Multipole Radiation

February 29, 2016

1 The electromagnetic field of an isolated, oscillating source

Consider a localized, oscillating source, located in otherwise empty space. We know that the solution for the vector potential (e.g. using the Green function for the outer boundary at infinity) is

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\int dt'\frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)$$

Let the source fields be confined in a region $d\ll\lambda$ where λ is the wavelength of the radiation, and let the time dependence be harmonic, with frequency ω,

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \mathbf{A}(\mathbf{x})e^{-i\omega t}\\ \mathbf{J}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x})e^{-i\omega t}\\ \rho(\mathbf{x},t) &= \rho(\mathbf{x})e^{-i\omega t}\end{aligned}$$

Then

$$\begin{aligned}\mathbf{A}(\mathbf{x})e^{-i\omega t} &= \frac{\mu_0}{4\pi}\int d^3x'\int dt'\frac{\mathbf{J}(\mathbf{x}')e^{-i\omega t'}}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\\ &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')e^{-i\omega\left(t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}}{|\mathbf{x}-\mathbf{x}'|}\end{aligned}$$

so that with $k=\frac{\omega}{c}$, we have

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

These considerations mean that the entire source oscillates very nearly coherently. The characteristic time of one oscillation of the source, $\sim\frac{\lambda}{c}$, is much longer than the time it takes light to cross the source, $\sim\frac{d}{c}$.

The electric and magnetic fields follow immediately. We know that $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, so

$$\mathbf{H}=\frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A}$$

while the Maxwell equation, $\boldsymbol{\nabla}\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=0$, shows that

$$-i\omega(\epsilon_0\mathbf{E})=\boldsymbol{\nabla}\times\mathbf{H}$$

Dividing by

$$\begin{aligned}-i\omega\epsilon_0 &= -ikc\epsilon_0\\ &= -\frac{ik\epsilon_0}{\sqrt{\mu_0\epsilon_0}}\\ &= -ik\sqrt{\frac{\epsilon_0}{\mu_0}}\end{aligned}$$

and defining the impedence of free space, $Z=\sqrt{\frac{\mu_0}{\epsilon_0}}$, gives the electric field in the form

$$\mathbf{E}=\frac{iZ}{k}\boldsymbol{\nabla}\times\mathbf{H}$$

Now we consider the radiation. The problem divides into three approximate regions, depending on the length scales d and λ. We assume $d\ll\lambda$. If r is the distance of the observation from the source, we will consider

$$\begin{aligned}
d &\ll r\ll\lambda \quad (\textit{static zone})\\
d &\ll r\sim\lambda \quad (\textit{induction zone})\\
d &\ll \lambda\ll r \quad (\textit{radiation zone})
\end{aligned}$$

or simply the near, intermediate, and far zones.

1.1 Near zone

For the near zone,

$$kr=\frac{2\pi r}{\lambda}\ll 1$$

implies

$$e^{ikr}\approx 1$$

and the potential becomes

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\lim_{kr\to 0}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\
&= \frac{\mu_0}{4\pi}\lim_{kr\to 0}\sum_{l,m}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\int d^3x'\,\mathbf{J}(\mathbf{x}')\,r'^{l}Y^*_{lm}(\theta',\varphi')
\end{aligned}$$

and the only time dependence is the sinusoidal oscillation, $e^{-i\omega t}$. The spatial integration depends strongly on the details of the source.

