Multipole radiation: higher multipoles

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We carry our solution for the vector potential, nmm

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

to the next higher order in the radiation zone. Before starting, we develop an identity relating moments of the charge density to moments of the current.

1 Current dipole contribution to the vector potential

Expand the exact expression for the vector potential above, and expand to the next order:

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}}{1-\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'} \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}')\left(1-ik\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\left(1+\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'\right) \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}') + \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}')\left(-ik\hat{\mathbf{r}}\cdot\mathbf{x}'+\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'\right)
\end{aligned}$$

We know that the first term on the right leads to electric dipole radiation. Suppose this term vanishes, so that the radiation is described by the next order terms:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int d^3x' \mathbf{J}(\mathbf{x}')\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)$$

This now involves the next higher moment of the current,

$$\int d^3x' \mathbf{J}(\mathbf{x}')\left(\hat{\mathbf{n}}\cdot\mathbf{x}'\right)$$

We can express this dipole moment of the current density in terms of moments of the charge distribution and the magnetic dipole moment density. This separates electric and magnetic parts of the radiation. Starting with the magnetic moment density

$$\boldsymbol{\mathcal{M}} = \frac{1}{2}\left(\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\right)$$

we take a second cross product with the unit vector in the direction of the observation,

$$\begin{aligned}
\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}} &= \frac{1}{2}\hat{\mathbf{r}}\times\left(\mathbf{x}'\times\mathbf{J}\right) \\
&= \frac{1}{2}\left(\left(\hat{\mathbf{r}}\cdot\mathbf{J}\right)\mathbf{x}'-\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\mathbf{J}\right)
\end{aligned}$$

so that

$$\mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}') = \mathbf{x}'(\hat{\mathbf{r}}\cdot\mathbf{J}) - 2\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}}$$

Writing half of $\mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}')$ using this identity, we have

$$\begin{aligned}\int d^3x'\mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}') &= \int d^3x'\frac{1}{2}\left(\mathbf{J}(\hat{\mathbf{r}}\cdot\mathbf{x}') + \mathbf{x}'(\hat{\mathbf{r}}\cdot\mathbf{J}) - 2\hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}}\right)\\ &= \int d^3x'\left(\frac{1}{2}\left(\mathbf{J}(\hat{\mathbf{r}}\cdot\mathbf{x}') + \mathbf{x}'(\hat{\mathbf{r}}\cdot\mathbf{J})\right) - \hat{\mathbf{r}}\times\boldsymbol{\mathcal{M}}\right)\end{aligned}$$

The first pair of terms give the integral of a symmetrized product,

$$\frac{1}{2}\hat{r}_k\int d^3x'\left(J_i x'_k + x'_i J_k\right)$$

and this integral may be related to the charge density by working with the integral of the divergence of $x'_i x'_j\mathbf{J}$. First expand the divergence,

$$\begin{aligned}\boldsymbol{\nabla}'\cdot\left(x'_i x'_j\mathbf{J}\right) &= \sum_{k=1}^{3}\frac{\partial}{\partial x'_k}\left(x'_i x'_j J_k\right)\\ &= \sum_{k=1}^{3}\left(\delta_{ik}x'_j J_k + x'_i\delta_{jk}J_k + x'_i x'_j\boldsymbol{\nabla}'\cdot\mathbf{J}\right)\\ &= x'_j J_i + x'_i J_j + i\omega x'_i x'_j\rho(\mathbf{x}')\end{aligned}$$

where in the last step we have used the continuity equation,

$$0 = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = -i\omega\rho + \boldsymbol{\nabla}\cdot\mathbf{J}$$

Now, since the integral of the divergence becomes a surface term at infinity where the current vanishes, we have

$$\begin{aligned}0 &= \int d^3x'\,\boldsymbol{\nabla}'\cdot\left(x'_i x'_j\mathbf{J}\right)\\ &= \int d^3x'\,\left(x'_j J_i + x'_i J_j + i\omega x'_i x'_j\rho(\mathbf{x}')\right)\end{aligned}$$

giving the symmetrized integral we seek in terms of the quadrupole moment of the charge distribution,

$$\frac{1}{2}\hat{r}_k\int d^3x'\left(J_i x'_k + x'_i J_k\right) = -i\omega\frac{1}{2}\hat{r}_k\int d^3x'\,x'_i x'_k\rho(\mathbf{x}')$$

