}\boldsymbol{J}}{\left|\mathbf{x}-\mathbf{x}'\right|}$$

We now assume that the fluctuations in ϵ and μ are localized in space and look at the scattered wave $\mathbf{D}^{(1)}$ in the radiation zone,

$$\begin{aligned}\mathbf{D} &= \mathbf{D}^{(0)}+\mathbf{D}^{(1)}\\ &= \mathbf{D}^{(0)}+\frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'e^{-ik\mathbf{n}\cdot\mathbf{x}'}\boldsymbol{J}\end{aligned}$$

The integral of $\boldsymbol{J}$ may be simplified by integration by parts by recalling the identity

$$\boldsymbol{\nabla}\times\left(f\mathbf{v}\right)=\boldsymbol{\nabla}f\times\mathbf{v}+f\boldsymbol{\nabla}\times\left(\mathbf{v}\right)$$

Applying this to the curl terms in the electric part of $\boldsymbol{J}$,

$$\begin{aligned}\mathbf{D}^{(1)}_1 &= \frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'e^{-ik\mathbf{n}\cdot\mathbf{x}'}\boldsymbol{\nabla}'\times\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\\ &= \frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'\boldsymbol{\nabla}'\times\left[e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right]\\ &\quad-\frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x'\left[\boldsymbol{\nabla}'e^{-ik\mathbf{n}\cdot\mathbf{x}'}\times\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right]\end{aligned}$$

The first term on the right may be integrated, giving a result that depends on the fields at large distance, where the perturbation, $\delta\epsilon$, vanishes. Then

$$\begin{aligned}
\mathbf{D}_1^{(1)} &= -\frac{e^{ikr}}{4\pi r}\int \mathrm{d}^3x' \left[-ike^{-ik\mathbf{n}\cdot\mathbf{x}'}\mathbf{n}\times\left(\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right] \\
&= \frac{ike^{ikr}}{4\pi r}\mathbf{n}\times\int \mathrm{d}^3x' \left[\left(e^{-ik\mathbf{n}\cdot\mathbf{x}'}\boldsymbol{\nabla}'\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right] \\
&= \frac{ike^{ikr}}{4\pi r}\mathbf{n}\times\int \mathrm{d}^3x' \left[\left(-\boldsymbol{\nabla}'e^{-ik\mathbf{n}\cdot\mathbf{x}'}\right)\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right] \\
&= -\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\mathbf{n}\times\left(\mathbf{n}\times\int \mathrm{d}^3x' \left[e^{-ik\mathbf{n}\cdot\mathbf{x}'}\delta\epsilon\mathbf{E}^{(0)}\right]\right)
\end{aligned}$$

where we have done a second integration by parts and again dropped the surface term. The electric dipole term is the lowest order term when we expand $e^{-ik\mathbf{n}\cdot\mathbf{x}'} = 1 - ik\mathbf{n}\cdot\mathbf{x}' + \ldots$, so the electric dipole field is

$$\mathbf{D}_1^{(1)} = -\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' \left[\mathbf{n}\times\left(\mathbf{n}\times\left(\delta\epsilon\mathbf{E}^{(0)}\right)\right)\right]$$

Comparing with the general electric dipole expression,

$$\mathbf{D}_{sc} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\right]$$

we see that we have an effective electric dipole source,

$$\mathbf{p} = \int \mathrm{d}^3x'\delta\epsilon\left(\mathbf{x}'\right)\mathbf{E}^{(0)}\left(\mathbf{x}'\right)$$

where $\mathbf{E}^{(0)}$ is the incident wave.

The second term in $\boldsymbol{J}$ works the same way,

$$\begin{aligned}
\mathbf{D}_2^{(1)} &= \frac{i\omega\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\boldsymbol{\nabla}'\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right] \\
&= -\frac{i\omega\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[-ik\mathbf{n}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right] \\
&= -\frac{k^2\epsilon_0}{4\pi c}\frac{e^{ikr}}{r}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{n}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right]
\end{aligned}$$

and this expression has the same form as the magnetic dipole field,

$$\mathbf{D}_{sc} = \frac{k^2\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\left[-\frac{1}{c}\mathbf{n}\times\mathbf{m}\right]$$

if we identify

$$\mathbf{m} = \int \mathrm{d}^3x'\left(\delta\mu\mathbf{H}^{(0)}\right)$$

The total scattering amplitude is $\mathbf{D}_{sc} = \mathbf{D}^{(1)} = \mathbf{D}_1^{(1)} + \mathbf{D}_2^{(1)}$, and the form is exactly the same as the form of the electric field for electric and magnetic dipole radiation,

$$\mathbf{D}_{sc} = \frac{k^2\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \frac{1}{c}\mathbf{n}\times\mathbf{m}\right]$$