1.2 Far zone

The exponential becomes important in the radiation zone, $kr\gg 1$. Setting $\mathbf{x}=r\hat{\mathbf{r}}$, we have

$$\begin{aligned}
|\mathbf{x}-\mathbf{x}'| &= \sqrt{r^2+x'^2-2r\hat{\mathbf{r}}\cdot\mathbf{x}'}\\
&= r\sqrt{1+\frac{x'^2}{r^2}-\frac{2}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'}
\end{aligned}$$

and since $x'^2\ll r^2$, we may drop the quadratic term and approximate the square root,

$$\begin{aligned}
|\mathbf{x}-\mathbf{x}'| &\approx r\sqrt{1-\frac{2}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'}\\
&\approx r\left(1-\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\\
&= r-\hat{\mathbf{r}}\cdot\mathbf{x}'
\end{aligned}$$

The potential is then

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')\,e^{ik\left(r-\hat{\mathbf{r}}\cdot\mathbf{x}'\right)}}{r-\hat{\mathbf{r}}\cdot\mathbf{x}'}$$

For the lowest order approximation, we neglect the x' term in the denominator,

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}}{1-\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'} \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}\left(1+\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}'\right) \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}
\end{aligned}$$

where the last step follows because $d \ll \lambda$ implies $\frac{1}{r}\hat{\mathbf{r}}\cdot\mathbf{x}' \ll k\hat{\mathbf{r}}\cdot\mathbf{x}'$.

For higher orders in $kx' \lessapprox kd \ll 1$, note that powers of kx' decrease rapidly in magnitude. We can carry a power series in kx' to higher order N in $kx' \sim kd$ as long as we can still neglect $\frac{d}{r}$,

$$\frac{d}{r} \ll (kd)^N$$

The expansion is then

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'} \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}\frac{(-ik)^n}{n!}\int d^3x' \mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}')^n
\end{aligned}$$

and because each term is smaller than the last by a factor on the order of kd, it is only the lowest nonvanishing moment of the current distribution

$$\int d^3x' \mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}')^n$$

that dominates the radiation field.

The radiation zone solution is characteristic of radiation. Returning to lowest order

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}$$

we note that the integral contributes only angular dependence of the field,

$$\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'} = \mathbf{f}(\theta,\varphi)$$

so the waveform is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{f}(\theta,\varphi)$$

The potential is therefore a harmonic, radially-expanding waveform, with amplitude decreasing as $\frac{1}{r}$,

$$\mathbf{A}(\mathbf{x},t) \sim \frac{e^{i(kr-\omega t)}}{r}$$

The magnetic field is given by

$$\begin{aligned}
\mathbf{B}(\mathbf{x}) &= \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x}) \\
&= \boldsymbol{\nabla}\times\left(\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}\right) \\
&= \frac{\mu_0}{4\pi}\left[\left(\boldsymbol{\nabla}\frac{e^{ikr}}{r}\right)\times\int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'} - \frac{e^{ikr}}{r}\int d^3x' \mathbf{J}(\mathbf{x}')\times\boldsymbol{\nabla}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}\right]
\end{aligned}$$

The first term is in the direction

$$\hat{\mathbf{r}} \times \int d^3x' \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}$$

and therefore transverse. Since the gradient includes a term

$$\nabla \frac{e^{ikr}}{r} \sim ik\hat{\mathbf{r}} \frac{e^{ikr}}{r}$$

the magnetic field will fall off as $\frac{1}{r}$.

For the second term

$$\int d^3x' \mathbf{J}(\mathbf{x}') \times \nabla e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'} = \int d^3x' \mathbf{J}(\mathbf{x}') \times \nabla e^{-ik\left(\mathbf{x}\cdot\mathbf{x}'\right)/r}$$

the gradient gives

$$\nabla e^{-ik\left(\mathbf{x}\cdot\mathbf{x}'\right)/r} = e^{-ik\left(\mathbf{x}\cdot\mathbf{x}'\right)/r} \left(-\frac{ik\mathbf{x}'}{r} + \frac{ik\hat{\mathbf{r}}\cdot\mathbf{x}'}{r}\hat{\mathbf{r}} \right)$$

The second term is also transverse. For the remaining term,

$$\int d^3x' \left(\mathbf{x}' \times \mathbf{J}\right) e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}$$

we may think of the exponential as giving a modified source,

$$\tilde{\mathbf{J}} = \mathbf{J} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}$$

Then this integral is just twice the magnetic dipole moment of that modified source. We know that the resulting magnetic field may be written as

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left[\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{m}\right) - \mathbf{m}}{r^3} \right]$$

Although this does have a radial component, the magnitude falls off as the *cube* of the distance, and is therefore negligible. The magnetic field is therefore transverse to the radial direction of propagation.

