Scattering

March 20, 2016

The scattering of waves of any kind, by a compact object, has applications on all scales, from the scattering of light from the early universe by intervening galaxies, to the scattering of protons into Higgs particles at the Large Hadron Collider. Between these extremes lie myriad phenomena from rainbows and the blue sky to geometric optics. The same basic principles apply to the description of all of these phenomena.

Scattering of light depends on the size of the scatterer relative to the wavelength of the light. For wavelengths much smaller than the scattering object, geometric optics gives an adequate description. At larger wavelengths, corrections to geometric optics may be found. At the other extreme, it is possible to do a treatment in terms of lowest-order multipoles. Between these extremes, a full multipole treatment is required.

1 Differential cross-section

We will be especially interested in the differential cross-section: the probability of scattering into a given solid angle.

In the radiation zone, the time-averaged radiated power per unit area is given by the Poynting vector, $\frac{dP}{dA} = \hat{\mathbf{r}} \cdot \mathbf{S}$. Writing the time-averaged radiated power as $\mathbf{S} = \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*)$, gives the time-averaged radiated power per unit area

$$\frac{dP}{dA} = \frac{1}{2}\left|\mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}^*)\right|$$

For electric dipole or quadrupole fields in the radiation zone we found that $\mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}(\mathbf{n} \times \mathbf{E})$, so this becomes

$$\begin{aligned}
\frac{dP}{dA} &= \frac{1}{2}\left|\mathbf{n} \cdot \left(\mathbf{E} \times \sqrt{\frac{\varepsilon_0}{\mu_0}}(\mathbf{n} \times \mathbf{E}^*)\right)\right| \\
&= \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left|\mathbf{E}\right|^2 \\
&= \frac{1}{2Z_0}\left|\mathbf{E}\right|^2
\end{aligned}$$

Finally, we write the area element $dA = r^2 d\Omega$ in terms of the solid angle $d\Omega$, to get the time-averaged radiated power per unit solid angle,

$$\frac{dP}{d\Omega} = \frac{1}{2Z_0} r^2 \left|\mathbf{E}\right|^2$$

If we desire the cross-section for a particular polarization, we consider only that component of the electric field,

$$\frac{dP}{d\Omega} = \frac{1}{2Z_0} r^2 \left|\boldsymbol{\varepsilon}^* \cdot \mathbf{E}\right|^2$$

In scattering experiments, a target is struck by many incoming waves, so we express the outgoing power as a probability for scattering in a given solid angle. To do this, we normalize by the incident power per unit area. The result is the *differential cross-section*

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &\equiv \frac{1}{dP_{incident}/dA}\frac{dP_{scattered}}{d\Omega} \\
&= \frac{\frac{1}{2Z_0}r^2\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{scattered}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{inc}\right|^2} \\
&= \frac{r^2\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{scattered}\right|^2}{\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{inc}\right|^2}
\end{aligned}$$

Because we divide by the incident power per unit *area*, the differential cross-section and the total cross-section have units of area. The total cross-section, σ, is found by integrating over all angles

$$\sigma = \int_0^{2\pi}\int_0^{\pi}\frac{d\sigma}{d\Omega}\sin\theta d\theta d\varphi$$

The total cross-section may be thought of as the effective cross-sectional area of the target.

For long wavelength or small scatterers, we may assume an incoming polarized plane wave,

$$\begin{aligned}
\mathbf{E}_{sc} &= \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\
\mathbf{H}_{sc} &= \frac{1}{Z_0}\mathbf{n}_0\times\mathbf{E}_{sc}
\end{aligned}$$

which induces electric and magnetic dipole moments, $\mathbf{p},\mathbf{m}$, in the scatterer. Then the scattered radiation is the resulting dipole radiation,

$$\begin{aligned}
\mathbf{E}_{sc} &= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}-\frac{1}{c}\mathbf{n}\times\mathbf{m}\right] \\
\mathbf{H}_{sc} &= \frac{1}{Z_0}\mathbf{n}\times\mathbf{E}_{sc}
\end{aligned}$$

