Waveguides and resonant cavities

February 26, 2016

Essentially, a waveguide is a conducting tube of uniform cross-section and a cavity is a waveguide with end caps. The dimensions of the guide or cavity are chosen to transmit, hold or amplify particular forms of electromagnetic wave.

We will consider the case of a hollow tube – a waveguide – extended in the z direction, with arbitrary but constant cross-sectional shape in the xy-plane. We consider possible wave solutions matching these boundary conditions.

1 Wave equations

As we have shown, the homogeneous Maxwell equations,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D} &= 0 \\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\
\boldsymbol{\nabla}\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0 \\
\boldsymbol{\nabla}\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t} &= 0
\end{aligned}$$

give rise to wave equations,

$$\begin{aligned}
\Box\mathbf{E} &= 0 \\
\Box\mathbf{B} &= 0
\end{aligned}$$

We assume that the electric and magnetic fields have spatial and time dependence of the form

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}(x,y)\,e^{\pm ikz-i\omega t} \\
\mathbf{B} &= \mathbf{B}(x,y)\,e^{\pm ikz-i\omega t}
\end{aligned}$$

for waves traveling in the $\pm z$ direction. Note that $\mathbf{E}(x,y)$ and $\mathbf{B}(x,y)$ still have components in all three spatial directions. Unlike our plane wave solutions, the fields must satisfy boundary conditions in the x and y directions, along the sides of the waveguide.

Separating the del operator into longitudinal (z) and transverse (xy) parts,

$$\boldsymbol{\nabla}=\boldsymbol{\nabla}_t+\hat{\mathbf{k}}\frac{\partial}{\partial z},$$

the d'Alembertian becomes

$$\begin{aligned}
\Box &= -\mu\epsilon\frac{\partial^2}{\partial t^2}+\nabla^2 \\
&= -\mu\epsilon\frac{\partial^2}{\partial t^2}+\nabla_t^2+\frac{\partial^2}{\partial z^2}
\end{aligned}$$

Substituting the assumed time and z-dependence into the wave equations gives

$$\begin{aligned} 0 &= \left(\nabla_t^2 - \mu\epsilon\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} \\ &= \left(\nabla_t^2 - \mu\epsilon\left(-i\omega\right)^2 + \left(ik\right)^2\right)\mathbf{E} \end{aligned}$$

and, with the corresponding result for $\mathbf{B}$,

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\mathbf{E} \\ 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\mathbf{B} \end{aligned}$$

2 General solution: Separating transverse and longitudinal components

We can simplify the problem by separating the transverse, $\mathbf{E}_t$, and longitudinal, $\hat{\mathbf{k}}E_z$, parts, then treating E_z and B_z as sources for the transverse parts. We can use the z-component of wave equation together with the boundary conditions to solve for the "sources", E_z, B_z. Along with $\boldsymbol{\nabla} = \boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}$, we define

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_t + \hat{\mathbf{k}}E_z \\ \mathbf{E}_t &= \left(\hat{\mathbf{k}} \times \mathbf{E}\right) \times \hat{\mathbf{k}} \end{aligned}$$

and similarly for $\mathbf{B}$, the source-free Maxwell equations become

$$\begin{aligned} \boldsymbol{\nabla}_t \cdot \mathbf{B}_t \pm ikB_z &= 0 \\ \boldsymbol{\nabla}_t \cdot \mathbf{E}_t \pm ikE_z &= 0 \\ \left(\boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) - i\omega\left(\mathbf{B}_t + \hat{\mathbf{k}}B_z\right) &= 0 \\ \left(\boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{B}_t + \hat{\mathbf{k}}B_z\right) + i\epsilon\mu\omega\left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) &= 0 \end{aligned} \tag{1}$$

The first two equations already have the form we are after,

$$\begin{aligned} \boldsymbol{\nabla}_t \cdot \mathbf{B}_t &= \mp ikB_z \\ \boldsymbol{\nabla}_t \cdot \mathbf{E}_t &= \mp ikE_z \end{aligned}$$

and we turn our attention to the curl equations.

The curl terms expand as

$$\begin{aligned} \left(\boldsymbol{\nabla}_t + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) &= \boldsymbol{\nabla}_t \times \mathbf{E}_t + \boldsymbol{\nabla}_t \times \hat{\mathbf{k}}E_z + \hat{\mathbf{k}} \times \frac{\partial \mathbf{E}_t}{\partial z} + \hat{\mathbf{k}} \times \hat{\mathbf{k}}\frac{\partial E_z}{\partial z} \\ &= \boldsymbol{\nabla}_t \times \mathbf{E}_t + \hat{\mathbf{i}} \times \hat{\mathbf{k}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{j}} \times \hat{\mathbf{k}}\frac{\partial E_z}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \mathbf{E}_t}{\partial z} \\ &= \boldsymbol{\nabla}_t \times \mathbf{E}_t - \hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \mathbf{E}_t}{\partial z} \end{aligned}$$

and similarly for the magnetic terms. Thus

$$\begin{aligned} \boldsymbol{\nabla}_t \times \mathbf{E}_t - \hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{E}_t - i\omega\left(\mathbf{B}_t + \hat{\mathbf{k}}B_z\right) &= 0 \\ \boldsymbol{\nabla}_t \times \mathbf{B}_t - \hat{\mathbf{j}}\frac{\partial B_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial B_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{B}_t + i\varepsilon\mu\omega\left(\mathbf{E}_t + \hat{\mathbf{k}}E_z\right) &= 0 \end{aligned}$$