If we define

$$\mathbf{D}_{sc} = \frac{e^{ikr}}{r}\mathbf{A}_{sc}$$

then the differential cross-section follows immediately as

$$\frac{d\sigma}{d\Omega} = \frac{\left|\boldsymbol{\varepsilon}^* \cdot \mathbf{A}_{sc}\right|^2}{\left|\mathbf{D}^{(0)}\right|^2}$$

If the initial wave has the form

$$\begin{aligned}
\mathbf{D}^{(0)} &= \boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\
\mathbf{B}^{(0)} &= \sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{n}_0 \times \mathbf{D}^{(0)}
\end{aligned}$$

then the general multipole expression for the scattered wave becomes

$$\begin{aligned}
\mathbf{A}_{sc} &= -\frac{k^2}{4\pi}\mathbf{n}\times\left(\mathbf{n}\times\int \mathrm{d}^3x'\left[e^{-ik\mathbf{n}\cdot\mathbf{x}'}\delta\epsilon\mathbf{E}^{(0)}\right]\right) - \frac{k^2\epsilon_0}{4\pi c}\int \mathrm{d}^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{n}\times\left(\delta\mu\mathbf{H}^{(0)}\right)\right] \\
&= \frac{k^2}{4\pi}D_0\int \mathrm{d}^3x' e^{ik\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\left(\mathbf{n}\times\boldsymbol{\varepsilon}_0\right)\times\mathbf{n} - \frac{\delta\mu}{\mu_0}\left[\mathbf{n}\times\left(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0\right)\right]\right]
\end{aligned}$$

so that $\frac{d\sigma}{d\Omega}$ is the square of

$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} = \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{ik\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\left[\left(\mathbf{n}\times\boldsymbol{\varepsilon}_0\right)\times\mathbf{n}\right] - \frac{\delta\mu}{\mu_0}\boldsymbol{\varepsilon}^*\cdot\left[\mathbf{n}\times\left(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0\right)\right]\right]$$

Rewriting

$$\begin{aligned}
\boldsymbol{\varepsilon}^*\cdot\left[\left(\mathbf{n}\times\boldsymbol{\varepsilon}_0\right)\times\mathbf{n}\right] &= \boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 \\
\boldsymbol{\varepsilon}^*\cdot\left[\mathbf{n}\times\left(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0\right)\right] &= \left(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0\right)\cdot\left(\boldsymbol{\varepsilon}^*\times\mathbf{n}\right) \\
&= -\left(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)
\end{aligned}$$

this becomes

$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} = \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 + \frac{\delta\mu}{\mu_0}\left(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right]$$

where $\mathbf{q} = k\left(\mathbf{n}_0 - \mathbf{n}\right)$. A simple approximation is to then take $\delta\epsilon$ constant inside a sphere of radius a, and set $\delta\mu = 0$. Then the integral becomes

$$\begin{aligned}
\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} &= \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right] \\
&= \frac{k^2}{4\pi}\frac{\delta\epsilon}{\epsilon_0}\left(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right)\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'} \\
&= \frac{k^2}{4\pi}\frac{\delta\epsilon}{\epsilon_0}\left(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right)\int r'^2\mathrm{d}r'\mathrm{d}\varphi'\mathrm{d}\left(\cos\theta'\right)e^{i\mathbf{q}\cdot\mathbf{x}'}
\end{aligned}$$