Now, substituting into the differential cross section, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0) &= \frac{\frac{1}{2Z_0}r^2\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{scattered}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}\right|^2} \\
&= \frac{r^2\left|\boldsymbol{\varepsilon}^*\cdot\frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}-\frac{1}{c}\mathbf{n}\times\mathbf{m}\right]\right|^2}{E_0^2} \\
&= \frac{k^4}{(4\pi\epsilon_0E_0)^2}\left|\left[\boldsymbol{\varepsilon}^*\cdot(\mathbf{p}-(\mathbf{n}\cdot\mathbf{p})\,\mathbf{n})-\frac{1}{c}\boldsymbol{\varepsilon}^*\cdot(\mathbf{n}\times\mathbf{m})\right]\right|^2 \\
&= \frac{k^4}{(4\pi\epsilon_0E_0)^2}\left|\left[\boldsymbol{\varepsilon}^*\cdot\mathbf{p}-(\mathbf{n}\cdot\mathbf{p})\,(\boldsymbol{\varepsilon}^*\cdot\mathbf{n})-\frac{1}{c}\mathbf{m}\cdot(\boldsymbol{\varepsilon}^*\times\mathbf{n})\right]\right|^2 \\
&= \frac{k^4}{(4\pi\epsilon_0E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}+\frac{1}{c}\mathbf{m}\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\right|^2
\end{aligned}$$

We have therefore reduced the scattering problem to finding the induced polarization and magnetization. These induced properties generically involve the direction, $\mathbf{n}_0$, and polarization, $\boldsymbol{\varepsilon}_0$ of the incident light. Notice that the differential cross section depends on k^4. This dependence, called Rayleigh's law, will be the case unless both dipole moments vanish - quadrupole radiation will depend on k^6, and so on.

2 Example 1: Scattering by a small dielectric sphere

2.1 The outgoing electric field

For a dielectric sphere much smaller than the wavelength, we may treat the electric field as momentarily constant across the sphere.

Recall the solution for a dielectric sphere in a constant field. We start with a pair of series solutions for the potential inside and out,

$$\begin{aligned}\Phi_{in} &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ \Phi_{out} &= -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)\end{aligned}$$

where the first term in Φ_{out} gives the constant field at "large" distances from the sphere. The remaining terms incorporate the boundary conditions at the origin and infinity. Then, equating the tangential **E** and normal **D** fields we equate like coefficients to find the solution

$$\begin{aligned}\Phi_{in} &= -\frac{3\epsilon_0}{\epsilon+2\epsilon_0} E_0 r\cos\theta \\ \Phi_{out} &= -E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^2}\cos\theta\end{aligned}$$

with the field outside being

$$\begin{aligned}\mathbf{E}_{out} &= -\nabla\Phi_{out} \\ &= -\hat{\mathbf{n}}\frac{\partial}{\partial r}\Phi_{out} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}(\Phi_{out}) \\ &= -\hat{\mathbf{n}}\frac{\partial}{\partial r}\left(-E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^2}\cos\theta\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(-E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^2}\cos\theta\right) \\ &= E_0\hat{\mathbf{k}} + \frac{2\epsilon-2\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\hat{\mathbf{n}}\cos\theta + \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\right)\hat{\boldsymbol{\theta}}\sin\theta \\ &= E_0\hat{\mathbf{k}} + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(2\hat{\mathbf{n}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \\ &= E_0\hat{\mathbf{k}} + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(3\hat{\mathbf{n}}\cos\theta - \left(\hat{\mathbf{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)\right) \\ &= E_0\hat{\mathbf{k}} + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(3\hat{\mathbf{n}}\cos\theta - \hat{\mathbf{k}}\right)\end{aligned}$$

where we used

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \hat{\mathbf{n}}\sin\theta - \hat{\mathbf{k}}\cos\theta \\ \hat{\mathbf{k}} &= \hat{\mathbf{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\end{aligned}$$

Notice that the potential inside is just proportional to $z = r\cos\theta$, so the induced electric field inside is parallel to the applied field, but changed (generally reduced) in magnitude by $\frac{3\epsilon_0}{\epsilon+2\epsilon_0}$.