We separate the transverse and longitudinal parts of each equation. Noting that $\boldsymbol{\nabla}_t \times \mathbf{E}_t$ lies in the z-direction, the first equation separates into

$$
\begin{aligned}
-\hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{E}_t - i\omega\mathbf{B}_t &= 0 \qquad (transverse) \\
\boldsymbol{\nabla}_t \times \mathbf{E}_t - i\omega B_z\hat{\mathbf{k}} &= 0 \qquad (longitudinal)
\end{aligned}
$$

while second gives

$$
\begin{aligned}
-\hat{\mathbf{j}}\frac{\partial B_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial B_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{B}_t + i\epsilon\mu\omega\mathbf{E}_t &= 0 \qquad (transverse) \\
\boldsymbol{\nabla}_t \times \mathbf{B}_t + i\epsilon\mu\omega E_z\hat{\mathbf{k}} &= 0 \qquad (longitudinal)
\end{aligned}
$$

We can simplify the transverse equations further. Since they are transverse, we lose no information by taking the cross product with $\hat{\mathbf{k}}$. Then the first becomes

$$
\begin{aligned}
0 &= \hat{\mathbf{k}} \times \left(-\hat{\mathbf{j}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{i}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times ik\mathbf{E}_t - i\omega\mathbf{B}_t\right) \\
&= \hat{\mathbf{i}}\frac{\partial E_z}{\partial x} + \hat{\mathbf{j}}\frac{\partial E_z}{\partial y} \pm \hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times ik\mathbf{E}_t\right) - i\omega\hat{\mathbf{k}} \times \mathbf{B}_t \\
&= \boldsymbol{\nabla}_t E_z \mp ik\mathbf{E}_t - i\omega\hat{\mathbf{k}} \times \mathbf{B}_t
\end{aligned}
$$

Similarly, the transverse equation for the magnetic field crossed with $\hat{\mathbf{k}}$

$$
0 = \boldsymbol{\nabla}_t B_z \mp ikB_t + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t
$$

Now we may complete the separation.

3 Generic solution for $\mathbf{E}_t$ and $\mathbf{B}_t$

3.1 Solving for the transverse components

In the resulting pair of equations,

$$
\begin{aligned}
\pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \mathbf{B}_t &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{B}_t - i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t &= \boldsymbol{\nabla}_t B_z
\end{aligned}
$$

the transverse $\mathbf{E}_t$ and $\mathbf{B}_t$ are still coupled. To separate them, solve the second for $\mathbf{B}_t$,

$$
\mathbf{B}_t = \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t\right)
$$

and substitute into the first,

$$
\begin{aligned}
\pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \mathbf{B}_t &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{E}_t + i\omega\hat{\mathbf{k}} \times \left(\pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t\right)\right) &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{E}_t \pm \frac{\omega}{k}\hat{\mathbf{k}} \times \left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t\right) &= \boldsymbol{\nabla}_t E_z \\
\pm ik\mathbf{E}_t \pm i\epsilon\mu\frac{\omega^2}{k}\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{E}_t\right) &= \boldsymbol{\nabla}_t E_z \mp \frac{\omega}{k}\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z
\end{aligned}
$$

and multiplying by k,

$$\pm\left(ik^2 - i\epsilon\mu\omega^2\right)\mathbf{E}_t \quad = \quad k\boldsymbol{\nabla}_t E_z \mp \omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z \tag{2}$$

Therefore, as long as $k^2 - \epsilon\mu\omega^2 \neq 0$, we may solve for the transverse electric field in terms of the longitudinal fields,

$$\mathbf{E}_t = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)$$

Now substitute this back into the expression for $\mathbf{B}_t$,

$$\begin{aligned}
\mathbf{B}_t &= \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z + i\epsilon\mu\omega\hat{\mathbf{k}}\times\mathbf{E}_t\right)\\
&= \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z - \frac{\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}\hat{\mathbf{k}}\times\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\right)\\
&= \pm\frac{1}{ik}\left(\boldsymbol{\nabla}_t B_z - \frac{\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}\left(\pm k\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z + \omega\boldsymbol{\nabla}_t B_z\right)\right)\\
&= \frac{1}{ik}\left(-\frac{\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}k\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \mp \frac{\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}\omega\boldsymbol{\nabla}_t B_z \pm \boldsymbol{\nabla}_t B_z\right)\\
&= \frac{i}{k}\left(\frac{\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}k\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \pm \left(\frac{\epsilon\mu\omega^2}{\epsilon\mu\omega^2 - k^2} - 1\right)\boldsymbol{\nabla}_t B_z\right)\\
&= \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\epsilon\mu\omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \pm k\boldsymbol{\nabla}_t B_z\right)
\end{aligned}$$

and we have solved for the transverse fields in terms of the longitudinal ones:

$$\begin{aligned}
\mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\\
\mathbf{B}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\epsilon\mu\omega\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t E_z \pm k\boldsymbol{\nabla}_t B_z\right)
\end{aligned} \tag{3}$$

This solution for the transverse fields holds in all cases except when $k^2 - \epsilon\mu\omega^2 = 0$.