Taking $\mathbf{n}_0$ in the z-direction,

$$\begin{aligned}
\int r'^2\mathrm{d}r'\mathrm{d}\varphi'\mathrm{d}\left(\cos\theta'\right)e^{i\mathbf{q}\cdot\mathbf{x}'} &= \int r'^2\mathrm{d}r'\mathrm{d}\varphi'\mathrm{d}\left(\cos\theta'\right)e^{iqr'\cos\theta'} \\
&= 2\pi\int r'^2\mathrm{d}r'\mathrm{d}\left(\cos\theta'\right)e^{iqr'\cos\theta'} \\
&= 2\pi\int_0^a r'^2\mathrm{d}r'\int_{-1}^1 \mathrm{d}x e^{iqr'x}
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_0^a r'^2 \mathrm{d}r' \left[\frac{e^{iqr'x}}{iqr'} \right]_{-1}^{+1} \\
&= 2\pi \int_0^a r'^2 \mathrm{d}r' \left[\frac{e^{iqr'} - e^{-iqr'}}{iqr'} \right] \\
&= 4\pi \int_0^a r'^2 \mathrm{d}r' \left[\frac{\sin qr'}{qr'} \right] \\
&= \frac{4\pi}{q} \int_0^a \mathrm{d}r' r' \sin qr' \\
&= \frac{4\pi}{q} \left(-\frac{d}{dq} \int_0^a \mathrm{d}r' \cos qr' \right) \\
&= \frac{4\pi}{q} \left(-\frac{d}{dq} \left[\frac{\sin qr'}{q} \right]_0^a \right) \\
&= \frac{4\pi}{q} \frac{d}{dq} \left(-\frac{\sin qa}{q} \right) \\
&= \frac{4\pi}{q^3} \left(\sin qa - qa \cos qa \right)
\end{aligned}
$$

In the long wavelength limit, $k, q \to 0$, and this becomes

$$
\begin{aligned}
4\pi \lim_{q\to 0} \left(\frac{\sin qa - qa\cos qa}{q^3} \right) &= 4\pi \lim_{q\to 0} \left(\frac{qa - \frac{1}{3!}q^3a^3 - qa\left(1 - \frac{1}{2}q^2a^2\right) + O\left(q^4\right)}{q^3} \right) \\
&= 4\pi \lim_{q\to 0} \left(\frac{-\frac{1}{3!}q^3a^3 + \frac{1}{2}q^3a^3 + O\left(q^4\right)}{q^3} \right) \\
&= 4\pi \lim_{q\to 0} \left(-\frac{1}{3!}a^3 + \frac{1}{2}a^3 + O\left(q^3\right) \right) \\
&= \frac{4}{3}\pi a^3
\end{aligned}
$$

The differential cross section is therefore

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left| \frac{k^2}{4\pi} \frac{\delta\epsilon}{\epsilon_0} \left(\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \right) \frac{4}{3}\pi a^3 \right|^2 \\
&= k^4 a^6 \left| \frac{\delta\epsilon}{3\epsilon_0} \right|^2 \left| \left(\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \right) \right|^2
\end{aligned}
$$

This agrees, to first order in $\delta\epsilon$, with our result for scattering from a dielectric sphere,

$$
\frac{d\sigma}{d\Omega} = \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right)^2 k^4 a^6 \sin^2 \varphi
$$

when we set $\epsilon - \epsilon_0 = \delta\varepsilon$ and $\epsilon + 2\epsilon_0 = 3\epsilon_0 + \delta\epsilon$, since

$$
\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} = \frac{\delta\epsilon}{3\epsilon_0 + \delta\epsilon}
$$

$$
\begin{aligned}
&= \frac{\delta\epsilon}{3\epsilon_0\left(1+\frac{\delta\epsilon}{3\epsilon_0}\right)} \\
&= \frac{\delta\epsilon}{3\epsilon_0}\left(1-\frac{\delta\epsilon}{3\epsilon_0}+\ldots\right) \\
&= \frac{\delta\epsilon}{3\epsilon_0}+O\left((\delta\epsilon)^2\right)
\end{aligned}
$$

2 The blue sky

Jackson gives two treatments of scattering in the atmosphere. The first is to approximate the atmosphere as a dilute gas with randomly distributed molecules. Then taking the molecules to have dipole moments

$$\mathbf{p}_i = \epsilon_0 \gamma_{mol} \mathbf{E}$$

where γ_{mol} is the molecular polarizability, the effective dielectric constant is

$$\delta\epsilon = \epsilon_0 \sum_i \gamma_{mol} \delta^3\left(\mathbf{x}-\mathbf{x}_i\right)$$