We know that a dipole $\mathbf{p} = p\hat{\mathbf{k}}$ at the origin produces an electric field

$$\begin{aligned}\mathbf{E} &= \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) - \mathbf{p}}{4\pi\epsilon_0 r^3} \\ &= \frac{p}{4\pi\epsilon_0 r^3}\left(3\hat{\mathbf{n}}\cos\theta - \hat{\mathbf{k}}\right)\end{aligned}$$

Comparing this dipole field to the non-constant part of the exterior field,

$$\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 a^3}{r^3}\left(3\hat{\mathbf{n}}\cos\theta-\hat{\mathbf{k}}\right)=\frac{p}{4\pi\epsilon_0 r^3}\left(3\hat{\mathbf{n}}\cos\theta-\hat{\mathbf{k}}\right)$$

we see that we can identify the dipole strength as

$$\mathbf{p} = 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{k}}$$

2.2 Differential cross section

Now return to the differential cross-section. Since there is no magnetic dipole moment, we set $\mathbf{m}=0$, leaving

$$\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0\right) = \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2$$

From here there are several cases, depending on the polarizations of the incoming and outgoing waves.

We have computed the dipole moment assuming the electric field is in the z-direction, but to find the angular distribution it is easier to let the incoming wave propagate in the z-direction. Rotating the coordinate system so that the incoming wave moves in the $\hat{\mathbf{k}}$-direction with polarization in the $\hat{\boldsymbol{\varepsilon}}_0$-direction, we have

$$\begin{aligned}\mathbf{E} &= E_0\hat{\boldsymbol{\varepsilon}}_0 e^{ikz-i\omega t}\\ \mathbf{p} &= 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\boldsymbol{\varepsilon}}_0\end{aligned}$$

where the polarization vector, $\hat{\boldsymbol{\varepsilon}}_0$, may be in any combination of the x and y directions. Notice that the induced polarization is parallel to the polarization of the incident wave. It makes no difference which direction we choose for the incident polarization, since changing it is simply a change in the origin of the φ coordinate. We may always choose the x-axis to be the polarization direction,

$$\boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}$$

For the outgoing wave, the possible linear polarizations are in the directions orthogonal to the outward radial unit vector, $\mathbf{n}$. It is convenient to write out the unit vectors for spherical coordinates:

$$\begin{aligned}\mathbf{n} &= \hat{\mathbf{i}}\sin\theta\cos\varphi+\hat{\mathbf{j}}\sin\theta\sin\varphi+\hat{\mathbf{k}}\cos\theta\\ \hat{\boldsymbol{\theta}} &= \hat{\mathbf{i}}\cos\theta\cos\varphi+\hat{\mathbf{j}}\cos\theta\sin\varphi-\hat{\mathbf{k}}\sin\theta\\ \hat{\boldsymbol{\varphi}} &= -\hat{\mathbf{i}}\sin\varphi+\hat{\mathbf{j}}\cos\varphi\end{aligned}$$

It is easy to check that these are orthonormal. The outward moving, scattered wave may have polarization in any combination of the $\hat{\boldsymbol{\theta}}$ and the $\hat{\boldsymbol{\varphi}}$ directions. We can now compute the differential cross-section. If the measured polarization is in the $\hat{\boldsymbol{\theta}}$ direction,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}},\mathbf{n}_0=\hat{\mathbf{k}},\boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right) &= \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}\right|^2\\ &= \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left|\hat{\boldsymbol{\theta}}\cdot 4\pi\epsilon_0 E_0 a^3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{i}}\right|^2\\ &= \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left(4\pi\epsilon_0 E_0 a^3\right)^2\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left|\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{i}}\right|^2\\ &= \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left(4\pi\epsilon_0 E_0 a^3\right)^2\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left|\cos\theta\cos\varphi\right|^2\\ &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6\cos^2\theta\cos^2\varphi\end{aligned}$$