3.2 Checking the remaining Maxwell equations

To have a complete solution, we must check the remaining Maxwell equations,

$$\begin{aligned}
\boldsymbol{\nabla}_t\cdot\mathbf{E}_t &= \mp ikE_z\\
\boldsymbol{\nabla}_t\cdot\mathbf{B}_t &= \mp ikB_z
\end{aligned}$$

and the longitudinal parts of the curl equations,

$$\begin{aligned}
\boldsymbol{\nabla}_t\times\mathbf{E}_t &= i\omega B_z\hat{\mathbf{k}}\\
\boldsymbol{\nabla}_t\times\mathbf{B}_t &= -i\epsilon\mu\omega E_z\hat{\mathbf{k}}
\end{aligned}$$

The divergence the transverse electric field is

$$\boldsymbol{\nabla}_t\cdot\mathbf{E}_t = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\pm k\boldsymbol{\nabla}_t\cdot\boldsymbol{\nabla}_t E_z - \omega\boldsymbol{\nabla}_t\cdot\left(\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\right)$$

Using the reduced form of the wave equation,

$$0 \quad = \quad \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\mathbf{E}$$

on the first term on the right, together with

$$
\begin{aligned}
\boldsymbol{\nabla}_t \cdot \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) &= \sum_{ijk} \nabla_i^t \varepsilon_{ijk} \left[\hat{\mathbf{k}}\right]_j \nabla_k^t B_z \\
&= \sum_{ijk} \nabla_i^t \varepsilon_{ijk} \delta_{j3} \nabla_k^t B_z \\
&= \sum_{ik} \nabla_i^t \varepsilon_{i3k} \nabla_k^t B_z \\
&= -\sum_{ik} \varepsilon_{3ik} \nabla_i^t \nabla_k^t B_z \\
&= -\sum_{ik} \left(\nabla_1^t \nabla_2^t - \nabla_2^t \nabla_1^t\right) B_z \\
&= 0
\end{aligned}
$$

to see that

$$
\begin{aligned}
\boldsymbol{\nabla}_t \cdot \mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2} \left(\pm k \left(k^2 - \mu\epsilon\omega^2\right) E_z\right) \\
&= \mp i k E_z
\end{aligned}
$$

as required.

For the curl of the electric field,

$$
\begin{aligned}
\boldsymbol{\nabla}_t \times \mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2} \boldsymbol{\nabla}_t \times \left(\pm k \boldsymbol{\nabla}_t E_z - \omega \hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\
&= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2} \boldsymbol{\nabla}_t \times \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right)
\end{aligned}
$$

Sorting out the double curl,

$$
\begin{aligned}
\left[\boldsymbol{\nabla}_t \times \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right)\right]_i &= \sum_{j,k} \varepsilon_{ijk} \nabla_j^t \left[\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right]_k \\
&= \sum_{j,k,m} \varepsilon_{ijk} \nabla_j^t \left(\varepsilon_{k3m} \nabla_m^t B_z\right) \\
&= \sum_{j,k,m} \varepsilon_{ijk} \varepsilon_{3mk} \nabla_j^t \nabla_m^t B_z \\
&= \sum_{j,m} \left(\delta_{i3}\delta_{jm} - \delta_{im}\delta_{j3}\right) \nabla_j^t \nabla_m^t B_z \\
&= \left(\delta_{i3} \sum_j \nabla_j^t \nabla_j^t B_z - \nabla_3^t \nabla_i^t B_z\right) \\
\boldsymbol{\nabla}_t \times \left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) &= \hat{\mathbf{k}} \boldsymbol{\nabla}_t^2 B_z
\end{aligned}
$$

The second term vanishes because $\boldsymbol{\nabla}_t$ has no z component, $\nabla_3^t = 0$. Therefore, using the wave equation, $0 = \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) \mathbf{B}$, again, the curl of $\mathbf{E}_t$ becomes

$$
\begin{aligned}
\boldsymbol{\nabla}_t \times \mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2} \hat{\mathbf{k}} \nabla_t^2 B_z \\
&= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2} \hat{\mathbf{k}} \left(k^2 - \mu\epsilon\omega^2\right) B_z \\
&= i\omega B_z \hat{\mathbf{k}}
\end{aligned}
$$

so the equations for the electric field are identically satisfied.

The corresponding divergence and curl of $\mathbf{B}_t$ are left as exercises for the reader.

4 Characteristics of solutions

Our general method of solution is now to solve

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) E_z \\ 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) B_z \end{aligned}$$

with the appropriate boundary conditions, then use these solutions to solve

$$\begin{aligned} \mathbf{E}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\pm k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\ \mathbf{B}_t &= \frac{i}{\epsilon\mu\omega^2 - k^2}\left(\epsilon\mu\omega\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t E_z \pm k\boldsymbol{\nabla}_t B_z\right) \end{aligned}$$

for the transverse parts. We consider three special cases, depending on one or both of E_z and B_z vanishing:

1. If both E_z and B_z vanish, then both the electric and magnetic fields are purely transverse. These solutions are called TEM waves (Transverse Electric and Magnetic). From eq.(2) and the corresponding equation for $\mathbf{B}_t$, we see that TEM waves have either $\epsilon\mu\omega^2 - k^2$, or the fields must vanish entirely, $\mathbf{E}_t = \mathbf{B}_t = 0$.