Finally, notice that

$$\hat{r}_k\int d^3x'\,\delta_{ik}r'^2\rho = (\nabla_i r)\int d^3x'\,r'^2\rho(\mathbf{x}')$$

Adding and subtracting a $\frac{i\omega}{6}\delta_{ij}$ times this gradient, gives

$$\begin{aligned}\hat{r}_k\int d^3x'\,\left(x'_k J_i + x'_i J_k\right) &= -i\omega\hat{r}_k\int d^3x'\,\frac{1}{6}\left(3x'_i x'_k - \delta_{ik}r'^2\right)\rho(\mathbf{x}') - \frac{i\omega}{6}\left(\int d^3x'\,\delta_{ik}r'^2\rho(\mathbf{x}')\right)\nabla_k r\\ &= -\frac{1}{6}i\omega\hat{r}_k\int d^3x'\,\left(3x'_i x'_k - \delta_{ik}r'^2\right)\rho(\mathbf{x}') - \frac{i\omega}{6}\left(\int d^3x'\,r'^2\rho(\mathbf{x}')\right)\nabla_i r\end{aligned}$$

so we may identify the first integral as the quadrupole moment,

$$Q_{ik} \equiv \int d^3x'\,\left(3x'_i x'_k - \delta_{ik}r'^2\right)\rho(\mathbf{x}')$$

To clean up the result, we define the vector $\mathbf{Q}(\hat{\mathbf{r}})$ with

$$[\mathbf{Q}(\hat{\mathbf{r}})]_i \equiv \sum_k \hat{r}_k Q_{ik}$$

and the second integral is the constant

$$T \equiv \int d^3x'\, r'^2 \rho(\mathbf{x}')$$

The integral over the magnetic dipole density is the magnetic dipole moment,

$$\mathbf{m} = \int d^3x \boldsymbol{\mathcal{M}}$$

so the vector potential becomes

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int d^3x' \mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}') \\ &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\frac{1}{6}i\omega\mathbf{Q}(\hat{\mathbf{r}}) - \hat{\mathbf{r}}\times\mathbf{m}\right) - \frac{i\omega T}{6}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\boldsymbol{\nabla} r\end{aligned}$$

Since the final term has the form $f(r)\boldsymbol{\nabla} r$, it is a total gradient; its curl vanishes and it makes no contribution to the fields. We may drop it without consequence to write the vector potential as

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\left(\frac{i\omega}{6}\mathbf{Q}(\hat{\mathbf{r}}) + \hat{\mathbf{r}}\times\mathbf{m}\right)$$

At this order of perturbation the vector potential has an electric quadrupole contribution and a magnetic dipole contribution.

2 The electric and magnetic fields

2.1 Magnetic dipole radiation

The fields are now found in the usual way. The magnetic dipole potential is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\hat{\mathbf{r}}\times\mathbf{m}$$

In calculating the *electric* dipole radiation, we showed that

$$\begin{aligned}\mathbf{E}(r) &= \frac{ik}{4\pi\epsilon_0}\boldsymbol{\nabla}\times\left(\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{p}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[k^2\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}}) + \frac{ik}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right]\end{aligned}$$

so that

$$\boldsymbol{\nabla}\times\left(ik\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{p}\right) = \frac{e^{ikr}}{r}\left[k^2\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}}) + \frac{ik}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right]$$

Now, using this same derivative with $\mathbf{p}$ replaced by $\mathbf{m}$ to find the magnetic field for dipole radiation,

$$\begin{aligned}\mathbf{H}(\mathbf{x}) &= \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A} \\ &= \frac{1}{4\pi}\boldsymbol{\nabla}\times\left(\frac{ike^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\hat{\mathbf{r}}\times\mathbf{m}\right) \\ &= \frac{1}{4\pi}\frac{e^{ikr}}{r}\left[k^2\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}}) + \frac{ik}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{m} - 3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right]\end{aligned}$$

At sufficiently large r, the magnetic field approaches

$$\mathbf{H}(\mathbf{x}) = \frac{1}{4\pi}\frac{k^2 e^{ikr}}{r}\hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}})$$

which is transverse to the radially propagating wave.