Notice that $k^4 a^6$ has units of area as expected. If we measure the polarization in the $\hat{\boldsymbol{\varphi}}$ direction, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right) &= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^* \cdot \mathbf{p}\right|^2 \\
&= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varphi}} \cdot 4\pi\epsilon_0 E_0 a^3 \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\hat{\mathbf{i}}\right|^2 \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \left|\hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{i}}\right|^2 \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \sin^2\varphi
\end{aligned}$$

If we want the initial polarization to be in the y-direction, it is simply a matter of replacing φ by $\varphi - \frac{\pi}{2}$ in the results above. For outgoing polarization in the $\hat{\boldsymbol{\theta}}$ direction this gives,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right) &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\theta\cos^2\left(\varphi-\frac{\pi}{2}\right) \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\theta\sin^2\varphi
\end{aligned}$$

while for the polarization in the $\hat{\boldsymbol{\varphi}}$ direction, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right) &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \sin^2\left(\varphi-\frac{\pi}{2}\right) \\
&= \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\varphi
\end{aligned}$$

2.3 Unpolarized incoming wave

When the incoming light is unpolarized, we *average* over the possible incoming polarizations. For outgoing polarization in the $\hat{\boldsymbol{\theta}}$ direction, this gives

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}\right) &= \frac{1}{2}\left[\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right) + \frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\theta\cos^2\varphi + \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\theta\sin^2\varphi\right] \\
&= \frac{1}{2}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\theta
\end{aligned}$$

which depends only on θ, while for the outgoing polarization in the $\hat{\boldsymbol{\varphi}}$ direction,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}\right) &= \frac{1}{2}\left[\frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{i}}\right) + \frac{d\sigma}{d\Omega}\left(\mathbf{n}, \boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}}, \mathbf{n}_0=\hat{\mathbf{k}}, \boldsymbol{\varepsilon}_0=\hat{\mathbf{j}}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \sin^2\varphi + \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4 a^6 \cos^2\varphi\right] \\
&= \frac{1}{2}k^4 a^6 \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2
\end{aligned}$$

which has no angular dependence. There is no preferred φ dependence in either case.

2.4 Final polarization not measured

As a final possibility, suppose we have unpolarized light coming in, and we do not measure the outgoing polarization. Then the result is the *sum* of the results for the outgoing radiation,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\mathbf{n}_0=\hat{\mathbf{k}}\right) &= \frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\theta}},\mathbf{n}_0=\hat{\mathbf{k}}\right)+\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon}=\hat{\boldsymbol{\varphi}},\mathbf{n}_0=\hat{\mathbf{k}}\right)\\ &= \frac{1}{2}k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left(1+\cos^2\theta\right)\end{aligned}$$

2.5 Total cross-section

The total light scattered gives an estimate of the size of the scatterer. In the present case, for unpolarized light, we integrate over all angles,

$$\begin{aligned}\sigma &= \int d\sigma\\ &= \int_0^{\pi}\int_0^{2\pi}\frac{1}{2}k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left(1+\cos^2\theta\right)\mathrm{d}\Omega\end{aligned}$$

The integral is

$$\iint\left(1+\cos^2\theta\right)\mathrm{d}\Omega = \frac{16\pi}{3}$$

so the total cross section is

$$\sigma=\frac{8}{3}\pi a^2\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2 k^4a^4$$

which in this case is the cross-sectional area, πa^2, times a dimensionless factor depending on the ratio $\frac{a}{\lambda}$.