2. If E_z vanishes, then the electric field is purely transverse. These solutions are called TE waves (Transverse Electric).

3. If B_z vanishes, then the magnetic field is purely transverse. These solutions are called TM waves (Transverse Magnetic).

Generic waves are a combination of all three. The method sketched above works for TE and TM waves (see below), but in the case where $E_z = B_z = 0$ we have $\epsilon\mu\omega^2 - k^2 = 0$ and this solution fails. We treat this special TEM case first, then use the generic solution to discusss TE and TM waves.

4.1 Transverse electromagnetic waves: TEM

Transverse electromagnetic waves in a waveguide are those with no z-component to the fields,

$$\begin{aligned} E_z &= 0 \\ B_z &= 0 \end{aligned}$$

and according to eq.(2) this requires $\epsilon\mu\omega^2 - k^2$. The generic solution, eqs.(3), does not hold and we return to the Maxwell equations, eqs.(1) with $E_z = B_z = 0$. Setting $\mathbf{E}_t = \mathbf{E}_{TEM}$ and $\mathbf{B}_t = \mathbf{B}_{TEM}$ and separating the longitudinal and transverse parts of the curl equations,

$$\begin{aligned} \boldsymbol{\nabla}_t \cdot \mathbf{E}_{TEM} &= 0 \\ \boldsymbol{\nabla}_t \cdot \mathbf{B}_{TEM} &= 0 \\ \boldsymbol{\nabla}_t \times \mathbf{B}_{TEM} &= 0 \\ \boldsymbol{\nabla}_t \times \mathbf{E}_{TEM} &= 0 \\ \pm k\hat{\mathbf{k}} \times \mathbf{E}_{TEM} - \omega\mathbf{B}_{TEM} &= 0 \\ \pm k\hat{\mathbf{k}} \times \mathbf{B}_{TEM} + \epsilon\mu\omega\mathbf{E}_{TEM} &= 0 \end{aligned}$$

The first four of these immediately imply the 2-dimensional Laplace equation for both the transverse electric and magnetic fields. The relevant solutions to the 2-dimensional Laplace equations is then determined purely by their boundary conditions. Since a closed conductor allows no field inside, TEM waves cannot exist inside a completely enclosed, perfectly conducting cavity, but TEM waves may exist in waveguides.

Combining the final two equations by substituting $\mathbf{B}_{TEM} = \pm\frac{k}{\omega}\hat{\mathbf{k}} \times \mathbf{E}_{TEM}$ into the final one,

$$\begin{aligned} \pm k\hat{\mathbf{k}} \times \left(\pm\frac{k}{\omega}\hat{\mathbf{k}} \times \mathbf{E}_{TEM}\right) + \epsilon\mu\omega\mathbf{E}_{TEM} &= 0 \\ \left(-k^2 + \epsilon\mu\omega^2\right)\mathbf{E}_{TEM} &= 0 \end{aligned}$$

so these are satisfied by the special case condition,

$$k = k_0 = \sqrt{\epsilon\mu}\omega$$

together with

$$\mathbf{B}_{TEM} = \pm\sqrt{\epsilon\mu}\hat{\mathbf{k}} \times \mathbf{E}_{TEM}$$

TEM waves are the dominant mode in a coaxial cable: inner and outer cylindrical conductors held at opposite potential lead to a radial electric field, while opposite currents on the conductors lead to an azimuthal magnetic field. These are transverse to the direction along the cable, so waves propagate along the cable between the conductors.

4.2 Boundary conditions for longitudinal modes: transverse electric and transverse magnetic modes

Modes driven by nonzero E_z and/or B_z fall into two categories: transverse electric (TE) and transverse magnetic (TM). To see why, consider a perfectly conducting boundary.

For a perfectly conducting waveguide, we find the boundary conditions using the assumption that free charges move instantly to produce whatever surface charge density, Σ, and surface current density, $\mathbf{K}$, are required to make the electric and magnetic fields vanish inside the conductor. With $\mathbf{n}$ normal to the conductor (hence perpendicular to $\mathbf{k}$ as well) the full boundary conditions are then

$$\begin{aligned} \mathbf{n} \cdot \mathbf{D} &= \Sigma \\ \mathbf{n} \times \mathbf{H} &= \mathbf{K} \\ \mathbf{n} \times \mathbf{E} &= 0 \\ \mathbf{n} \cdot \mathbf{B} &= 0 \end{aligned}$$

There can therefore be no tangential component of the electric field at the surface, and no normal component of the magnetic field.