The Maxwell equations for harmonic sources tell us that the electric field is given in terms of the potentials by

$$\mathbf{E} = i\omega\mathbf{A} - \boldsymbol{\nabla}\Phi$$

and the absence of lower multipoles shows that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0$$

so that $\Phi = 0$. Therefore,

$$\mathbf{E} = \frac{i\omega\mu_0}{4\pi}\frac{ike^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\hat{\mathbf{r}} \times \mathbf{m}$$

Using $Zc = \frac{1}{\epsilon_0}$ and $\omega = kc$,

$$\mathbf{E} = -\frac{Z}{4\pi}\frac{k^2 e^{ikr-i\omega t}}{r}\left(1 - \frac{1}{ikr}\right)\hat{\mathbf{r}} \times \mathbf{m}$$

This gives the fields produced by magnetic dipole radiation are therefore

$$\begin{aligned}
\mathbf{H} &= \frac{1}{4\pi}\frac{e^{ikr-i\omega t}}{r}\left[k^2\hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}}) + \frac{ik}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{m} - 3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})\right] \\
\mathbf{E} &= -\frac{Z_0}{4\pi}k^2\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}} \times \mathbf{m}
\end{aligned}$$

in complete analogy to the electric dipole field, but with magnetic and electric parts interchanged. In the radiation zone, these become

$$\begin{aligned}
\mathbf{H} &= \frac{1}{4\pi}k^2\frac{e^{ikr-i\omega t}}{r}[\hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}})] \\
\mathbf{E} &= -\frac{Z_0}{4\pi}k^2\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}} \times \mathbf{m}
\end{aligned}$$

and the fields become totally transverse.

2.2 Electric quadrupole radiation

For the quadrupole fields, we begin with the quadrupole piece of the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{i\omega\mu_0}{24\pi}\frac{ike^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\mathbf{Q}(\hat{\mathbf{r}})$$

The magnetic field is then

$$\mathbf{H}(\mathbf{x}, t) = \frac{1}{\mu_0}\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})$$

Keeping only terms of order $\frac{1}{r}$, this gives

$$\begin{aligned}
\mathbf{H}(\mathbf{x}, t) &= \frac{1}{\mu_0}\boldsymbol{\nabla} \times \left[\frac{i\omega\mu_0}{24\pi}\frac{ike^{ikr}}{r}\mathbf{Q}(\hat{\mathbf{r}})\right] \\
&= \frac{1}{\mu_0}\left[-\frac{i\omega k^2\mu_0}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \mathbf{Q}(\hat{\mathbf{r}})\right] \\
&= -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \mathbf{Q}(\hat{\mathbf{r}})
\end{aligned}$$

since the only derivative term that does not increase the power of $\frac{1}{r}$ is $\left[\frac{i\omega\mu_0}{24\pi}\frac{ik}{r}\left(\boldsymbol{\nabla}e^{ikr}\right)\times\mathbf{Q}(\hat{\mathbf{r}})\right]$. The electric field in this approximation is then

$$\begin{aligned}
\mathbf{E} &= Z_0\mathbf{H}\times\hat{\mathbf{r}} \\
&= Z_0\left(\frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\times\hat{\mathbf{r}} \\
&= -\frac{ik^3}{24\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}
\end{aligned}$$

3 Radiated power

The energy per unit area carried by an electromagnetic wave is given by the Poynting vector,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