The electric field is also dominated by the

$$\nabla e^{ikr} \sim ik\mathbf{n} e^{ikr}$$

term, falling off as $\frac{1}{r}$, and is therefore also transverse to the radial propagation.

1.3 Intermediate zone

In the intermediate zone, $r \sim \lambda$, an exact expansion of the Green function is required. This is found by expanding

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{ik\left|\mathbf{x}-\mathbf{x}'\right|}}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}$$

in spherical harmonics,

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l,m} g_{lm}(r, r') Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta', \varphi')$$

and solving the rest of the Helmholtz equation for the radial function. The result is spherical Bessel functions, $j_l(x), n_l(x)$, and the related spherical Henkel functions, $h_l^{(1)}, h_l^{(2)}$, which are essentially Bessel function times $\frac{1}{\sqrt{r}}$. They are discussed in Jackson, Section 9.6. The vector potential then takes the form

$$\mathbf{A}(\mathbf{x}) = ik\mu_0 \sum_{l,m} h_l^{(1)}(kr) Y_{lm}(\theta, \varphi) \int d^3x' \mathbf{J}(\mathbf{x}') j_l(kr') Y^*_{lm}(\theta', \varphi')$$

The spherical Bessel function may be expanded in powers of kr to recover the previous approximations.

2 Explicit multipoles: n = 0 and n = 1

For higher multipoles ($n > 0$), we require the vector potential in order to get both magnetic and electric fields. We can then find both fields using

$$\mathbf{H} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A}$$
$$\mathbf{E} = \frac{iZ}{k}\boldsymbol{\nabla}\times\mathbf{H}$$

Since there is no magnetic monopole field, we may use the scalar potential to demonstrate the absence of monopole radiation.

2.1 Monopole field

The lowest order far field is the electric monopole field. For this it is easiest to use the solution for the scalar potential, in terms of charge density,

$$\begin{aligned}\phi(\mathbf{x},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\int dt'\frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\\ &= \frac{1}{4\pi\varepsilon_0 r}\int d^3x'\int dt'\rho(\mathbf{x}',t')\,\delta\left(t'-t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\end{aligned}$$

However, all charge is confined to a central region, and total charge is conserved. This means that the spatial integral is independent of time,

$$q_{tot} = \int d^3x'\rho(\mathbf{x}',t')$$

and the time integral of the delta function simply gives one. Therefore,

$$\phi(\mathbf{x},t) = \frac{q_{tot}}{4\pi\varepsilon_0 r}$$

The electric field is a static Coulomb field; there is no radiation.

2.2 Dipole field

If the first nonvanishing term in the multipole expansion,

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}(\mathbf{x}')\,e^{-ik\hat{\mathbf{r}}\cdot\mathbf{x}'}\\ &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}\frac{(-ik)^n}{n!}\int d^3x'\mathbf{J}(\mathbf{x}')(\hat{\mathbf{r}}\cdot\mathbf{x}')^n\end{aligned}$$

is the lowest ($n = 0$), then we have

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}(\mathbf{x}')$$

From the continuity equation, we have

$$\begin{aligned}0 &= \frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{J}\\ &= -i\omega\rho(\mathbf{x})+\boldsymbol{\nabla}\cdot\mathbf{J}\end{aligned}$$

Now, since the current vanishes at infinity, the integral of the divergence of $x_j \mathbf{J}(\mathbf{x}')$ must vanish:

$$\begin{aligned}\int d^3x' \boldsymbol{\nabla}' \cdot \left(x_j' \mathbf{J}(\mathbf{x}')\right) &= \int d^2x' \hat{\mathbf{r}}' \cdot \left(x_j' \mathbf{J}(\mathbf{x}')\right) \\ &= 0\end{aligned}$$