For the longitudinal electric field, the boundary condition is

$$\left(\mathbf{n} \times \hat{\mathbf{k}}\right) E_z = 0$$

so that $E_z = 0$ at the boundary. For the boundary condition on B_z we start with our separation of the Maxwell equations into longitudinal and transverse components, where we found

$$ik\mathbf{B}_t - i\epsilon\mu\omega\hat{\mathbf{k}} \times \mathbf{E}_t = \boldsymbol{\nabla}_t B_z$$

Consider the normal component of this equation,

$$\begin{aligned} ik\mathbf{n} \cdot \mathbf{B}_t - i\epsilon\mu\omega\mathbf{n} \cdot \left(\hat{\mathbf{k}} \times \mathbf{E}_t\right) &= \mathbf{n} \cdot \boldsymbol{\nabla}_t B_z \\ ik\left(\mathbf{n} \cdot \mathbf{B}_t\right) + i\epsilon\mu\omega\hat{\mathbf{k}} \cdot \left(\mathbf{n} \times \mathbf{E}_t\right) &= \frac{\partial B_z}{\partial n} \end{aligned}$$

The left side of this equation vanishes by the boundary conditions, so we must have

$$\frac{\partial B_z}{\partial n} = 0$$

at the surface as well.

Therefore, we seek solutions to the 2-dimensional wave equations

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) E_z \\ 0 &= \left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) B_z \end{aligned} \tag{4}$$

with boundary conditions

$$\begin{aligned} E_z\big|_S &= 0 \\ \left.\frac{\partial B_z}{\partial n}\right|_S &= 0 \end{aligned} \tag{5}$$

These two boundary conditions are sufficient for a unique solution, since the current and charge density modify freely to satisfy the remaining conditions. The system, eqs.(4) and (5), constitute a well-defined eigenvalue problems for E_z and B_z. Since the transverse direction is a bounded region, we expect a discrete set of eigenvalues. Because the boundary conditions are different but the equations the same, uniqueness guarantees that the spectrum of allowed values will be different for E_z and B_z. This means that at a given resonant frequency, in general only one or the other source field will be excited. This divides the solutions into two types,

1. TE waves: The electric field is transverse, i.e., $E_z = 0$ everywhere while B_z satisfies the boundary condition $\left.\frac{\partial B_z}{\partial n}\right|_S = 0$.
2. TM waves: The magnetic field is transverse so that $B_z = 0$ everywhere while E_z satisfies the boundary condition $E_z\big|_S = 0$.

A general solution including a superposition of frequencies for the field in a waveguide or cavity is a superposition of TE, TM and, for waveguides, TEM waves.

4.3 The transverse fields

Now we may study the transverse fields in the TE and TM cases. First, we simplify our solutions for transverse fields in each of the two cases.

4.3.1 TM waves

For TM waves, we set $B_z = 0$. Then the solution eqs.(3) satisfies

$$\begin{aligned} \mathbf{E}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\boldsymbol{\nabla}_t E_z \\ \mathbf{B}_t &= \frac{i\epsilon\mu\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t E_z\right) \end{aligned}$$

Taking the cross product of $\hat{\mathbf{k}}$ with the transverse electric field,

$$\begin{aligned} \hat{\mathbf{k}} \times \mathbf{E}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t E_z \\ &= \pm\frac{ik}{i\epsilon\mu\omega}\mathbf{B}_t \\ &= \pm\frac{k}{\epsilon\omega}\mathbf{H}_t \end{aligned}$$

so that

$$\mathbf{H}_t = \pm\frac{\epsilon\omega}{k}\hat{\mathbf{k}}\times\mathbf{E}_t$$

4.3.2 TE waves

For TE waves, $E_z = 0$ we get a similar result,

$$\begin{aligned}\mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\\ \mathbf{B}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\left(\boldsymbol{\nabla}_t B_z\right)\end{aligned}$$

so substituting the second equation into the first,

$$\begin{aligned}\mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\\ &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}}\times\frac{\epsilon\mu\omega^2 - k^2}{\pm ik}\mathbf{B}_t\right)\\ &= \mp\frac{\omega}{k}\hat{\mathbf{k}}\times\mathbf{B}_t\end{aligned}$$

so taking the cross product with $\hat{\mathbf{k}}$,

$$\begin{aligned}\hat{\mathbf{k}}\times\mathbf{E}_t &= \mp\frac{\omega}{k}\hat{\mathbf{k}}\times\left(\hat{\mathbf{k}}\times\mathbf{B}_t\right)\\ &= \pm\frac{\omega}{k}\mathbf{B}_t\end{aligned}$$

so that

$$\mathbf{H}_t = \pm\frac{k}{\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}_t$$

4.3.3 Wave impedence

Both of these relationships, for TM and TE waves, have the form

$$\mathbf{H}_t = \pm\frac{1}{Z}\hat{\mathbf{k}}\times\mathbf{E}_t$$

where, using $k_0 = \sqrt{\varepsilon\mu}\omega$,

$$Z = \begin{cases}\frac{k}{\epsilon\omega} = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}} & TM\ modes\\ \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}} & TE\ modes\end{cases}$$

The quantity Z is called the *wave impedence*.

This gives solutions of the form:

$$\begin{aligned}\mathbf{E}_t &= \pm\frac{ik}{\epsilon\mu\omega^2 - k^2}\boldsymbol{\nabla}_t E_z\\ \mathbf{H}_t &= \pm\frac{1}{Z}\hat{\mathbf{k}}\times\mathbf{E}_t\end{aligned} \tag{6}$$

with $Z = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}}$ for TM and

$$\begin{aligned}\mathbf{E}_t &= -\frac{i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}}\times\boldsymbol{\nabla}_t B_z\right)\\ \mathbf{H}_t &= \pm\frac{1}{Z}\hat{\mathbf{k}}\times\mathbf{E}_t\end{aligned} \tag{7}$$

with $Z = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}}$ for TE.