For a plane wave, we have

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t) \\
\mathbf{H} &= \frac{1}{\mu}\sqrt{\mu\varepsilon}\frac{1}{k}\mathbf{k}\times\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{S} &= \frac{c}{8\pi}\mathbf{E}\times\mathbf{H}^* \\
\mathbf{S} &= \mathbf{E}\times\mathbf{H} \\
&= \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)\times\left(\frac{1}{\mu}\sqrt{\mu\varepsilon}\frac{1}{k}\mathbf{k}\times\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)\right) \\
&= \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}\mathbf{E}_0\times(\mathbf{k}\times\mathbf{E}_0)\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t) \\
&= \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}E_0^2\mathbf{k}\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t)
\end{aligned}$$

with real part

$$\mathbf{S} = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}E_0^2\mathbf{k}\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t)$$

with time average

$$\mathbf{S} = \left(\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}E_0^2\right)\hat{\mathbf{k}}$$

For a complex representation of the wave,

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_0e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
\mathbf{H} &= \frac{1}{\mu}\sqrt{\mu\varepsilon}\frac{1}{k}\mathbf{k}\times\mathbf{E}_0e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}
\end{aligned}$$

we may write the same quantity as

$$\begin{aligned}
\frac{1}{2}Re(\mathbf{E}\times\mathbf{H}^*) &= \frac{1}{2}Re\left(\mathbf{E}_0e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\times\frac{1}{\mu}\sqrt{\mu\varepsilon}\hat{\mathbf{k}}\times\mathbf{E}_0^*e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right) \\
&= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|E_0|^2\hat{\mathbf{k}}
\end{aligned}$$

so the time-averaged energy flow per unit area per unit time is

$$\mathbf{S} = \frac{1}{2} Re\left(\mathbf{E} \times \mathbf{H}^*\right)$$

Now consider the average power carried off by electric dipole, electric quadrupole, and magnetic dipole radiation.

3.1 Electric dipole

The radiation zone fields were found above to be

$$\begin{aligned} \mathbf{H}(\mathbf{x}, t) &= \frac{k^2 c}{4\pi} \frac{e^{ikr - i\omega t}}{r} \hat{\mathbf{r}} \times \mathbf{p} \\ \mathbf{E}(\mathbf{x}, t) &= \frac{k^2}{4\pi\varepsilon_0} \frac{e^{ikr - i\omega t}}{r} \hat{\mathbf{r}} \times (\mathbf{p} \times \hat{\mathbf{r}}) \end{aligned}$$

so that

$$\begin{aligned} \frac{dP}{dA} &= \hat{\mathbf{r}} \cdot \mathbf{S} \\ &= \frac{1}{2} Re\left(\hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{H}^*)\right) \\ &= \frac{1}{2} Re\left(\frac{k^2}{4\pi\varepsilon_0} \frac{1}{r} \hat{\mathbf{r}} \cdot \left([\hat{\mathbf{r}} \times (\mathbf{p} \times \hat{\mathbf{r}})] \times \frac{k^2 c}{4\pi} \frac{1}{r} [\hat{\mathbf{r}} \times \mathbf{p}]\right)\right) \\ &= \frac{ck^4}{32\pi^2\varepsilon_0} \frac{1}{r^2} |\mathbf{p}|^2 \sin^2\theta \\ &= \frac{c^2 k^4 \sqrt{\mu_0 \varepsilon_0}}{32\pi^2 \varepsilon_0} \frac{1}{r^2} |\mathbf{p}|^2 \sin^2\theta \\ &= \frac{c^2 Z_0}{32\pi^2} \frac{1}{r^2} k^4 |\mathbf{p}|^2 \sin^2\theta \end{aligned}$$

This is the power per unit area. Since the area element at large distances is $dA = r^2 d\Omega$, where Ω is the solid angle, we may write the differential power radiated per unit solid angle using

$$\frac{dP}{dA} = \frac{1}{r^2} \frac{dP}{d\Omega}$$

so that

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta$$

3.2 Magnetic dipole

The radiation zone fields for magnetic dipole radiation are

$$\begin{aligned} \mathbf{H}(\mathbf{x}, t) &= \frac{k^2}{4\pi} \frac{e^{ikr - i\omega t}}{r} \hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}}) \\ \mathbf{E}(\mathbf{x}, t) &= -\frac{Z_0 k^2}{4\pi} \frac{e^{ikr - i\omega t}}{r} \hat{\mathbf{r}} \times \mathbf{m} \end{aligned}$$

so the result is the same as for the electric dipole with the substitution $\mathbf{p} \longrightarrow \mathbf{m}/c$,