Then

$$\begin{aligned}0 &= \int d^3x' \boldsymbol{\nabla}' \cdot \left(x_j' \mathbf{J}(\mathbf{x}')\right) \\ &= \sum_i \int d^3x' \nabla_i' \left(x_j' J_i(\mathbf{x}')\right) \\ &= \sum_i \int d^3x' \left(\left(\nabla_i' x_j'\right) J_i(\mathbf{x}') + x_j' \nabla_i' J_i(\mathbf{x}')\right) \\ &= \sum_i \int d^3x' \left(\delta_{ij} J_i(\mathbf{x}') + x_j' \nabla_i' J_i(\mathbf{x}')\right) \\ &= \int d^3x' \left(J_j(\mathbf{x}') + x_j' \boldsymbol{\nabla}' \cdot \mathbf{J}\right)\end{aligned}$$

where the first integral is the one we require. Using the continuity equation to replace the divergence, we have

$$\begin{aligned}\int d^3x' J_j(\mathbf{x}') &= -\int d^3x'\, x_j' \boldsymbol{\nabla}' \cdot \mathbf{J} \\ &= -i\omega \int d^3x'\, x_j' \rho(\mathbf{x}')\end{aligned}$$

This integral is the electric dipole moment,

$$\mathbf{p} = \int d^3x'\, \mathbf{x}' \rho(\mathbf{x}')$$

and the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\frac{i\omega\mu_0}{4\pi} \frac{e^{ikr}}{r} \mathbf{p}$$

The magnetic field is the curl of this,

$$\begin{aligned}\mathbf{H}(\mathbf{x}) &= \frac{1}{\mu_0} \boldsymbol{\nabla} \times \mathbf{A} \\ &= -\frac{i\omega}{4\pi} \boldsymbol{\nabla} \times \left(\frac{e^{ikr}}{r} \mathbf{p}\right) \\ &= -\frac{i\omega}{4\pi} \left(\boldsymbol{\nabla} \frac{e^{ikr}}{r} \times \mathbf{p}\right) \\ &= -\frac{i\omega}{4\pi} \frac{\partial}{\partial r} \left(\frac{e^{ikr}}{r}\right) \hat{\mathbf{r}} \times \mathbf{p} \\ &= -\frac{i\omega}{4\pi} \left(\frac{ike^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right) \hat{\mathbf{r}} \times \mathbf{p} \\ &= \frac{\omega k}{4\pi} \left(1 - \frac{1}{ikr}\right) \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times \mathbf{p} \\ &= \frac{k^2 c}{4\pi} \left(1 - \frac{1}{ikr}\right) \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times \mathbf{p}\end{aligned}$$

This is transverse to the radially propagating wave. For the electric field,

$$\begin{aligned}\mathbf{E} &= \frac{iZ}{k}\boldsymbol{\nabla}\times\mathbf{H} \\ &= \frac{iZkc}{4\pi}\boldsymbol{\nabla}\times\left(\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{p}\right)\end{aligned}$$

Using

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b}$$

with $\mathbf{a} = \left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}$ and $\mathbf{b} = \mathbf{p}$, this becomes

$$\mathbf{E} = \frac{iZkc}{4\pi}\left[-\mathbf{p}\left(\boldsymbol{\nabla}\cdot\left[\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right]\right) + (\mathbf{p}\cdot\boldsymbol{\nabla})\left[\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right]\right]$$