3 Example 2: Scattering by a small, perfectly conducting sphere

For a perfectly conducting sphere, the boundary conditions change. In problem 1 you are asked to work out this case, so here we simply state the results for the electric and magnetic dipole strengths:

$$\begin{aligned}\mathbf{p} &= 4\pi\epsilon_0a^3\mathbf{E}_{inc}\\ \mathbf{m} &= -2\pi a^3\mathbf{H}_{inc}\end{aligned}$$

For linear polarization, $\mathbf{E}_{inc}$ and $\mathbf{H}_{inc}$ are orthogonal and orthogonal to the direction of propagation,

$$\begin{aligned}\mathbf{E}_{inc} &= E_{inc}\hat{\boldsymbol{\varepsilon}}_0e^{i\mathbf{n}_0\cdot\mathbf{x}-i\omega t}\\ \mathbf{H}_{inc} &= \frac{1}{\mu_0c}\mathbf{n}_0\times\mathbf{E}_{inc}\end{aligned}$$

so the dipole strengths are perpendicular as well. We can immediately write the differential cross-section,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0\right) &= \frac{k^4}{\left(4\pi\epsilon_0E_{inc}\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}+\frac{1}{c}\mathbf{m}\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2\\ &= \frac{k^4}{\left(4\pi\epsilon_0E_{inc}\right)^2}\left|\boldsymbol{\varepsilon}^*\cdot\left(4\pi\epsilon_0a^3\mathbf{E}_{inc}\right)-\frac{2\pi a^3}{c}\mathbf{H}_{inc}\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{k^4}{(4\pi\epsilon_0 E_{inc})^2}\left|\boldsymbol{\varepsilon}^*\cdot\left(4\pi\epsilon_0 a^3\mathbf{E}_{inc}\right) - 2\pi a^3\frac{1}{\mu_0 c^2}\left(\mathbf{n}_0\times\mathbf{E}_{inc}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2 \\
&= \frac{k^4}{(4\pi\epsilon_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\left(4\pi\epsilon_0 a^3\hat{\boldsymbol{\varepsilon}}_0\right) - 2\pi a^3\epsilon_0\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_0\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2 \\
&= k^4 a^6\left|\boldsymbol{\varepsilon}^*\cdot\hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_0\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2
\end{aligned}
$$

Now compute the differential cross-section in the case of unpolarized incident light. This means we *average* over incident polarizations, $\hat{\boldsymbol{\varepsilon}}_{0\|}$ and $\hat{\boldsymbol{\varepsilon}}_{0\perp}$, where

$$
\begin{aligned}
\hat{\boldsymbol{\varepsilon}}_{0\perp} &= \frac{1}{\sin\theta}\left(\mathbf{n}_0\times\mathbf{n}\right) \\
\hat{\boldsymbol{\varepsilon}}_{0\|} &= \frac{1}{\sin\theta}\left(\mathbf{n}_0\times\mathbf{n}\right)\times\mathbf{n}_0
\end{aligned}
$$

to find

$$
\frac{d\sigma}{d\Omega}\left(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0\right) = \frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\|},\hat{\boldsymbol{\varepsilon}}_{0\perp}} k^4 a^6\left|\boldsymbol{\varepsilon}^*\cdot\hat{\boldsymbol{\varepsilon}}_0 - \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_0\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}^*\right)\right|^2
$$

For the outgoing polarization perpendicular to the plane of incidence we still have

$$
\boldsymbol{\varepsilon}_\perp = \frac{1}{\sin\theta}\mathbf{n}_0\times\mathbf{n}
$$

but for the outgoing polarization perpendicular to the plane of incidence we need

$$
\boldsymbol{\varepsilon}_\| = \frac{1}{\sin\theta}\left(\mathbf{n}_0\times\mathbf{n}\right)\times\mathbf{n}
$$

Notice that

$$
\begin{aligned}
\hat{\boldsymbol{\varepsilon}}_{0\perp}\cdot\boldsymbol{\varepsilon}_\perp &= 1 \\
\hat{\boldsymbol{\varepsilon}}_{0\perp}\cdot\boldsymbol{\varepsilon}_\| &= 0 \\
\hat{\boldsymbol{\varepsilon}}_{0\|}\cdot\boldsymbol{\varepsilon}_\perp &= 0 \\
\hat{\boldsymbol{\varepsilon}}_{0\|}\cdot\boldsymbol{\varepsilon}_\| &= \cos\theta
\end{aligned}
$$