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2} k^4 |\mathbf{m}|^2 \sin^2\theta$$

3.3 Electric quadrupole moment

For electric quadrupole radiation the fields are given by

$$\begin{aligned}\mathbf{H}(\mathbf{x},t) &= \frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}}) \\ \mathbf{E}(\mathbf{x},t) &= -\frac{ik^3}{24\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\end{aligned}$$

giving an average power per unit solid angle of

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{r^2}{2}\left|Re\left(\mathbf{E}\times\mathbf{H}^*\right)\right| \\ &= \frac{r^2}{2}\left|Re\left[\left(-\frac{ik^3}{24\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right)\times\left(\frac{ick^3}{24\pi}\frac{e^{-ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\right]\right| \\ &= \frac{1}{2}\frac{k^3}{24\pi\varepsilon_0}\frac{ck^3}{24\pi}\left|\left(\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right)\times\left(\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\right| \\ &= \frac{ck^6}{1152\pi^2\varepsilon_0}\left|\left(\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right)\times\left(\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right)\right| \\ &= \frac{Z_0c^2}{1152\pi^2}k^6\left|\left[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})\right]\times\hat{\mathbf{r}}\right|^2\end{aligned}$$

Notice that the power radiated by the quadrupole moment depends on k^6, whereas the power radiated by the dipole moments both go as k^4. This pattern continues for higher moments.

4 Higher moments of the current density

Higher multipole moments of the fields, will depend on higher moments of the current density. Here we show that these higher moments may always be expressed in terms of moments of the charge density, and moments of the magnetic moment density.

In our discussion of electric dipole radiation, we used the continuity equation to find the identity

$$\int d^3x\, J_j(\mathbf{x}) = i\omega\int d^3x\, x_j\rho(\mathbf{x})$$

relating the zeroth moment of the current density to the first moment of the charge density. In the example above, we found that the first moment of the current density may be written in terms of the magnetic dipole moment density, and the quadrupole moment of charge density. We show here that in general that moments of these two types of distributions are necessary: the moments of the current density may always be expressed in terms of moments of the charge density and moments of the magnetic dipole moment density.

Define the n^{th} moment of the charge density to be

$$p_{k_1\ldots k_n} \equiv \int d^3x\, x_{k_1}x_{k_2}\ldots x_{k_n}\rho(\mathbf{x})$$

and recall that the magnetic dipole moment is defined to be $\boldsymbol{\mathcal{M}} = \frac{1}{2}\mathbf{x}\times\mathbf{J}$. It is convenient to extract the components of $\boldsymbol{\mathcal{M}}$,

$$\mathcal{M}_{ij} \equiv \frac{1}{2}\left(x_iJ_j - x_jJ_i\right)$$

so $\mathcal{M}_i = \frac{1}{2}\sum_{j,k}\varepsilon_{ijk}\mathcal{M}_{jk}$. This lets us exchange indices between $\mathbf{J}$ and $\mathbf{x}$ when computing moments,

$$x_mJ_n = x_nJ_m + 2\mathcal{M}_{mn}$$

Then we define the n^{th} moment of the magnetic dipole moment to be

$$m_{k_1 k_2 \ldots k_n} \equiv \int d^3x \, \mathcal{M}_{k_1 k_2} x_{k_3} \ldots x_{k_n}$$