Then

$$
\begin{aligned}
\frac{\boldsymbol{\varepsilon}^* \cdot \mathbf{A}_{sc}}{D_0} &= \frac{k^2}{4\pi}\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left[\frac{\delta\epsilon}{\epsilon_0}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right] \\
&= \frac{k^2}{4\pi}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\gamma_{mol}\sum_i\int \mathrm{d}^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\delta^3\left(\mathbf{x}-\mathbf{x}_i\right) \\
&= \frac{k^2}{4\pi}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\gamma_{mol}\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left|\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0}\right|^2 \\
&= \frac{k^4}{16\pi^2}\left|\gamma_{mol}\right|^2\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2\left|\sum e_i^{i\mathbf{q}\cdot\mathbf{x}'}\right|^2
\end{aligned}
$$

When we sum over all particles, the structure function

$$\mathcal{F} = \left|\sum e_i^{i\mathbf{q}\cdot\mathbf{x}'}\right|^2$$

will be the total number of particles because of the randomness of the gas. For a single particle, we may therefore drop the phase – the average differential cross-section per particle is just

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}\left|\gamma_{mol}\right|^2\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2$$

Now, for a dilute gas,

$$\epsilon_r \approx 1 + N\gamma_{mol}$$

where ϵ_r is the relative dielectric constant and N is the number density of molecules. Therefore,

$$\left|\gamma_{mol}\right|^2 = \frac{\left|\epsilon_r - 1\right|^2}{N^2}$$

Rewriting this in terms of the index of refraction,

$$\begin{aligned}
n &= \sqrt{\frac{\mu}{\mu_0}\frac{\epsilon}{\epsilon_0}} \\
&= \sqrt{\epsilon_r} \\
n-1 &= \sqrt{\epsilon_r}-1 \\
n-1 &= \sqrt{1+(\epsilon_r-1)}-1 \\
n-1 &\approx 1+\frac{1}{2}(\epsilon_r-1)-1 \\
2(n-1) &= \epsilon_r-1
\end{aligned}$$

so that the differential cross section becomes

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{k^4}{16\pi^2}\left|\gamma_{mol}\right|^2\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 \\
&= \frac{k^4}{16\pi^2}\frac{\left|\epsilon_r-1\right|^2}{N^2}\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 \\
&= \frac{k^4}{4\pi^2}\frac{\left|n-1\right|^2}{N^2}\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2
\end{aligned}$$

For unpolarized initial and final states, we average over initial and sum over final polarizations, so the polarization factor becomes

$$\left(\frac{1}{2}\sum_{\boldsymbol{\varepsilon}_0=\mathbf{i},\mathbf{j}}\right)\left(\sum_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2}\right)\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2=\frac{1}{2}\sum_{\boldsymbol{\varepsilon}_0=\mathbf{i},\mathbf{j}}\sum_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2}\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2$$

which we can find by dropping the magnetic dipole part of our earlier calculation for the sphere. Averaging over

$$\begin{aligned}
\boldsymbol{\varepsilon}_\parallel &= \frac{1}{\sin\theta}(\mathbf{n}_0\times\mathbf{n})\times\mathbf{n} \\
\boldsymbol{\varepsilon}_\perp &= \frac{1}{\sin\theta}\mathbf{n}_0\times\mathbf{n}
\end{aligned}$$

we first find the parallel and perpendicular cases separately:

$$\begin{aligned}
\frac{d\sigma_\parallel}{d\Omega} &= \left(\frac{k^4}{4\pi^2}\frac{\left|n-1\right|^2}{N^2}\right)\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\boldsymbol{\varepsilon}_\parallel^*\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2 \\
\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\boldsymbol{\varepsilon}_\parallel^*\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2 &= \sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\frac{1}{\sin\theta}((\mathbf{n}_0\times\mathbf{n})\times\mathbf{n})\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2 \\
&= \frac{1}{2}\frac{1}{\sin^2\theta}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|((\mathbf{n}_0\times\mathbf{n})\times\mathbf{n})\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2 \\
&= \frac{1}{2}\frac{1}{\sin^2\theta}\left|((\mathbf{n}_0\times\mathbf{n})\times\mathbf{n})\cdot\hat{\boldsymbol{\varepsilon}}_{0\parallel}\right|^2 \\
&= \frac{1}{2}\cos^2\theta
\end{aligned}$$