Outgoing polarization parallel to the plane of incidence For the parallel case, working through the vector products, we will need

$$
\begin{aligned}
\boldsymbol{\varepsilon}_\|^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\|} &= \cos\theta \\
\boldsymbol{\varepsilon}_\|^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\perp} &= 0 \\
\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\perp}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\|^*\right) &= \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\perp}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\|\right) \\
&= 0 \\
\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\|}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\|^*\right) &= \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\|}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\|\right) \\
&= \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\|}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\|\right) \\
&= \frac{1}{2}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\frac{d\sigma_\parallel}{d\Omega} &= \frac{1}{2}\sum_{\hat{\boldsymbol{\varepsilon}}_{0\parallel},\hat{\boldsymbol{\varepsilon}}_{0\perp}} k^4a^6\left|\boldsymbol{\varepsilon}_\parallel^*\cdot\hat{\boldsymbol{\varepsilon}}_0-\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_0\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\parallel^*\right)\right|^2\\
&= \frac{1}{2}k^4a^6\left|\boldsymbol{\varepsilon}_\parallel^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\parallel}-\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\parallel}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\parallel^*\right)\right|^2+\frac{1}{2}k^4a^6\left|\boldsymbol{\varepsilon}_\parallel^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\perp}-\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\perp}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\parallel^*\right)\right|^2\\
&= \frac{1}{2}k^4a^6\left|\cos\theta-\frac{1}{2}\right|^2
\end{aligned}$$

Outgoing polarization perpendicular to the plane of incidence For the perpendicular case, we need the vector products

$$\begin{aligned}
\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\parallel} &= 0\\
\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\perp} &= 1\\
\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\perp}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\perp^*\right) &= \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\perp}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\perp\right)\\
&= \frac{1}{2}\cos\theta\\
\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\parallel}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\perp^*\right) &= \frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\parallel}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\perp\right)\\
&= 0
\end{aligned}$$

so the perpendicular contribution to the differential cross-section is

$$\begin{aligned}
\frac{d\sigma_\perp}{d\Omega} &= \frac{1}{2}k^4a^6\left|\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\parallel}-\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\parallel}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\perp^*\right)\right|^2+\frac{1}{2}k^4a^6\left|\boldsymbol{\varepsilon}_\perp^*\cdot\hat{\boldsymbol{\varepsilon}}_{0\perp}-\frac{1}{2}\left(\mathbf{n}_0\times\hat{\boldsymbol{\varepsilon}}_{0\perp}\right)\cdot\left(\mathbf{n}\times\boldsymbol{\varepsilon}_\perp^*\right)\right|^2\\
&= \frac{1}{2}k^4a^6\left|1-\frac{1}{2}\cos\theta\right|^2\\
&= \frac{1}{2}k^4a^6\left|1-\frac{1}{2}\cos\theta\right|^2
\end{aligned}$$

If the final polarization is unmeasured, we sum these, giving

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma_\parallel}{d\Omega}+\frac{d\sigma_\perp}{d\Omega}\\
&= \frac{1}{2}k^4a^6\left|\cos\theta-\frac{1}{2}\right|^2+\frac{1}{2}k^4a^6\left|1-\frac{1}{2}\cos\theta\right|^2\\
&= \frac{1}{2}k^4a^6\left[\left(\cos^2\theta-\cos\theta+\frac{1}{4}\right)+\left(1-\cos\theta+\frac{1}{4}\cos^2\theta\right)\right]\\
&= \frac{1}{2}k^4a^6\left[\frac{5}{4}-2\cos\theta+\frac{5}{4}\cos^2\theta\right]\\
&= k^4a^6\left[\frac{5}{8}-\cos\theta+\frac{5}{8}\cos^2\theta\right]
\end{aligned}$$