We now show that all moments of the current density may be expressed in terms of moments of the charge density $p_{k_1 \ldots k_n}$ and moments of the magnetic dipole density, $m_{k_1 \ldots k_n}$. The proof is by induction. We know how the zeroth moment of the current may be expressed in terms of the first moment of the charge density,

$$\int d^3x J_j(\mathbf{x}) = i\omega \int d^3x \, x_j \rho(\mathbf{x})$$

Now, suppose we know that the first $n-1$ moments of a current distribution, $\mathbf{j}_{k_1 \ldots k_{n-1}}$, may be expressed in terms of moments of the charge density, and moments of the magnetic dipole density, and consider the n^{th} moment of the current density,

$$\mathbf{j}_{k_1 \ldots k_n} \equiv \int d^3x \, x_{k_1} \ldots x_{k_n} \mathbf{J}(\mathbf{x})$$

with components

$$j_{m k_1 \ldots k_n} = \int d^3x \, x_{k_1} \ldots x_{k_n} J_m$$

Since we are dealing with isolated systems, $\mathbf{J}(\mathbf{x})$ vanishes at infinity, so the integral of a total divergence involving $\mathbf{J}(\mathbf{x})$ vanishes as well,

$$\begin{aligned} 0 &= \int d^3x \boldsymbol{\nabla} \cdot \left(x_{k_1} x_{k_2} \ldots x_{k_n} \mathbf{J}(\mathbf{x}) \right) \\ &= \sum_i \int d^3x \nabla_i \left(x_{k_1} x_{k_2} \ldots x_{k_n} J_i \right) \\ &= \sum_i \int d^3x \left[\left(\delta_{ik_1} x_{k_2} \ldots x_{k_n} + x_{k_1} \delta_{ik_2} \ldots x_{k_{n+1}} + \ldots + x_{k_1} x_{k_2} \ldots \delta_{ik_n} \right) J_i + x_{k_1} x_{k_2} \ldots x_{k_n} \nabla_i J_i \right] \\ &= \int d^3x \left[J_{k_1} x_{k_2} \ldots x_{k_{n+1}} + x_{k_1} J_{k_2} \ldots x_{k_{n+1}} + \ldots + x_{k_1} x_{k_2} \ldots J_{k_n} + i\omega x_{k_1} x_{k_2} \ldots x_{k_n} \rho \right] \end{aligned}$$

Now, use the rearrangement result on the first n terms,

$$J_{k_1} x_{k_2} \ldots x_{k_{n+1}} + \ldots + x_{k_1} x_{k_2} \ldots J_{k_n} = n J_{k_1} x_{k_2} \ldots x_{k_n} + 2\mathcal{M}_{k_1 k_2} x_{k_3} \ldots x_{k_{n+1}} + \ldots + 2\mathcal{M}_{k_1 k_{n+1}} x_{k_2} \ldots x_{k_n}$$

while the final term is the n^{th} moment of the charge density, as desired.

Therefore,

$$\begin{aligned} 0 &= \int d^3x \left[n J_{k_1} x_{k_2} \ldots x_{k_n} + 2\mathcal{M}_{k_1 k_2} x_{k_3} \ldots x_{k_{n+1}} + \ldots + 2\mathcal{M}_{k_1 k_{n+1}} x_{k_2} \ldots x_{k_n} + i\omega x_{k_1} x_{k_2} \ldots x_{k_n} \rho \right] \\ &= n j_{m k_1 \ldots k_n} + 2\mathcal{M}_{k_1 k_2 k_3 \ldots k_n} + 2\mathcal{M}_{k_1 k_3 k_2 k_4 \ldots k_n} + \ldots + 2\mathcal{M}_{k_1 k_n k_2 k_3 \ldots k_{n-1}} + i\omega p_{k_1 \ldots k_n} \end{aligned}$$

and we have shown that the n^{th} moment of the current density may be expressed in terms of moments of the magnetic dipole moment density and the electric charge density,

$$j_{m k_1 \ldots k_n} = \frac{2}{n} \left(\mathcal{M}_{k_1 k_2 k_3 \ldots k_n} + \mathcal{M}_{k_1 k_3 k_2 k_4 \ldots k_n} + \ldots + \mathcal{M}_{k_1 k_n k_2 k_3 \ldots k_{n-1}} \right) + \frac{i}{n} \omega p_{k_1 \ldots k_n}$$

Therefore, the result holds for all current density moments.