For the first term,

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\hat{\mathbf{r}}f(r)) &= f\boldsymbol{\nabla}\cdot\hat{\mathbf{r}} + \hat{\mathbf{r}}\cdot\boldsymbol{\nabla}f \\ &= \frac{2}{r}f + \frac{\partial f}{\partial r}\end{aligned}$$

so that with

$$\begin{aligned}\frac{\partial}{\partial r}\left(\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\right) &= \frac{1}{ikr^2}\frac{e^{ikr}}{r} + \left(1-\frac{1}{ikr}\right)\left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right) \\ &= \left(\frac{1}{ikr^2} + ik - \frac{1}{r} - \frac{1}{r} + \frac{1}{ikr^2}\right)\frac{e^{ikr}}{r} \\ &= \left(ik - \frac{2}{r} + \frac{2}{ikr^2}\right)\frac{e^{ikr}}{r}\end{aligned}$$

we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left[\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right] &= \frac{2}{r}\left(\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\right) + \left(ik - \frac{2}{r} + \frac{2}{ikr^2}\right)\frac{e^{ikr}}{r} \\ &= \left(\frac{2}{r} - \frac{2}{ikr^2} + ik - \frac{2}{r} + \frac{2}{ikr^2}\right)\frac{e^{ikr}}{r} \\ &= \frac{ike^{ikr}}{r}\end{aligned}$$

We compute the second term in the brackets for $\mathbf{E}$ using the identity

$$\begin{aligned}(\mathbf{a}\cdot\boldsymbol{\nabla})(\hat{\mathbf{r}}f(r)) &= \hat{\mathbf{r}}(\mathbf{a}\cdot\boldsymbol{\nabla}f) + f(\mathbf{a}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} \\ &= \hat{\mathbf{r}}(\mathbf{a}\cdot\hat{\mathbf{r}})\frac{\partial f}{\partial r} + f(\mathbf{a}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}}\end{aligned}$$

Since $\hat{\mathbf{r}}$ is constant in the r direction, $(\mathbf{a}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}}$ depends only on the angular derivatives,

$$\begin{aligned}(\mathbf{a}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} &= \frac{1}{r}a_\theta\frac{\partial}{\partial\theta}(\sin\theta\cos\varphi\mathbf{i} + \sin\theta\sin\varphi\mathbf{j} + \cos\theta\mathbf{k}) + \frac{1}{r\sin\theta}a_\varphi\frac{\partial}{\partial\varphi}(\sin\theta\cos\varphi\mathbf{i} + \sin\theta\sin\varphi\mathbf{j} + \cos\theta\mathbf{k}) \\ &= \frac{1}{r}a_\theta(\cos\theta\cos\varphi\mathbf{i} + \cos\theta\sin\varphi\mathbf{j} - \sin\theta\mathbf{k}) + \frac{1}{r\sin\theta}a_\varphi(-\sin\theta\sin\varphi\mathbf{i} + \sin\theta\cos\varphi\mathbf{j}) \\ &= \frac{1}{r}a_\theta(\cos\theta\hat{\boldsymbol{\rho}} - \sin\theta\mathbf{k}) + \frac{1}{r}a_\varphi\hat{\boldsymbol{\varphi}} \\ &= \frac{1}{r}\left(a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\hat{\boldsymbol{\varphi}}\right) \\ &= \frac{1}{r}\mathbf{a}_\perp \\ &= \frac{1}{r}(\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\end{aligned}$$

Thus, the second term is

$$
\begin{aligned}
(\mathbf{p}\cdot\boldsymbol{\nabla})\left[\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right] &= \hat{\mathbf{r}}\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\frac{\partial}{\partial r}\left(\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\right)+\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\frac{1}{r}\left(\mathbf{p}-\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right)\\
&= \hat{\mathbf{r}}\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\left(ik-\frac{2}{r}+\frac{2}{ikr^2}\right)\frac{e^{ikr}}{r}+\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\frac{1}{r}\left(\mathbf{p}-\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right)\\
&= \left(\left(ik-\frac{3}{r}+\frac{3}{ikr^2}\right)\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}+\left(\frac{1}{r}-\frac{1}{ikr^2}\right)\mathbf{p}\right)\frac{e^{ikr}}{r}
\end{aligned}
$$