4.3.4 The eigenvalue problem

We want to find solutions for TE and TM modes. These will differ from the TEM mode due to the presence of either nonzero E_z or nonzero B_z, which provide the source for the transverse fields.

Denote either of these source fields by ψ,

$$\psi = \begin{cases} E_z & TM\ modes \\ B_z & TE\ modes \end{cases}$$

and define

$$\gamma^2 \equiv \mu\epsilon\omega^2 - k^2$$

Then the reduced wave equation for ψ becomes an eigenvalue problem,

$$\nabla_t^2 \psi = -\gamma^2 \psi$$

with boundary conditons

$$\begin{aligned} \psi|_S &= 0 \quad TM \\ \left.\frac{\partial\psi}{\partial n}\right|_S &= 0 \quad TE \end{aligned}$$

Because these boundary conditions are periodic, there will be a discrete spectrum of allowed values for γ, $\gamma_1 < \gamma_2 < \gamma_3 < \ldots$. For each value of n, we have

$$k^2 = \mu\varepsilon \left(\omega^2 - \frac{1}{\mu\epsilon} \gamma_n^2 \right)$$

Since we have assumed waves of the form $e^{\pm ikz - i\omega t}$, waves will propagate only if $k^2 > 0$, and therefore if

$$\omega \geq \omega_n \equiv \frac{\gamma_n}{\sqrt{\mu\epsilon}}$$

Now fix a frequency, ω_0. As n increases, γ_n and ω_n increase, so for some n_0 we have $\omega_0 < \omega_{n_0}$ and waves will no longer propagate, instead decaying as

$$\exp\left(i\sqrt{\mu\epsilon}\sqrt{\omega_0^2 - \omega_{n_0}^2} z \right) = \exp\left(-\sqrt{\mu\epsilon} \left(\sqrt{\omega_{n_0}^2 - \omega_0^2} \right) z \right)$$

This means that, for a given frequency ω_0, only a finite number of wavelengths are possible, lying in the range

$$\mu\epsilon \left(\omega_0^2 - \omega_1^2 \right) \geq k^2 \geq \mu\epsilon \left(\omega_0^2 - \omega_{n_0}^2 \right)$$

and equal to

$$k^2 = \mu\varepsilon \left(\omega_0^2 - \frac{1}{\mu\epsilon} \gamma_n^2 \right)$$

for any n in the range $1 \leq n \leq n_0$. If we choose $\omega_1 < \omega < \omega_2$ then only the wavelength with wave number

$$k^2 = \mu\varepsilon \left(\omega_0^2 - \omega_1^2 \right)$$

will propagate in the waveguide. More commonly we desire a particular frequency or frequency range, and we choose the dimensions of the waveguide so that at the desired frequency, only the single value of $n = 1$ is allowed and only a single wavelength propagates.

Once we have solved for ψ, we can find the transverse fields from the expressions above.

4.4 Example: TE modes in a rectangular waveguide

4.4.1 Eigenfunctions and eigenvalues

Suppose we have TE modes in a rectangular waveguide, so that $E_z = 0$ and we solve the wave equation for H_z. Let the cross-section of the guide run from $x = 0$ to $x = a$, and from $y = 0$ to $y = b$. Then, (with $\psi = H_z$)

$$\begin{aligned} 0 &= \left(\nabla_t^2 + \gamma^2\right) H_z \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) H_z \end{aligned}$$

with boundary conditon

$$\left.\frac{\partial H_z}{\partial n}\right|_S = 0 \quad TE$$

The boundary condition in each direction may be satisfied at the origin by a cosine, so we have

$$H_z = H_0 \cos \alpha x \cos \beta y$$

and fitting the boundary conditions at $x = a$ and at $y = b$ requires $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$. Therefore, the eigenfunctions are

$$\psi_{mn} = H_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{8}$$

with eigenvalues

$$\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The cutoff frequency is

$$\begin{aligned} \omega_{mn} &= \frac{\gamma_{mn}}{\sqrt{\mu\varepsilon}} \\ &= \frac{\pi}{\sqrt{\mu\varepsilon}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \end{aligned}$$

for each choice of m, n.

4.4.2 Selecting a wavelength

Suppose for concreteness that $b = \sqrt{\frac{2}{3}}a$. Then the lowest frequency mode is,

$$\omega_{10} = \frac{\pi}{a\sqrt{\mu\epsilon}}$$

with subsequent modes,

$$\begin{aligned} \omega_{01} &= \sqrt{\frac{3}{2}}\omega_{10} \\ \omega_{20} &= 2\omega_{10} \\ \omega_{11} &= \sqrt{\frac{5}{2}}\omega_{10} \\ \omega_{21} &= \sqrt{\frac{11}{2}}\omega_{10} \\ \omega_{02} &= \sqrt{6}\omega_{10} \end{aligned}$$

and so on.