We define the *polarization* of the scattered radiation to be the difference between the parallel and perpendicular cross-sections, normalized by the total differential cross-section,

$$\Pi \equiv \frac{1}{\frac{d\sigma}{d\Omega}}\left[\frac{d\sigma_\perp}{d\Omega}-\frac{d\sigma_\parallel}{d\Omega}\right]$$

$$\begin{aligned}
&= \frac{1}{k^4 a^6 \left[\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}\right]} \left[\frac{1}{2}k^4 a^6 \left|1 - \frac{1}{2}\cos\theta\right|^2 - \frac{1}{2}k^4 a^6 \left|\cos\theta - \frac{1}{2}\right|^2\right] \\
&= \frac{1}{\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}} \left[\frac{1}{2}\left(1 - \cos\theta + \frac{1}{4}\cos^2\theta\right) - \frac{1}{2}\left(\cos^2\theta - \cos\theta + \frac{1}{4}\right)\right] \\
&= \frac{1}{\frac{5}{8}\cos^2\theta - \cos\theta + \frac{5}{8}} \left[\frac{3}{8} - \frac{3}{8}\cos^2\theta\right] \\
&= \frac{3\sin^2\theta}{5\cos^2\theta - 8\cos\theta + 5}
\end{aligned}$$

4 Collections of scatterers

When light travels through a medium, it encounters many scatterers, so the scattering is a superposition of the results of many scatterings. Suppose there are scatterers located at positions, $\mathbf{x}_i$. Then since the fields vary as $e^{i\mathbf{k}\cdot\mathbf{x}}$ there will be factors

$$\mathbf{E}(\mathbf{x}_i), \mathbf{B}(\mathbf{x}_i) \sim e^{ik\mathbf{n}_0\cdot\mathbf{x}_i}$$

associated with the corresponding induced dipole moments,

$$\mathbf{p}_i, \mathbf{m}_i \sim e^{ik\mathbf{n}_0\cdot\mathbf{x}_i}$$

Recalling that the total differential cross-section depends on $\mathbf{E}\times\mathbf{B}^*$, we will have a sum over conjugate pairs of phase factors:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\sum_{i,j}\left[\boldsymbol{\varepsilon}^*\cdot\mathbf{p}_i + \frac{1}{c}\mathbf{m}_i\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\right] e^{ik\mathbf{n}_0\cdot\mathbf{x}_i} e^{-ik\mathbf{n}\cdot\mathbf{x}_i}\right|^2$$

Define

$$ik\mathbf{q}\cdot\mathbf{x}_i \equiv ik(\mathbf{n}_0 - \mathbf{n})\cdot\mathbf{x}_i$$

and assume all the scatterers are identical, so that

$$\begin{aligned}
\mathbf{p}_i &= \mathbf{p} \\
\mathbf{m}_i &= \mathbf{m}
\end{aligned}$$

Then the sum applies only to the phase factor, giving an overall factor of

$$\begin{aligned}
\mathscr{F}(\mathbf{q}) &= \left|\sum_i e^{ik\mathbf{q}\cdot\mathbf{x}_i}\right|^2 \\
&= \sum_i e^{ik\mathbf{q}\cdot\mathbf{x}_i}\sum_j e^{-ik\mathbf{q}\cdot\mathbf{x}_j} \\
&= \sum_{i,j} e^{ik\mathbf{q}\cdot(\mathbf{x}_i - \mathbf{x}_j)}
\end{aligned}$$

There are two important limiting cases of this. When the scatterers are randomly distributed, as in a gas, the different phases tend to cancel, so only the diagonal terms contribute,

$$\begin{aligned}
\mathscr{F}(\mathbf{q}) &= \sum_{i=j} e^{ik\mathbf{q}\cdot(\mathbf{x}_i - \mathbf{x}_j)} \\
&= \sum_{i=j} 1 \\
&= N
\end{aligned}$$

where N is the total number of scatterers. The second limiting case is when the scatterers form some sort of regular lattice. For a perfect lattice, the same thing happens, with the effect of a scatterer at $\mathbf{x}_i$ cancelling the effect of another scatterer at $-\mathbf{x}_i$. The wave progresses only in the forward direction (picture a clear crystal of pure quartz, for example). Scatterings do occur as a result of thermal vibrations which make the lattice imperfect. Jackson gives and explicit example of an exact result in eq.(10.20).