and

$$\frac{d\sigma_\perp}{d\Omega} = \left(\frac{k^4}{4\pi^2}\frac{\left|n-1\right|^2}{N^2}\right)\frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}}\left|\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$
\begin{aligned}
\frac{1}{2}\sum_{\hat{\varepsilon}_{0\parallel},\hat{\varepsilon}_{0\perp}} \left|\boldsymbol{\varepsilon}_{\perp}^{*}\cdot\hat{\varepsilon}_{0}\right|^{2} &= \frac{1}{2}\sum_{\hat{\varepsilon}_{0\parallel},\hat{\varepsilon}_{0\perp}} \left|\frac{1}{\sin\theta}\left(\mathbf{n}_{0}\times\mathbf{n}\right)\cdot\hat{\varepsilon}_{0}\right|^{2} \\
&= \frac{1}{2}\left|\frac{1}{\sin\theta}\left(\mathbf{n}_{0}\times\mathbf{n}\right)\cdot\hat{\varepsilon}_{0\perp}\right|^{2} \\
&= \frac{1}{2}
\end{aligned}
$$

If the final polarization is unmeasured, we sum these, giving

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma_{\parallel}}{d\Omega}+\frac{d\sigma_{\perp}}{d\Omega} \\
&= \left(\frac{k^{4}}{4\pi^{2}}\frac{\left|n-1\right|^{2}}{N^{2}}\right)\frac{1}{2}\left(1+\cos^{2}\theta\right)
\end{aligned}
$$

The total cross section is now given by integrating over angles

$$
\begin{aligned}
\sigma &= \frac{k^{4}}{8\pi^{2}}\frac{\left|n-1\right|^{2}}{N^{2}}\int\left(1+\cos^{2}\theta\right)\mathrm{d}\Omega \\
&= \frac{k^{4}}{8\pi^{2}}\frac{\left|n-1\right|^{2}}{N^{2}}2\pi\int_{-1}^{1}\left(1+x^{2}\right)\mathrm{d}x \\
&= \frac{k^{4}}{8\pi^{2}}\frac{\left|n-1\right|^{2}}{N^{2}}2\pi\left[x+\frac{1}{3}x^{3}\right]_{-1}^{1} \\
&= \frac{k^{4}}{8\pi^{2}}\frac{\left|n-1\right|^{2}}{N^{2}}2\pi\left[\frac{8}{3}\right] \\
&= \frac{2k^{4}}{3\pi}\frac{\left|n-1\right|^{2}}{N^{2}}
\end{aligned}
$$

This is the total scattered flux per incident flux from a single molecule. Remembering that σ is the effective area of scatterers, the total fraction of light scattered, $\left|\frac{dI}{I}\right|$, in a travel distance dx is therefore $N\sigma\mathrm{d}x$. Suppose the incident beam has intensity I_0. Then

$$
\begin{aligned}
\frac{dI}{I} &= -N\sigma\mathrm{d}x \\
I &= I_{0}e^{-N\sigma x}
\end{aligned}
$$

The absorption coefficient, $\alpha = N\sigma$ is therefore

$$
\alpha = \frac{2k^{4}}{3\pi}\frac{\left|n-1\right|^{2}}{N}
$$

This result describes Rayleigh scattering.

And improved version, due to Einstein, starts with the density fluctuations ΔN_j in cells of volume v, as given by the Clausius-Mossotti relation (eq.4.70),

$$
\delta\epsilon_{j} = \frac{\left(\epsilon_{r}-1\right)\left(\epsilon_{r}+2\right)}{3Nv}\Delta N_{j}
$$

Carrying out the cross-section calculation again, and summing over all cells, gives the attenuation coefficient as

$$
\alpha = \frac{k^{4}}{6\pi N}\left|\frac{\left(\epsilon_{r}-1\right)\left(\epsilon_{r}+2\right)}{3}\right|^{2}\frac{\Delta N_{V}^{2}}{NV}
$$

where ΔN_V^2 is the mean-square number fluctuation per unit volume,

$$\Delta N_V^2 = \left\langle N_i^2 \right\rangle - \left\langle N_i \right\rangle^2$$

The final ratio may be expressed in terms of the isothermal compressibility, $\beta_T = -\frac{1}{V}\left(\frac{dV}{dP}\right)_T$,

$$\frac{\Delta N_V^2}{NV} = NkT\beta_T$$

so that the absorption coefficient becomes

$$\alpha = \frac{kT\beta_T}{6\pi} k^4 \left| \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3} \right|^2$$

This is called the Einstein-Smoluchowski formula.

With $NkT\beta_T = 1$ and the approximation

$$(\epsilon_r - 1)\frac{(\epsilon_r + 2)}{3} \approx 2(n-1) \cdot 1$$

as above, we recover the previous result.