If we want to design a waveguide with only one allowed mode, we may choose a frequency in the range

$$\omega_{10} \le \omega_0 < \omega_{01}$$

This restricts the wavelength to lie in the range

$$\begin{aligned} \mu\epsilon\left(\omega_0^2 - \omega_{10}^2\right) &> k^2 > \mu\epsilon\left(\omega_0^2 - \omega_{01}^2\right) \\ \mu\epsilon\omega_0^2 - \frac{\pi^2}{a^2} &> k^2 > \mu\epsilon\omega_0^2 - \frac{3\pi^2}{2a^2} \end{aligned}$$

Furthermore, the wavelength must be such that $k^2 = \mu\varepsilon\left(\omega_0^2 - \frac{1}{\mu\epsilon}\gamma_{mn}^2\right) = \mu\varepsilon\left(\omega_0^2 - \omega_{mn}^2\right)$, and this gives the possible values

$$\begin{aligned} k_1^2 &= \mu\varepsilon\left(\omega_0^2 - \omega_{10}^2\right) \\ k_2^2 &= \mu\varepsilon\left(\omega_0^2 - \omega_{01}^2\right) \\ &\vdots \end{aligned}$$

and only the first of these lies in the propagating range.

4.4.3 Solution for the fields

For this mode, we have the fields,

$$H_z = \psi_{10} = H_0 \cos\frac{\pi x}{a} e^{i(kz-\omega t)}$$

and therefore, from

$$\begin{aligned} \mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\ \mathbf{B}_t &= \frac{\pm ik}{\epsilon\mu\omega^2 - k^2}\left(\boldsymbol{\nabla}_t B_z\right) \end{aligned}$$

we find

$$\begin{aligned} \mathbf{E}_t &= \frac{-i\omega}{\epsilon\mu\omega^2 - k^2}\left(\hat{\mathbf{k}} \times \boldsymbol{\nabla}_t B_z\right) \\ &= \frac{i\omega_0}{\gamma_{10}^2}\hat{\mathbf{k}} \times \hat{\mathbf{i}}\frac{H_0\pi}{a}\sin\frac{\pi x}{a}e^{i(kz-\omega t)} \\ &= \frac{i\omega_0 a H_0}{\pi}\sin\frac{\pi x}{a}e^{i(kz-\omega t)}\hat{\mathbf{j}} \end{aligned}$$

for the electric field and

$$\begin{aligned} \mathbf{H}_t &= \frac{k}{\mu\omega_0}\hat{\mathbf{k}} \times \mathbf{E}_t \\ \mathbf{B}_t &= \frac{k}{\omega_0}\frac{i\omega_0 a H_0}{\pi}\sin\frac{\pi x}{a}e^{i(kz-\omega t)}\hat{\mathbf{k}} \times \hat{\mathbf{j}} \\ &= -\frac{ikaH_0}{\pi}\sin\frac{\pi x}{a}e^{i(kz-\omega t)}\hat{\mathbf{i}} \end{aligned}$$

4.5 Example: Half-circular tube

Consider TM modes propagating in a perfectly conducting waveguide with semicircular cross section of radius R, and the remaining side flat. Pick a frequency such that only a single wavelength will propagate, and find that wavelength. Find the resulting electric and magnetic fields.

4.5.1 Solving the wave equation

Choose the $z$axis along the axis of the guide. Then (with $B_z = 0$), we must first solve the eigenvalue problem for the electric field,

$$\left(\nabla_t^2 + \gamma^2\right) E_z = 0$$

with boundary conditions $\left.E_z\left(r, \varphi\right)\right|_S = 0$:

$$\begin{aligned} E_z\left(r, 0\right) &= 0 \qquad 0 \le r \le R \\ E_z\left(r, \pi\right) &= 0 \qquad 0 \le r \le R \\ E_z\left(R, \varphi\right) &= 0 \qquad 0 \le \varphi \le \pi \end{aligned}$$

Writing the wave equation in polar coordinates we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \gamma^2 E_z = 0$$

Setting $E_z = E\left(r\right) e^{in\varphi}$, this becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \left(\gamma^2 - \frac{n^2}{r^2}\right) E_z = 0$$

Now divide by γ^2 and define $\rho = \gamma r$,

$$\frac{1}{\gamma^2}\frac{1}{r}\frac{d}{dr}\left(r\frac{dE}{dr}\right) + \left(\frac{1}{\gamma^2}\gamma^2 - \frac{1}{\gamma^2}\frac{n^2}{r^2}\right) E = 0$$

to get the Bessel equation,

$$\frac{d^2 E}{d\rho^2} + \frac{1}{\rho}\frac{dE}{d\rho} + \left(1 - \frac{n^2}{\rho^2}\right) E = 0$$

with solutions

$$E_z = \sum_{n=-\infty}^{\infty}\left(a_n J_n\left(\gamma r\right) + b_n K_n\left(\gamma r\right)\right) e^{in\varphi}$$

4.5.2 Imposing the boundary conditions

The boundary condition at the origin immediately shows that $b_n = 0$. Also, at $r = 0, \varphi = 0$, we have

$$\begin{aligned} 0 &= \sum_{n=-\infty}^{\infty} a_n J_n\left(0\right) \\ &= a_0 J_0\left(0\right) \end{aligned}$$

so we must set $a_0 = 0$.