Combining these results, and using $Zc=\sqrt{\frac{\mu_0}{\epsilon_0}}\sqrt{\frac{1}{\mu_0\epsilon_0}}=\frac{1}{\epsilon_0}$,

$$
\begin{aligned}
\mathbf{E} &= \frac{ik}{4\pi\epsilon_0}\left[-\mathbf{p}\left(\boldsymbol{\nabla}\cdot\left[\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right]\right)+(\mathbf{p}\cdot\boldsymbol{\nabla})\left[\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}\hat{\mathbf{r}}\right]\right]\\
&= \frac{ik}{4\pi\epsilon_0}\left[-\frac{ike^{ikr}}{r}\mathbf{p}+\left(\left(ik-\frac{3}{r}+\frac{3}{ikr^2}\right)\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}+\left(\frac{1}{r}-\frac{1}{ikr^2}\right)\mathbf{p}\right)\frac{e^{ikr}}{r}\right]\\
&= \frac{ik}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\left(\frac{1}{r}-\frac{1}{ikr^2}-ik\right)\mathbf{p}+\left(ik-\frac{3}{r}+\frac{3}{ikr^2}\right)\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right]\\
&= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\left(\frac{ik}{r}-\frac{1}{r^2}+k^2\right)\mathbf{p}+\left(-k^2-\frac{3ik}{r}+\frac{3}{r^2}\right)\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right]
\end{aligned}
$$

Rearranging terms,

$$
\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[k^2\left(\mathbf{p}-\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right)+\frac{ik}{r}\left(1-\frac{1}{ikr}\right)\left(\mathbf{p}-3\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right)\right]
$$

Finally, noting that

$$
\hat{\mathbf{r}}\times\left(\mathbf{p}\times\hat{\mathbf{r}}\right)=\mathbf{p}-\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}
$$

we see that this is equivalent to the expression in Jackson. Notice that the first term is transverse, but not the second.

The electric and magnetic fields for an oscillating dipole field are therefore,

$$
\begin{aligned}
\mathbf{H}\left(\mathbf{x},t\right) &= \frac{k^2c}{4\pi}\left(1-\frac{1}{ikr}\right)\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}\\
\mathbf{E}\left(\mathbf{x},t\right) &= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\left[k^2\hat{\mathbf{r}}\times\left(\mathbf{p}\times\hat{\mathbf{r}}\right)+\frac{ik}{r}\left(1-\frac{1}{ikr}\right)\left(\mathbf{p}-3\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right)\right]
\end{aligned}
$$

In the radiation zone, $kr\gg 1$, these simplify to

$$
\begin{aligned}
\mathbf{H}\left(\mathbf{x},t\right) &= \frac{k^2c}{4\pi}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\mathbf{p}\\
\mathbf{E}\left(\mathbf{x},t\right) &= \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\hat{\mathbf{r}}\times\left(\mathbf{p}\times\hat{\mathbf{r}}\right)\\
&= Z_0\mathbf{H}\times\hat{\mathbf{r}}
\end{aligned}
$$

while in the near zone, $kr\ll 1$,

$$
\begin{aligned}
\mathbf{H}\left(\mathbf{x},t\right) &= \frac{ikc}{4\pi}\frac{1}{r^2}e^{ikr-i\omega t}\hat{\mathbf{r}}\times\mathbf{p}\\
\mathbf{E}\left(\mathbf{x},t\right) &= \frac{1}{4\pi\epsilon_0 r^3}e^{ikr-i\omega t}\left(3\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{p}\right)
\end{aligned}
$$

Notice that in the near zone, the electric field is just $e^{-i\omega t}$ times a static dipole field, while

$$\begin{aligned} H &= \frac{kr}{Z}\frac{1}{4\pi\epsilon_0}\frac{p}{r^3} \\ E &= \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left|3\left(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\hat{\mathbf{p}}\right| \end{aligned}$$

so that $H \ll E$.