Now consider the full set of boundary conditions,

$$\begin{aligned} E_z\left(r, 0\right) &= 0 \qquad 0 \le r \le R \\ E_z\left(r, \pi\right) &= 0 \qquad 0 \le r \le R \\ E_z\left(R, \varphi\right) &= 0 \qquad 0 \le \varphi \le \pi \end{aligned}$$

Notice that the first two conditions will be satisfied if our series depends only on $\sin n\varphi$. To achieve this, we note that $J_{-n}\left(\rho\right) = \left(-1\right)^n J_n\left(\rho\right)$, so that we may write

$$\begin{aligned} E_z &= \sum_{n=1}^{\infty}\left(a_n J_n\left(\gamma r\right) e^{in\varphi} + a_{-n} J_{-n}\left(\gamma r\right) e^{-in\varphi}\right) \\ &= \sum_{n=1}^{\infty} J_n\left(\gamma r\right)\left(a_n e^{in\varphi} + \left(-1\right)^n a_{-n} e^{-in\varphi}\right) \end{aligned}$$

and the boundary condition shows that we need

$$\left(a_n e^{in\varphi} + (-1)^n a_{-n} e^{-in\varphi}\right) = 2ia_n \sin n\varphi$$

and therefore

$$a_{-n} = (-1)^{n+1} a_n$$

Absorbing the constant $2i$ into a new constant c_n,

$$E_z = \sum_{n=1}^{\infty} c_n J_n(\gamma r) \sin n\varphi$$

The remaining boundary condition,

$$E_z = \sum_{n=1}^{\infty} c_n J_n(\gamma R) \sin n\varphi$$

must hold for all φ in the range, so γR must be a root of the Bessel function.Let

$$J_n(x_{nk}) = 0$$

give the roots of the Bessel functions, where $k = 1, 2, \ldots$. Then at $r = R$, we must have

$$\begin{aligned} \gamma_{nk} R &= x_{nk} \\ \gamma_{nk} &= \frac{x_{nk}}{R} \end{aligned}$$

4.5.3 Restricting the wavelength

The first roots of the various Bessel functions are increasing, that is, $x_{n+1,1} > x_{n,1}$. Therefore, the lowest root of all the x_{nk} is x_{11}, and we want to select for the corresponding γ_{11}. Let the cutoff frequencies be defined by

$$\omega_{nk} = \frac{1}{\sqrt{\epsilon\mu}} \gamma_{nk}$$

and choose $\omega = \omega_0$ in the interval

$$\omega_{11} < \omega_0 < \omega_{12}, \omega_{21}, etc.$$

for all remaining n, k. Then the wave number,

$$k_{nk} = \sqrt{\epsilon\mu}\sqrt{\omega_0^2 - \omega_{nk}^2}$$

is imaginary except for the lowest cutoff frequency,

$$k_{11} = \sqrt{\epsilon\mu}\sqrt{\omega_0^2 - \omega_{11}^2}$$

and the field in the waveguide is given by only J_1,

$$E_z(r, \varphi) = E_0 J_1\left(\frac{x_{01} r}{R}\right) \sin n\varphi e^{i(\pm kz - \omega t)}$$

where the wavelength is

$$\lambda = \frac{2\pi}{\sqrt{\epsilon\mu\omega_0^2 - \frac{x_{11}}{R^2}}}$$

4.5.4 The electric and magnetic fields

For TM waves the fields are given by

$$\begin{aligned}\mathbf{E}_t &= \pm\frac{ik}{\epsilon\mu\omega^2 - k^2}\boldsymbol{\nabla}_t E_z \\ \mathbf{H}_t &= \pm\frac{1}{Z}\hat{\mathbf{k}}\times\mathbf{E}_t\end{aligned}$$

where $\gamma^2 = \epsilon\mu\omega^2 - k^2$ and $Z = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}}$. With $E_z = E_0 J_1\left(\frac{x_{01}r}{R}\right)\sin n\varphi e^{i(kz-\omega t)}$, the transverse derivative gives

$$\begin{aligned}\mathbf{E}_t &= \pm\frac{ik}{\gamma^2}\boldsymbol{\nabla}_t\left(E_0 J_1\left(\frac{x_{01}r}{R}\right)\sin n\varphi e^{i(\pm kz-\omega t)}\right) \\ &= \pm\frac{ikE_0}{\gamma^2}\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial}{\partial\varphi}\right)\left(J_1\left(\frac{x_{01}r}{R}\right)\sin n\varphi\right)e^{i(\pm kz-\omega t)} \\ &= \pm\frac{ikE_0}{\gamma^2}\left(\frac{x_{01}}{R}\hat{\mathbf{r}}\frac{\partial J_1(x)}{\partial x}\sin n\varphi + \hat{\boldsymbol{\varphi}}\frac{n}{r}J_1\left(\frac{x_{01}r}{R}\right)\cos n\varphi\right)e^{i(\pm kz-\omega t)}\end{aligned}$$

and the magnetic field follows as

$$\mathbf{H}_t = \frac{ikE_0}{Z\gamma^2}\left(\hat{\boldsymbol{\varphi}}\frac{x_{01}}{R}\frac{\partial J_1(x)}{\partial x}\sin n\varphi - \hat{\mathbf{r}}\frac{n}{r}J_1\left(\frac{x_{01}r}{R}\right)\cos n\varphi\right)e^{i(\pm kz-\omega t)}$$